A TREATISE ON THE INTEGRAL CALCULUS

VOLUME I.

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A TREATISE

ON THE

GRAL CALCULUS

TH APPLICATIONS, EXAMPLES
AND PROBLEMS

BY

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ABBREVIATIONS USED IN THE REFERENCES.

Ox. I. P. or Ox. II. P., etc. = First or Second Public Examination, Oxford University.

Math. Trip. I. or Math. Trip. II. = Mathematical Tripos Examination, Cambridge University, Parts I. or II.

Ox. J. M. S. = Oxford, Junior Mathematical Scholarship.

Colleges a, etc.

To indicate the sources from which many of the Examples are derived in cases where a group of Cambridge Colleges have held an examination in common, the references are abbreviated as follows:

(α) = St. Peter's, Pembroke, Corpus Christi, Queen's and St. Catharine's.

(β) = Clare, Caius, Trinity Hall and King's.

(γ) = Jesus, Christ's, Magdalen, Emmanuel and Sidney Sussex.

(δ) = Jesus, Christ's, Emmanuel and Sidney Sussex.

(ϵ) = Clare, Caius and King's.

I.C.S. = Examination for the Indian Civil Service and Home Office Clerkships, Grade I.

(R. P.) = Set in problem paper to his classes by the late Dr. Routh, possibly taken from examination papers or possibly original. Source unknown to the present author.

L. = London University Examinations.

E. F. = *Elliptic Functions.*

C. I. = *Calcul Intégral.*

References to *Diff. Calc.* are to the author's larger *Treatise on the Differential Calculus.*

CHAPTER I.

NATURE OF THE PROBLEM. PRELIMINARY CONSIDERATIONS.

1. INTEGRATION is a reversal of the operation of Differentiation, the finding of a function of x when the differential coefficient is known. Thus the differential coefficient of x^2e^x, say, is $(2x+x^2)e^x$. We require a method of retracing our steps, and having given the expression $(2x+x^2)e^x$, we aim at the formulation of a method of arriving at the original function x^2e^x. The result of integrating a function of x is called the integral of the function.

2. In the language of the early writers on the subject, a differential coefficient was called a "fluxion." The original expression regarded as derived from the differential coefficient was called the "fluent."

Thus, in Kinetics, if s be the space described by a particle moving with a uniform acceleration f in time t, and with initial velocity u, $s=ut+\frac{1}{2}ft^2$, and the velocity at any time is given by $v=u+ft$. We obtain, by differentiating these expressions,

$$\frac{dv}{dt}=f, \quad \frac{ds}{dt}=u+ft.$$

So f is the differential coefficient (or "fluxion") of v with regard to t,

$u+ft$ is the differential coefficient (or "fluxion") of s with regard to t.

Regarding $u+ft$ and f as the original quantities, their integrals with regard to t (*i.e.* their "fluents") are respectively $ut+\frac{1}{2}ft^2$, *i.e.* s, and $ft+u$, *i.e.* v.

3. It will be noted that, as a constant quantity has no "rate of variation," all unattached constants, *i.e.* constants which do not multiply variables, as for instance u in the formula $v=u+ft$, disappear on differentiation. We may therefore expect constants to reappear upon integration.

Thus it appears that the differential coefficient with regard to time (or "fluxion") of a length, or distance, is a velocity or rate of change of the length. The integral with regard to time (or "fluent") of a velocity is a length. In other words, the problem of the Differential Calculus is, given any quantity which is changing its value continuously, to find the rate of that change; whilst the problem to be attacked in the Integral Calculus is the converse, viz., given the rate of change, to find what the nature of the varying quantity must be.

4. The general character of integration is necessarily tentative. Newton remarked in his *Method of Fluxions*, "It may not be amiss to take notice, that in the Science of Computation all the Operations are of two kinds, either Compositive or Resolutative. The Compositive or Synthetic Operations proceed necessarily and directly, in computing their several *quaesita*, and not tentatively or by way of tryal. Such are Addition, Multiplication, Raising of Powers, and taking of Fluxions. But the Resolutative or Analytical Operations, as Subtraction, Division, Extraction of Roots, and finding of Fluents, are forced to proceed indirectly and tentatively, by long deduction, to arrive at their several *quaesita*; and suppose or require the contrary Synthetic Operations, to prove and compare every step of the process. The Compositive Operations, always when the *data* are finite and terminated, and often when they are interminate or infinite, will produce finite conclusions; whereas, very often in the Resolutative Operations, tho' the *data* are in finite Terms, yet the *quaesita* cannot be obtain'd without an infinite Series of Terms."

5. We have illustrated the object of integration from the fundamental equations of motion of a particle moving with a constant acceleration and with a given initial velocity. This is sufficient for the present. But it will be seen later that the reversal of the operation of differentiation will also enable us to calculate with precision the areas bounded by curved lines, the lengths of such curved lines, the volumes contained by curved surfaces, the areas of such surfaces and many other quantities which it is necessary to find in both Pure and Applied Mathematics.

6. Before embarking upon the general problem of the reversal of a differential operation, it will be instructive to the student to consider how such a reversal could be used in such a problem as the discovery of the area of a space bounded by curved lines.

The plan adopted for this purpose is to imagine the area divided into a very large number of very small elements according to some fixed principle of division. We have then to devise some method of obtaining the limit of the sum of all these elements when each is ultimately infinitesimally small, and at the same time their number is indefinitely increased. And when once such a method of summation is discovered it will be found to be applicable also to many other problems, such as those already mentioned of finding the lengths of specified portions of curves, volumes bounded by specific surfaces, the positions of centroids, etc.

7. In some elementary cases it will be found that the requisite summation can be performed by ordinary algebraical or trigonometrical means. But such processes will be generally tedious and almost always inadequate to the treatment of any but the simplest examples.

A fundamental theorem will, however, be established showing how this summation depends upon the *reversal of a differentiation*. We shall therefore, after a few illustrations, confine our attention for several chapters mainly to the purely analytical problem of reversing the fundamental operation of the Differential Calculus, with the end explained in view. And when the student is well equipped with this powerful weapon we shall proceed to discuss more fully the uses to which the process may be applied.

8. To avoid constant repetition, we may state that throughout the book all coordinate axes will be supposed rectangular, all angles will be supposed measured in circular measure, all logarithms will be supposed Napierian except where otherwise expressly stated, and for the present all variables will be supposed real and all functions will be considered continuous functions of a real variable.

9. Newton's Second Lemma.

In the First Section of the *Principia* (Lemma II.), Newton enunciates and proves the following Theorem:*

If in any figure Aab ... kL bounded by the straight lines Aa, AL and the curve abc ... kL any number of parallelograms Ab, Bc, Cd, etc., be inscribed upon equal bases AB, BC, CD, etc., and having sides Bb, Cc, Dd, etc., parallel to the side Aa of the figure, and the parallelograms aPbp, bQcq, cRdr, etc., be completed; then, if the breadth of these parallelograms be diminished and the number increased indefinitely, the ultimate ratios which the inscribed figure APbQcRdS ... kK, the circumscribed figure Aapbqcrd ... ykzL and the curvilinear figure Aabcde ... kL have to one another are ratios of equality.

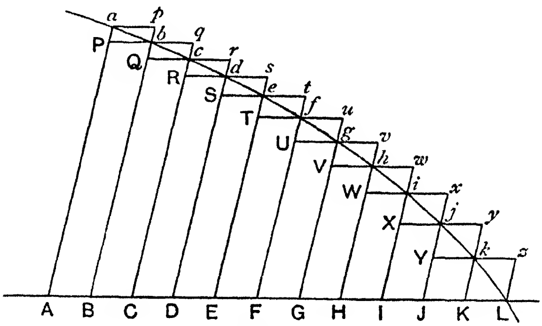


Fig. 1.

To prove this statement it may be observed that the difference of the sums of the inscribed and circumscribed rectilineal figures is the sum of the parallelograms *Pp*, *Qq*, *Rr*, ..., *Kz*; and as the bases *Pb*, *Qc*, ... , *KL* of these parallelograms are all equal and their aggregate altitude is the sum of their individual altitudes, the sum of these parallelograms is equal to the parallelogram *Ap*. And in the limit, when the bases *AB BC*, ... , are diminished indefinitely, the area of this parallelogram which has a finite altitude and indefinitely small breadth becomes less than anything conceivable, however small. Hence the inscribed and circumscribed figures, and therefore also the curvilinear figure whose area is intermediate between the areas of these figures, in the limit become ultimately equal

* See Frost's *Newton's Principia*, pages 17, 18.

10. Newton devotes the next Lemma (III.) to proving that "the same ultimate ratios are also ratios of equality when the breadths of the parallelograms, $AB, BC, CD, \ldots$ are *unequal*, and are all diminished indefinitely."

This is proved in like manner, and may be established by the student.

It follows that the limit of the sum of either the inscribed parallelograms or of the parallelograms which make up the circumscribed figure ultimately coincides in area with that of the curvilinear figure itself.

11. **Analytical expression of the above result.**

We shall now obtain an analytical expression for the sum of such a system of inscribed parallelograms.

Suppose it be required to find the area of the portion of space bounded by a given curve AB, whose Cartesian Equation is $y=\phi(x)$, the ordinates AL and BM, and the axis of x, the axes being rectangular, and all ordinates from A to B being finite, and for the purposes of this article, increasing or decreasing from A to B.

Following the method of Newton's Second Lemma, let LM be divided into n equal small parts $LQ_1, Q_1Q_2, Q_2Q_3, \ldots$, each of length h; and let a and b be the abscissae of A and B, *i.e.* $OL=a$, $OM=b$. Then $b-a=nh$.

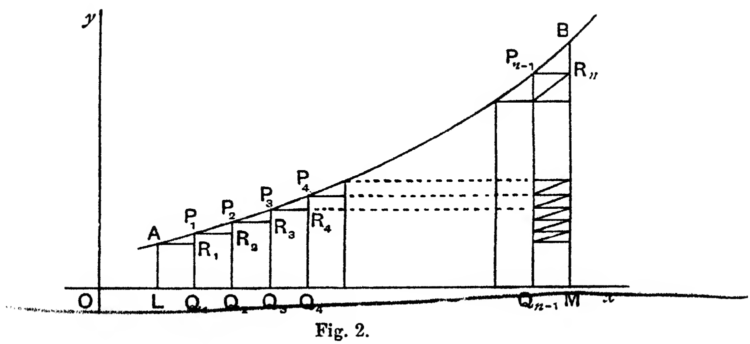


Fig. 2.

The ordinates LA, Q_1P_1, Q_2P_2, etc., $Q_{n-1}P_{n-1}$, MB at the points $L, Q_1, Q_2, \ldots, Q_{n-1}, M$ are respectively

$\phi(a)$, $\phi(a+h)$, $\phi(a+2h)$, $\phi(a+3h)$, $\ldots$, $\phi\{a+(n-1)h\}$, $\phi(b)$.

Complete the rectangles $AQ_1, P_1Q_2, P_2Q_3, \ldots$.

Now the sum of these n rectangles falls short of the area sought by the sum of the n small figures AR_1P_1, $P_1R_2P_2$, etc. Let each of these be supposed to slide parallel to the x-axis into a corresponding position upon the longest strip, say $P_{n-1}Q_{n-1}MB$. Their sum is then less than the area of this strip, *i.e.* in the limit less than an infinitesimal of the first order, for the breadth $Q_{n-1}M$ is h and is ultimately an infinitesimal of the first order, and the length MB is supposed finite.

Hence the area required is the limit when h is zero (and therefore n infinite) of the sum of the n infinitesimal terms of the first order,

$$h\phi(a)+h\phi(a+h)+h\phi(a+2h)+\ldots+h\phi[a+(n-1)h].$$

This sum may be denoted by

$$\mathop{S}_{a+rh=a}^{a+rh=b-h}\phi(a+rh)\,h \quad\text{or}\quad \sum_{a}^{b-h}\phi(a+rh)\,h,$$

where S or Σ denotes the "sum" between the limits indicated.

Regarding $a+rh$ as a variable x, the infinitesimal increment h may be written as δx or dx. It is customary also upon taking the limit to replace the symbol S by the more convenient sign $\int$, which is, as a matter of fact, merely only another way of writing the same letter, and the limit of the above summation when h is diminished indefinitely is then written

$$\int_a^b \phi(x)\,dx,$$

and read as "the integral of $\phi(x)$ with respect to x [or of $\phi(x)\,dx$] between the limits $x=a$ and $x=b$"; or more shortly "the integral of $\phi(x)$ from a to b."

b is called the "upper" or "superior" limit,

a is called the "lower" or "inferior" limit.

12. The sum of $(n+1)$ terms of the same series, viz.,

$$h\phi(a)+h\phi(a+h)+h\phi(a+2h)+\ldots$$
$$+h\phi[a+(n-1)h]+h\phi(a+nh),$$

differs from the above series merely in the addition of the term $h\phi(a+nh)$, *i.e.* $h\phi(b)$, which being an infinitesimal of

the first order vanishes when the limit is taken. Hence the limit of this series may also be written

$$\int_a^b \phi(x)\,dx.$$

13. In the same way, if in fig. 2, Art. 11, LQ_1, Q_1Q_2, Q_2Q_3, ..., $Q_{n-1}M$ are not necessarily equal, but are respectively h_1, h_2, h_3, ..., h_n, the ordinates at the several points L, Q_1, Q_2, ..., Q_{n-1} are respectively,

$$\phi(a), \quad \phi(a+h_1), \quad \phi(a+h_1+h_2), \ldots, \quad \phi(b-h_n)$$

and the sum of the inscribed rectangles is

$$h_1\phi(a)+h_2\phi(a+h_1)+h_3\phi(a+h_1+h_2)+\ldots+h_n\phi(b-h_n),$$

and the sum of the residuary areas AR_1P_1, $P_1R_2P_2$, $P_2R_3P_3$, etc., is less than the area of a rectangle whose breadth is the greatest of the quantities h_1, h_2, h_3 ... h_n, and whose height is the greatest ordinate of the given curve; and as in the last article, this sum therefore vanishes in the limit when h_1, h_2, h_3, ... h_n are each made infinitesimally small, provided that the curve has no infinite ordinate either at A, B or between A and B.

Hence the limit of

$$h_1\phi(a)+h_2\phi(a+h_1)+h_3\phi(a+h_1+h_2)+\ldots+h_n\phi(b-h_n),$$

is also the area of the portion $LABM$ described in Art. 11.

[See also Art. 1875, Vol. II.]

14. The quantities h_1, h_2, h_3, ... h_n may clearly be either independent, or equal, or connected by any arbitrary law, provided only that they each and all become infinitesimally small in the limit when their number is increased indefinitely.

These arbitrary infinitesimals will be chosen equal to each other in general, and the series to be summed will therefore be that of Art. 11.

15. We postpone till later in the chapter the explanation of how this summation is connected with the reversal of a differentiation, and illustrate what has been stated as to the finding of areas by a few elementary cases in which the limit of the summation may be found by elementary processes without undue difficulty.

16. ILLUSTRATIVE EXAMPLES.

Ex. 1. To calculate $\int_a^b ce^{mx}\,dx$, that is to find the area of the space bounded by the x-axis, the logarithmic curve $y=ce^{mx}$ and two ordinates $x=a$ and $x=b$.

Here we have to evaluate

$$Lt_{h=0}\,ch\left[e^{ma}+e^{m(a+h)}+e^{m(a+2h)}+\ldots+e^{m(a+\overline{n-1}h)}\right]$$

where $b=a+nh$.

$$\text{This expression} = Lt_{h=0}\,che^{ma}\frac{e^{nmh}-1}{e^{mh}-1}$$

$$= Lt_{h=0}\,c\frac{e^{ma}}{m}\cdot\frac{mh}{e^{mh}-1}\cdot\left[e^{m(b-a)}-1\right]$$

$$= c\frac{e^{ma}}{m}\cdot 1\cdot\left[e^{m(b-a)}-1\right], \text{ by } \textit{Diff. Cal.} \text{ (Art. 21)},$$

$$= c\frac{e^{mb}-e^{ma}}{m}.$$

$\therefore$ the area sought is equal to the rectangle contained by $\frac{1}{m}$ (which is of the dimension of a line) and the difference of the initial and final ordinates.

E.g. if now $\frac{1}{m}=1$ inch and $a=0$, $b=1$, $c=2$,

the area in question $=2(e-1)=2\times 1{\cdot}71828\ldots$ square inches
$=3{\cdot}43656\ldots$ square inches,
i.e. a little less than $3\frac{1}{2}$ square inches.

Ex. 2. Shew that in the last result, *i.e.* $y=ce^{mx}$, if $A_1, A_2, A_3, \ldots$ be the areas between

$x=0$ and $x=1$, $x=1$ and $x=2$, $x=2$ and $x=3$, etc.,

then $A_1, A_2, A_3, \ldots$ form a G.P. whose common ratio is e^m.

Ex. 3. Calculate the area bounded by the *curve of sines* $y=c\sin mx$, the x-axis and two ordinates $x=a$ and $x=b$ $\left(0<a<b<\frac{\pi}{m}\right)$.

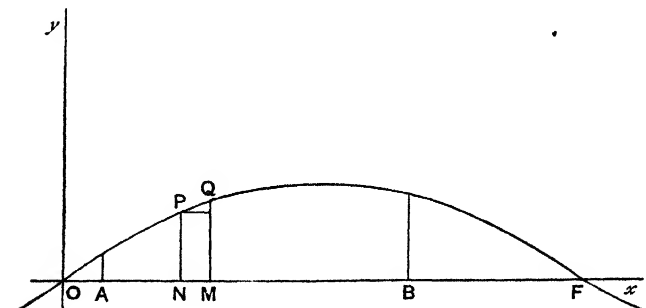


Fig. 3.

Here we are to evaluate $\int_a^b c \sin mx\, dx$,

that is $Lt_{h=0} ch[\sin ma + \sin m(a+h) + \sin m(a+2h) \ldots \text{ to } n \text{ terms}]$

where $nh = b - a.$

This expression $= Lt_{h=0} ch \dfrac{\sin\left\{ma + (n-1)\dfrac{mh}{2}\right\} \sin n\dfrac{mh}{2}}{\sin\dfrac{mh}{2}}$

$$= Lt_{h=0} c\left[\cos m\left(a - \frac{h}{2}\right) - \cos m\left\{a + (2n-1)\frac{h}{2}\right\}\right] \frac{\frac{mh}{2}}{\sin\frac{mh}{2}} \cdot \frac{1}{m}$$

$$= c\frac{\cos ma - \cos mb}{m}.$$

Thus, if the limits are such as to take in one half wave length, *i.e.* the portion above the x-axis from $x=0$ to $mx=\pi$, and if $c=1$ inch, the area sought is

$$\frac{\cos 0 - \cos \pi}{m} = \frac{2}{m},$$

or if, say, $m = \frac{1}{10}$, the area is 20 square inches.

Ex. 4. Find the value of $\int_a^b \frac{x^3}{c^2} dx$; that is the area bounded by the cubical parabola $c^2 y = x^3$, the x-axis and two ordinates $x=a$ and $x=b$.

Here we have to evaluate

$$Lt_{h=0} \frac{1}{c^2} \sum_{r=0}^{r=n-1} (a+rh)^3 h,$$

where $nh = b - a$.

Now
$$\frac{h}{c^2}[a^3 + (a+h)^3 + (a+2h)^3 + \ldots + (a+\overline{n-1}h)^3]$$
$$= \frac{h}{c^2}\left[na^3 + 3a^2 h\frac{(n-1)n}{2} + 3ah^2\frac{(n-1)n(2n-1)}{6} + h^3\frac{(n-1)^2 n^2}{4}\right]$$
$$= \frac{(b-a)}{c^2}\left[a^3 + 3a^2\frac{(b-a)}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{2}a(b-a)^2\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right.$$
$$\left. + \frac{1}{4}(b-a)^3\left(1 - \frac{1}{n}\right)^2\right],$$

and when n becomes infinite this becomes

$$= \frac{(b-a)}{c^2}[a^3 + \tfrac{3}{2}a^2(b-a) + a(b-a)^2 + \tfrac{1}{4}(b-a)^3]$$
$$= \frac{(b-a)}{4c^2}(b^3 + b^2 a + ba^2 + a^3)$$
$$= \frac{b^4 - a^4}{4c^2}.$$

Ex. 5. Find $$\int_a^b \frac{1}{x^2} dx.$$

We have to evaluate

$$Lt_{h=0} h\left[\frac{1}{a^2} + \frac{1}{(a+h)^2} + \frac{1}{(a+2h)^2} + \ldots + \frac{1}{b^2}\right].$$

This is $> Lt\left[\frac{1}{a(a+h)}+\frac{1}{(a+h)(a+2h)}+\dots+\frac{1}{b(b+h)}\right]h,$

i.e. $> Lt\left[\left(\frac{1}{a}-\frac{1}{a+h}\right)+\left(\frac{1}{a+h}-\frac{1}{a+2h}\right)+\dots+\left(\frac{1}{b}-\frac{1}{b+h}\right)\right],$

i.e. $> Lt\left(\frac{1}{a}-\frac{1}{b+h}\right),$

and $< Lt\left[\frac{1}{(a-h)a}+\frac{1}{a(a+h)}+\dots+\frac{1}{(b-h)b}\right]h$

i.e. $< Lt\left[\left(\frac{1}{a-h}-\frac{1}{a}\right)+\left(\frac{1}{a}-\frac{1}{a+h}\right)+\dots+\left(\frac{1}{b-h}-\frac{1}{b}\right]\right),$

i.e. $< Lt\left(\frac{1}{a-h}-\frac{1}{b}\right),$

and when h diminishes without limit, each of these expressions becomes $\frac{1}{a}-\frac{1}{b}$. Thus the value is entrapped between two ultimately equal expressions, and $\int_a^b \frac{1}{x^2}dx=\frac{1}{a}-\frac{1}{b}$.

Ex. 6. **Integration of x^m**, from the definition, between limits a and b $(m \neq -1)$.

Here we have to consider

$$Lt_{h=0}h[a^m+(a+h)^m+(a+2h)^m+\dots+(a+\overline{n-1}h)^m],$$

where $\frac{b-a}{n}=h$ and n is indefinitely large, $m+1$ not being zero.

In the *Differential Calculus for Beginners* (Art. 13) it is proved without the aid of the Binomial Theorem [which was purposely avoided, as it was then proposed later to apply Taylor's Theorem to the expansion of $(x+h)^n$] that

$$Lt_{z=1}\frac{z^{m+1}-1}{z-1}=m+1.$$

Writing $$z=1+\frac{h}{y},$$

we have $$Lt_{h=0}\frac{\left(1+\frac{h}{y}\right)^{m+1}-1}{\frac{h}{y}}=m+1,$$

or $$Lt_{h=0}\frac{(y+h)^{m+1}-y^{m+1}}{hy^m}=m+1.$$

In this result put y successively $a, a+h, a+2h, \dots, a+(n-1)h$, and we get

$$Lt_{h=0}\frac{(a+h)^{m+1}-a^{m+1}}{ha^m}=Lt_{h=0}\frac{(a+2h)^{m+1}-(a+h)^{m+1}}{h(a+h)^m}=\dots$$
$$=Lt_{h=0}\frac{(a+nh)^{m+1}-(a+\overline{n-1}h)^{m+1}}{h(a+\overline{n-1}h)^m}=m+1,$$

or, adding numerators for a new numerator and denominators for a new denominator,

$$Lt\,\frac{(a+nh)^{m+1}-a^{m+1}}{h[a^m+(a+h)^m+(a+2h)^m+\dots+(a+\overline{n-1}h)^m]}=m+1,$$

i.e. $Lt_{h=0}\,h[a^m+(a+h)^m+(a+2h)^m+\dots+(a+\overline{n-1}h)^m]=\dfrac{b^{m+1}-a^{m+1}}{m+1}$,

i.e. in accordance with the notation of Art. 11,

$$\int_a^b x^m\,dx=\frac{b^{m+1}-a^{m+1}}{m+1}.$$

The letters a and b may represent any finite quantities whatever, provided x^m does not become ∞ between $x=a$ and $x=b$.

When a is taken exceedingly small and ultimately zero it is necessary in the proof to suppose h an infinitesimal of higher order, for it has been assumed that in the limit $\dfrac{h}{y}$ is zero for all the values given to y.

When $b=1$ and $a=0$, the theorem ultimately becomes

$$\int_0^1 x^m\,dx=\frac{1}{m+1}\quad\text{if } (m+1) \text{ be positive,}$$

$$\text{or } =\infty\quad\text{if } (m+1) \text{ be negative.}$$

This result may be written also

$$Lt_{n=\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^m+\left(\frac{2}{n}\right)^m+\left(\frac{3}{n}\right)^m+\dots+\left(\frac{n-1}{n}\right)^m\right]=\frac{1}{m+1},\quad\text{or }\infty$$

according as $m+1$ is positive or negative.

The Limit $\qquad Lt_{n=\infty}\dfrac{1}{n}\left[\left(\dfrac{1}{n}\right)^m+\left(\dfrac{2}{n}\right)^m+\dots+\left(\dfrac{n}{n}\right)^m\right]$,

or, which is the same thing,

$$Lt_{n=\infty}\frac{1^m+2^m+3^m+\dots+n^m}{n^{m+1}}$$

differs from the former by $\dfrac{1}{n}$, *i.e.* by 0 in the limit, and is therefore also $\dfrac{1}{m+1}$, or ∞, according as $m+1$ is positive or negative.

The case when $m+1=0$ needs special consideration. It is at once derivable from the result

$$\int_a^b x^m\,dx=\frac{b^{m+1}-a^{m+1}}{m+1}$$

as a limiting form.

$$\begin{aligned}Lt_{m+1=0}\int_a^b x^m\,dx&=Lt_{m+1=0}\frac{b^{m+1}-a^{m+1}}{m+1}\\&=Lt\,\frac{b^{m+1}-1}{m+1}-Lt\,\frac{a^{m+1}-1}{m+1}\\&=\log b-\log a\quad(\textit{Diff. Cal.}\text{ Art. 21})\\&=\log\frac{b}{a}.\end{aligned}$$

EXAMPLES.

1. Find the values of $\int_a^b x\,dx$ and $\int_a^b x^2\,dx$, and interpret the results geometrically.

2. Find the area of the portion of the parabola $x^2=4ay$ cut off by the latus rectum.

3. Prove by summation that

$$(\alpha)\ \int_a^b \sinh x\,dx=\cosh b-\cosh a\,;$$

$$(\beta)\ \int_a^b \frac{1}{\sqrt{x}}\,dx=2(\sqrt{b}-\sqrt{a})\,;$$

$$(\gamma)\ \int_a^b \cos mx\,dx=\frac{1}{m}(\sin mb-\sin ma).$$

4. In a right circular cone of height h and semivertical angle α, the axis is divided into a large number, n, of equal portions, and planes are drawn through the points of division perpendicular to the axis, the cone being thus divided into a large number of circular laminae. If x be the distance from the vertex of any of these laminae, show that to the first order of small quantities its volume may be written

$\pi x^2\tan^2\alpha\,\delta x$, δx being the thickness of the lamina.

Find, by taking the limit of the summation of such quantities, the volume of the cone.

Show also that the volume of a frustum of thickness T is

$$\frac{T}{3}(A+\sqrt{AB}+B),$$

where A and B are the areas of the two ends.

5. A quantity y is an unknown function of another quantity x. When x has the values

5 8 10 12 14 16

y is found by observation to be

2·0 2·6 3·2 3·8 5·0 6·5

respectively, and the errors of observation cannot be more than 5 per cent.; draw the simplest continuous curve which can represent y, and estimate its slope when $x=15$.

Find also the value of x for which the slope of the curve is equal to $\frac{y}{x}$. Estimate the value of the definite integral $\int_{11}^{15} y\,dx$.

17. THE FUNDAMENTAL PROPOSITION.

Let $\phi(x)$ be any function of a real variable x, finite, continuous and single valued, for all values of x from $x=a$ to $x=b$ inclusive. Let a be less than b, each being finite, and

suppose the difference $b-a$ to be divided into n portions each equal to h, so that $b-a=nh$. It is required to find the *limit of the sum* of the series

$$h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b-h)+\phi(b)]$$

when h is diminished indefinitely, and therefore n increased without limit, keeping the product $nh=b-a$.

That this limit is finite may at once be made clear.

For if $h\phi(a+rh)$, say, be the greatest term, the sum is

$$<(n+1)h\phi(a+rh),$$

$$i.e. \quad <(b-a)\phi(a+rh)+h\phi(a+rh),$$

which is finite, since by hypothesis $\phi(x)$ is finite for all values of x intermediate between b and a.

Let $\psi(x)$ be another function of x such that $\phi(x)$ is its differential coefficient, *i.e.* such that

$$\phi(x)=\frac{d}{dx}\psi(x)=\psi'(x).$$

We shall then prove that

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b)]=\psi(b)-\psi(a).$$

By definition,

$$\phi(a)=Lt_{h=0}\frac{\psi(a+h)-\psi(a)}{h},$$

and therefore $\qquad \phi(a)=\dfrac{\psi(a+h)-\psi(a)}{h}+a_1,$

where a_1 is a quantity whose limit is zero when h diminishes indefinitely; thus

$$h\phi(a) \qquad =\psi(a+h) \quad -\psi(a) \qquad +ha_1.$$

Similarly,

$$h\phi(a+h) \qquad =\psi(a+2h)-\psi(a+h) \qquad +ha_2,$$

$$h\phi(a+2h) \qquad =\psi(a+3h)-\psi(a+2h) \qquad +ha_3,$$

etc.,

$$h\phi\{a+(n-1)h\}=\psi(a+nh)-\psi\{a+(n-1)h\}+ha_n,$$

where the quantities $a_2, a_3, \ldots, a_n$ are all, like a_1, quantities whose limits are zero when h diminishes indefinitely.

By addition,

$$h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b-h)]$$

$$=\psi(a+nh)-\psi(a)+h[a_1+a_2+\ldots+a_n].$$

Let a be the greatest of the quantities $a_1, a_2, \dots, a_n$.

Then $\quad h[a_1+a_2+\dots+a_n]$ is $<nha$,

that is $\quad <(b-a)a$,

and therefore vanishes in the limit.

Thus

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\dots+\phi(b-h)]$$
$$=\psi(b)-\psi(a).$$

The term $h\phi(b)$ is itself also in the limit zero; hence, if we desire, it may be added to the left-hand member of this result, without affecting it; and it may then be stated that

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\dots+\phi(b-h)+\phi(b)]$$
$$=\psi(b)-\psi(a),$$

$$i.e.\ \int_a^b \phi(x)\,dx=\psi(b)-\psi(a),$$

where $$\frac{d\psi}{dx}=\phi(x).$$

The result $\psi(b)-\psi(a)$ is frequently denoted by

$$\Big[\psi(x)\Big]_a^b.$$

From this result it appears that when the form of the function $\psi(x)$, of which $\phi(x)$ is the differential coefficient, is obtained, the *process of algebraic or trigonometric summation to obtain* $\int_a^b\phi(x)dx$ *may be avoided.*

18. The letters b and a are supposed in the above work to denote *finite* quantities. We shall now *extend the notation* so as to let $\int_a^\infty\phi(x)dx$ express the limit when b becomes infinitely large of $\psi(b)-\psi(a)$, *i.e.*

$$\int_a^\infty \phi(x)dx=Lt_{b=\infty}\int_a^b\phi(x)dx.$$

Similarly, by $\int_\infty^b\phi(x)dx$ we shall be understood to mean

$$Lt_{a=\infty}[\psi(b)-\psi(a)] \quad\text{or}\quad Lt_{a=\infty}\int_a^b\phi(x)dx.$$

ILLUSTRATIVE EXAMPLES.

Taking the same examples as have been already considered otherwise in Art. 16,

1. ce^{mx} is the differential coefficient of $\frac{c}{m}e^{mx}$.

Therefore $$\int_a^b ce^{mx}\,dx = \frac{c}{m}(e^{mb} - e^{ma}),$$
the result obtained in Ex. 1, p. 8.

2. $c \sin mx$ is the differential coefficient of $-\frac{c}{m}\cos mx$.

Therefore $$\int_a^b c\sin mx\,dx = \left[-\frac{c}{m}\cos mx\right]_a^b = \frac{c}{m}(\cos ma - \cos mb),$$
the result of Ex. 3, p. 9.

3. $\frac{x^3}{c^2}$ is the differential coefficient of $\frac{x^4}{4c^2}$.

Therefore $$\int_a^b \frac{x^3}{c^2}dx = \frac{b^4 - a^4}{4c^2},$$
the result of Ex. 4 of p. 9.

4. $\frac{1}{x^2}$ is the differential coefficient of $-\frac{1}{x}$.

Therefore $$\int_a^b \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_a^b = \frac{1}{a} - \frac{1}{b},$$
the result of Ex. 5 of p. 10.

Comparing these solutions with those of the same problems of Art. 16, the student will at once see the advantage derived from a use of the fundamental proposition of Art. 17.

5. $\frac{1}{x}$ is the differential coefficient of $\log x$.

Therefore $$\int_a^b \frac{1}{x}dx = \Big[\log x\Big]_a^b = \log b - \log a = \log\frac{b}{a}.$$

6. $+e^{-x}$ is the differential coefficient of $-e^{-x}$.

Therefore $$\int_0^\infty e^{-x}\,dx = Lt_{\substack{b=\infty\\a=0}}\Big[-e^{-x}\Big]_a^b = (-e^{-\infty}) - (-e^0) = 1$$

EXAMPLES.

1. Write down the values of

(1) $\int_0^1 x\,dx$, $\int_0^1 x^2\,dx$, $\int_0^1 x^5\,dx$, $\int_0^1 x^n\,dx$;

(2) $\int_0^{\frac{\pi}{2}} \sin x\,dx$, $\int_0^{\frac{\pi}{2}} \cos x\,dx$, $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$, $\int_0^{\frac{\pi}{4}} \sec x\tan x\,dx$;

(3) $\int_0^1 \frac{1}{1+x^2}dx$, $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$, $\int_0^1 \frac{1}{1+x}dx$, $\int_0^1 e^x\,dx$;

and interpret each result geometrically as the evaluation of an area.

19. Geometrical Illustration of Proof.

The proof of the above theorem of Art. 17 may be interpreted geometrically thus:

Let AB be a portion of a curve, of which the ordinate is finite and continuous at all points between A and B, as also the tangent of the angle which the tangent to the curve makes with the x-axis.

Let the abscissae of A and B be a and b respectively. Draw the ordinates AN, BM. Let the portion NM be divided into n equal parts, each of length h. Erect ordinates at each of these points of division, cutting the curve in P, Q, R, ... , etc. Draw the successive tangents AP_1, PQ_1, QR_1, etc., and the lines AP_2, PQ_2, QR_2, etc., *parallel* to the x-axis, and let the equation of the curve be $y=\psi(x)$, where $\psi'(x)=\phi(x)$.

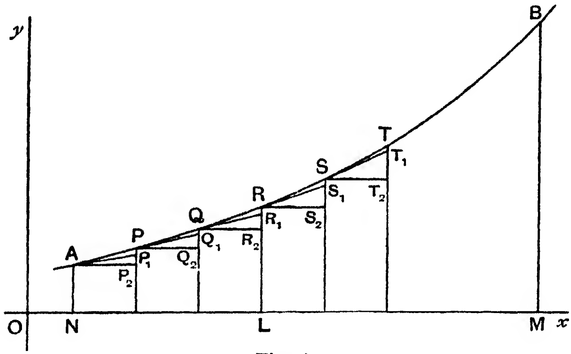


Fig. 4.

Then $\qquad \phi(a), \qquad \phi(a+h), \qquad \phi(a+2h)$, etc.,
are respectively

$$\psi'(a), \qquad \psi'(a+h), \qquad \psi'(a+2h), \text{ etc.,}$$

$$i.e. \quad \tan P_2AP_1, \quad \tan Q_2PQ_1, \quad \tan R_2QR_1, \text{ etc.,}$$

and $h\phi(a)$, $h\phi(a+h)$, etc., are respectively

the lengths P_2P_1, Q_2Q_1, R_2R_1, etc.

Now, it is clear that the algebraical sum of

$$P_2P, \quad Q_2Q, \quad R_2R, \ \dots,$$

is $\qquad MB-NA, \quad i.e. \ \psi(b)-\psi(a).$

Hence

$$(P_2P_1+Q_2Q_1+R_2R_1+\dots)+[P_1P+Q_1Q+\dots]=\psi(b)-\psi(a).$$

Now, the portion between square brackets may be shown to diminish indefinitely with h. For if R_1R, for instance, be the greatest of the several quantities P_1P, Q_1Q, etc., the sum

$$[P_1P+Q_1Q+\ldots] \text{ is } < nR_1R, \quad i.e. < (b-a)\frac{R_1R}{h}.$$

But if the abscissa of Q be called x, then

$$LR_2=\psi(x), \quad R_2R_1=h\psi'(x),$$

and $$LR=\psi(x+h)=\psi(x)+h\psi'(x)+\frac{h^2}{\underline{|2}}\psi''(x+\theta h)$$

(*Diff. Cal.* Art. 130),

so that $$R_1R=\frac{h^2}{\underline{|2}}\psi''(x+\theta h)=\frac{h^2}{2}\phi'(x+\theta h),$$

and $$(b-a)\frac{R_1R}{h}=\frac{b-a}{2}h\phi'(x+\theta h),$$

which is an infinitesimal in general of the first order.

Thus $\quad Lt_{h=0}(P_2P_1+Q_2Q_1+R_2R_1+\ldots)=\psi(b)-\psi(a)$,

or

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b-h)]=\psi(b)-\psi(a).$$

Also, since $Lt_{h=0}h\phi(b)=0$, we have, by addition,

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b)]=\psi(b)-\psi(a).$$

20. Case of an Unknown Curve passing through a given system of Points.

In a certain graph, such, for instance, as the graph on a temperature chart, the temperature being noted at stated intervals, the following table gives the corresponding abscissae and ordinates of eleven points on the curve:

x	1	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2
y	·900	·879	·856	·831	·804	·775	·744	·711	·676	639	·600

On the assumption that the graph is that of a continuous function of x, and the ordinate continually decreasing in the intervals between the several stated values, it is required to calculate $\int_1^2 y\,dx$, *i.e.* to find the area bounded by the curve, the x-axis and the extreme ordinates.

Constructing the inscribed and circumscribed parallelograms as explained in Art. 9,

The sum of the circumscribed figures is

$$\cdot1\times[\cdot9+\cdot879+\cdot856+\ldots+639]=\cdot7815;$$

The sum of the inscribed figures is

$$\cdot 1 \times [\cdot 879 + \cdot 856 + \ldots + \cdot 639 + \cdot 600] = \cdot 7515.$$

The first is clearly too large by the sum of the ten small triangular-shaped elements outside the area to be found.

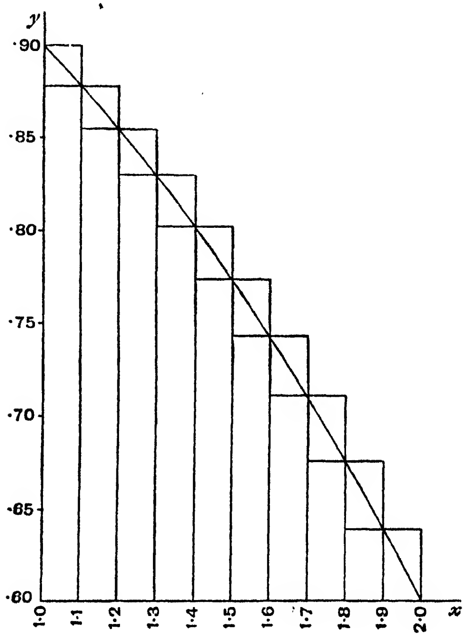


FIG. 5.

The second is too small by the sum of the ten triangular-shaped elements which are omitted.

The mean of these results, viz. $\frac{\cdot 7815 + \cdot 7515}{2} = \cdot 7665$, will be a much closer approximation, but will be a little too small, because it omits the very small areas which lie between the *chords* which join successive points on the graph and the corresponding arcs.

Hence, as a closer approximation, we may take

$$\int_1^2 y\,dx = \cdot 7665 \text{ square units.}$$

[From a finite number of ordinates it is impossible to assign the equation to the curve, but it is customary to take the *simplest algebraic* curve which satisfies the prescribed conditions. In the present case the simplest curve to fit the data will be found to be $y = 1 - \frac{x^2}{10}$.

Any other curve of the form

$$y=1-\frac{x^2}{10}+(x-1)(x-1\cdot1)(x-1\cdot2)\quad(x-2)\phi(x)$$

where $\phi(x)$ is any integral algebraic expression, would go through the same points, but is much more complicated.

The true area on the supposition of the curve being $y=1-\frac{x^2}{10}$ will be found by the result of Art. 16, Ex. 6, to be $\left[x-\frac{x^3}{30}\right]_1^2$, *i.e.* $\frac{23}{30}$, or ·7666..., which shows errors as follows:

In the first estimate, - ·0148 in excess, *i.e.* a 1·9 % error in excess,
" second " - ·0152 in defect, *i.e.* a 2·0 % error in defect,
" mean " - ·0002 in defect, *i.e.* a 0·03 % error in defect.]

21. Simpson's Rule.

If a curve be partially defined as passing through an odd number of points whose abscissae are in arithmetical progression, *e.g.* the points

$$(a, y_1),\quad (a+h, y_2),\quad (a+2h, y_3)\ldots(a+\overline{n-1}h, y_n),$$

and if the same assumptions be made as in the last article as to continuity, etc., it is possible to find a very close approximation to the area of the curve, which is useful in many practical cases, as follows:

Consider first the case of the parabola whose equation is

$$y=a+bx+cx^2,$$

and let a, b, c be chosen so as to make this curve go through

$$(-h, y_1),\quad (0, y_2),\quad (h, y_3).$$

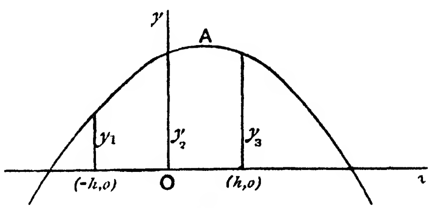


Fig. 6.

Then

$$\left.\begin{aligned} a-bh+ch^2&=y_1,\\ a\qquad\quad&=y_2,\\ a+bh+ch^2&=y_3.\end{aligned}\right\}$$

So that $a=y_2$, $b=\frac{y_3-y_1}{2h}$, $c=\left(\frac{y_1+y_3}{2}-y_2\right)/h^2$.

Now the area bounded by the x-axis, the parabola and the ordinates y_1 and y_2 is, by Art. 16, Ex. 6,

$$\int_{-h}^{h}(a+bx+cx^2)\,dx=\left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right]_{-h}^{h}$$
$$=2ah+\frac{2c}{3}h^3$$
$$=h\{2y_2+\tfrac{1}{3}(y_1-2y_2+y_3)\}$$
$$=\frac{h}{3}(y_1+4y_2+y_3).$$

If we apply this rule to the case in question, passing parabolic arcs through the (1st, 2nd, 3rd points), (3rd, 4th, 5th), (5th, 6th, 7th), etc., we have the following approximative rule, viz.

$$\begin{aligned}A=\frac{h}{3}[y_1+4y_2+y_3&\\ &+y_3+4y_4+y_5\\ &\qquad+y_5+4y_6+y_7\\ &\qquad\qquad+\dots+y_{n-2}+4y_{n-1}+y_n]\end{aligned}$$
$$=\frac{h}{3}[y_1+4y_2+2y_3+4y_4+2y_5+4y_6+\dots+4y_{n-1}+y_n]$$
$$=\frac{h}{3}[y_1+y_n+2(y_3+y_5+y_7+\dots)+4(y_2+y_4+y_6+\dots)],$$

i.e. $\frac{h}{3}$ (sum of first and last + twice sum of all other odd ordinates + four times the sum of the even ordinates).

This is known as Simpson's Rule. It will be noticed that it consists in the division of the area by an odd number of equidistant ordinates, and the substitution of *parabolic* arcs for the *actual* but *unknown* arcs passing through consecutive groups of 3 points.

Other approximations can be found. Thus we may take a curve $y=a+bx+cx^2+dx^3$ to pass through 4 consecutive points, or $y=a+bx+cx^2+dx^3+ex^4$ to pass through 5 consecutive points, and so on, and thus build up similar rules. Simpson's Rule, however, in most cases gives a sufficiently close approximation for ordinary purposes. (See Examples 27, 28, page 33.)

22. The Trapezoidal Rule and Weddle's Rule.

The approximation previously adopted in Art. 20 of the mean of the inscribed and circumscribed rectangles may be expressed in similar manner, as

$$h\left(\frac{y_1+y_2}{2}+\frac{y_2+y_3}{2}+\frac{y_3+y_4}{2}+\dots+\frac{y_{n-1}+y_n}{2}\right)$$

$$=\frac{h}{2}(y_1+2y_2+2y_3+2y_4+\dots+2y_{n-1}+y_n)$$

$$=\frac{h}{2}(\text{sum of first and last ordinates}+\text{twice the sum of all the rest}),$$

which is a convenient form, but not usually so accurate as Simpson's Rule.

It consists, as already explained, of substituting chords joining consecutive points for their arcs, and as we are summing a series of Trapezoids this is known as the Trapezoidal Rule.

23. **Other Approximative Rules.**

Other rules will be found in Examples 27, 28 at the end of this chapter, and in Examples 24, 25, 26, page 61.

A very convenient rule was given by Weddle, *Math. Journal*, vol. ix., for the case where there are *seven equidistant ordinates*, $y_1, y_2, y_3, \dots, y_7$ at mutual distances h, viz.

$$\tfrac{3}{10}h[y_1+y_3+y_5+y_7+5(y_2+y_4+y_6)+y_4],$$

i.e. $\frac{3}{10}\times$ mutual distance $[\Sigma$ odds $+\,5\,\Sigma$ evens $+$ middle$]$.

(Weddle's Rule.)

We transcribe this for convenience, but the proof is one most conveniently treated by finite difference methods. It will be found in Boole's *Finite Differences*, pages 47-48.

Boole remarks that in all applications of such approximate formulae "it is desirable to avoid extreme differences among the ordinates."

Ex. Apply the Trapezoidal Rule, Simpson's Rule and Weddle's Rule to find the area bounded by the x-axis, the extreme ordinates and the arc of a circle through the seven points:

$x=$	$-\frac{3}{6}$	$-\frac{2}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$
$y=$	·86602	·94281	·98614	1	·98614	·94281	·86602

First and last.	2^{nd}, 4^{th} and 6^{th}.	3^{rd} and 5^{th}.
·86602	·94281	·98614
·86602	1·00000	·98614
1·73204	·94281	1·97228
	2·88562	2
	4	3·94456
	11·54248	

For Trapezoidal Rule, $\text{Area} = \frac{1}{6}(\cdot 86602 + 2\cdot 88562 + 1\cdot 97228)$
$= \frac{1}{6}(5\cdot 72392) = \underline{\cdot 95398.}$

For Simpson's Rule, $\text{Area} = \frac{1}{18}(1\cdot 73204 + 3\cdot 94456 + 11\cdot 54248)$
$= \frac{1}{18}(17\cdot 21908)$
$= \underline{\cdot 95661.}$

For Weddle's Rule, $\text{Area} = \frac{1}{20}(3\cdot 70432 + 14\cdot 42810 + 1\cdot 00000)$
$= \underline{\cdot 95662}$

This area, being the area of that part of a semicircle whose centre is at the origin and radius unity bounded by two ordinates $x = \cdot 5$, $x = -\cdot 5$, may be seen to have its area correctly $= \frac{\pi}{6} + \frac{\sqrt{3}}{4} = \cdot 956611\ldots$, and therefore Simpson's Rule gives a result accurate to the last figure.

[See Boole, *Finite Differences*, p. 49.]

The approximation by Weddle's Rule does not appreciably differ from that by Simpson's Rule.

The Trapezoidal Rule errs in defect by $\cdot 00263$, *i.e.* by about $\cdot 3$ % of the whole.

24. Determination of a Volume of Revolution.

Let it be required to find the volume formed by the revolution of a given curve AB about an axis in its own plane which it does not cut.

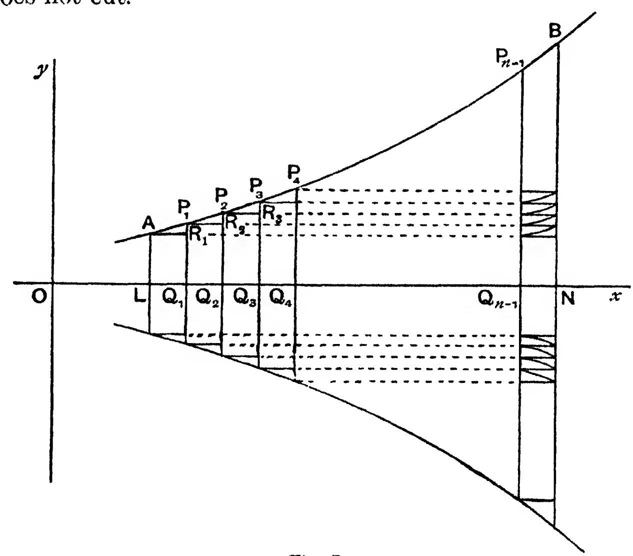


Fig. 7.

Taking the axis of revolution as the x-axis, the figure may be described exactly as in Art. 11. The elementary rectangles

AQ_1, P_1Q_2, P_2Q_3, etc., trace in their revolution circular discs of equal thickness and of volumes $\pi AL^2 . LQ_1$, $\pi P_1Q_1^2 . Q_1Q_2$, etc. The several annular portions formed by the revolution of the portions AR_1P_1, $P_1R_2P_2$, $P_2R_3P_3$, etc., may be considered to slide parallel to the x-axis into a corresponding position upon the disc of greatest radius, say that formed by the revolution of the figure $P_{n-1}Q_{n-1}NB$. Their sum is less than this disc, *i.e.* in the limit less than an infinitesimal of the first order, for the breadth $Q_{n-1}N$ is h, according to the notation of Art. 11, and is ultimately an infinitesimal of the first order, and the radius NB is, as in that article, supposed finite, as also all other ordinates of the curve from A to B.

Hence the volume required is the limit when $h=0$ (and therefore $n=\infty$) of the sum of the series

$$\pi[\phi(a)]^2h+\pi[\phi(a+h)]^2h+\pi[\phi(a+2h)]^2h+\dots$$
$$+\pi[\phi(a+\overline{n-1}h)]^2h,$$

or, as it may be written,

$$\pi\int_a^b[\phi(x)]^2dx \quad \text{or} \quad \pi\int_a^b y^2dx,$$

the equation of the curve being $y=\phi(x)$ and the extreme ordinates $x=a$ and $x=b$, as in the article cited.

25. Illustrative Examples.

Ex. 1. The portion of the parabola $y^2=4ax$ bounded by the line $x=c$ revolves about the axis. Find the volume generated.

Let the portion required be that formed by the revolution of the area APM about the axis, being bounded by the curve, the axis and an ordinate MP. (See Fig. 8.)

Dividing as in Art. 24 into elementary circular laminae, we have

$$\text{Vol.}=\int_0^c \pi y^2dx=4a\pi\int_0^c x\,dx=4a\pi\frac{c^2}{2}=2\pi ac^2 \text{ (Art. 16, Ex. 6)}$$
$$=\tfrac{1}{2}\pi PM^2 . AM$$
$$=\tfrac{1}{2} \text{ cylinder of radius } PM \text{ and height } AM$$
$$=\tfrac{1}{2} \text{ vol. of circumscribing cylinder.}$$

[Or, if expressed as a series,

$$4a\pi\int_0^c x\,dx=4a\pi\, Lt\,\frac{1}{n}\left[\left(\frac{1}{n}\right)+\left(\frac{2}{n}\right)+\left(\frac{3}{n}\right)+\dots+\left(\frac{n-1}{n}\right)\right]c^2$$
$$=4\pi a\frac{c^2}{2}=2\pi ac^2 \text{ as before.]}$$

Ex. 2. Find the area of the portion PAP' of the same parabola, PP' being the double ordinate through P.

$$\text{Area } PAM = \int_0^c y\,dx = 2\sqrt{a}\int_0^c x^{\frac{1}{2}}\,dx = 2\sqrt{a}\,\frac{c^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \tfrac{4}{3}\sqrt{ac^3} = \tfrac{2}{3}c\sqrt{4ac} = \tfrac{2}{3}AM \,.\, MP$$

$\therefore$ Area $PAP' = \frac{2}{3}$ of the circumscribing rectangle $RPP'R'$.

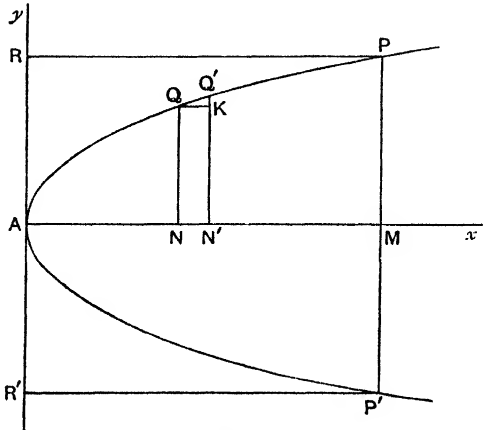


Fig. 8.

[Or we may proceed thus: Divide c into n equal portions, and erect ordinates. Let QN be the ordinate at $x = \frac{r}{n}c$.

Then $\quad$ Area $PAM = Lt \sum_{r=0}^{r=n-1} \sqrt{4arh}\,.\,h$, $\quad$ where $h = \frac{c}{n}$,

$$= 2a^{\frac{1}{2}}c^{\frac{3}{2}}\,Lt\,\frac{1}{n^{\frac{3}{2}}}\left[1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}\right]$$

$$= 2a^{\frac{1}{2}}c^{\frac{3}{2}}\,.\,\tfrac{2}{3} = \tfrac{2}{3}c\sqrt{4ac}, \text{ as before.]}$$

Ex. 3. The portion of a circle $x^2+y^2=a^2$ between ordinates $x=h_1$, $x=h_2$ rotates about the x-axis. Find the volume of the frustum of the sphere generated.

Let the portion required be that formed by the portion $N_1P_1P_2N_2$ of the circle revolving about N_1N_2 (Fig. 9).

Here we are to evaluate

$$\pi\int_{h_1}^{h_2} y^2\,dx = \pi\int_{h_1}^{h_2}(a^2-x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{h_1}^{h_2} \quad \text{(Art. 16, Ex. 6)}$$

$$= \pi a^2(h_2-h_1) - \frac{\pi(h_2^3-h_1^3)}{3}.$$

It is convenient for mensuration purposes to express this in terms of the radii of the ends of the frustum and its thickness.

Let T be the thickness $=h_2-h_1$ and $y_1^2=a^2-h_1^2$,

$y_2^2=a^2-h_2^2$.

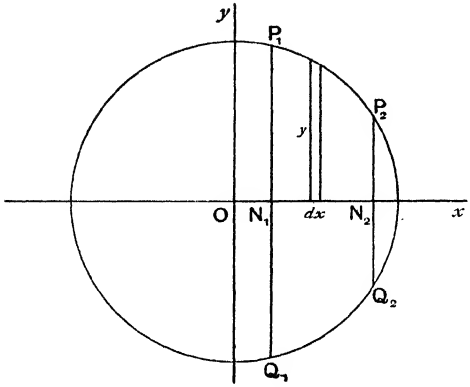


Fig. 9.

Then $$\begin{aligned}\text{Vol.} &= \tfrac{1}{6}\pi T(6a^2-2h_1^2-2h_2^2-2h_1h_2)\\ &= \tfrac{1}{6}\pi T\{3(a^2-h_1^2)+3(a^2-h_2^2)+(h_2-h_1)^2\}\\ &= \tfrac{1}{6}\pi T(3y_1^2+3y_2^2+T^2)\\ &= \frac{T}{6}(3\pi y_1^2+3\pi y_2^2+\pi T^2)\end{aligned}$$

$=\frac{1}{6}$ thickness $\times$ [3 sum of circular faces + circle on thickness as radius].

Cor. For the whole sphere

$$T=2a,\quad y_1=y_2=0,\quad V=\tfrac{4}{3}\pi a^3.$$

EXAMPLES.

1. Find the volume of the prolate spheroid formed by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis.

2. Find the mass of a rod whose density varies as the m^{th} power of the distance from one end. $\left(\rho=D\frac{x^m}{c^m}\right.$, say, where D and c are constants.$\left.\right)$

Let a be the length of the rod,

ω the sectional area.

Divide as before into n equal elementary portions.

The volume of the $(r+1)^{\text{th}}$ element from the end of zero density is $\omega\frac{a}{n}$. Its density varies from $\frac{D}{c^m}\left(\frac{ra}{n}\right)^m$ to $\frac{D}{c^m}\left[\frac{(r+1)a}{n}\right]^m$. Its mass is therefore intermediate between

$$D\frac{\omega a^{m+1}}{c^m}\frac{r^m}{n^{m+1}} \quad\text{and}\quad D\frac{\omega a^{m+1}}{c^m}\frac{(r+1)^m}{n^{m+1}},$$

and the mass of the rod lies between

$$D\frac{\omega a^{m+1}}{c^m}\frac{1^m+2^m+3^m+\ldots+(n-1)^m}{n^{m+1}} \quad \text{and} \quad D\frac{\omega a^{m+1}}{c^m}\frac{2^m+3^m+\ldots+n^m}{n^{m+1}},$$

and in the limit, when n is increased indefinitely, becomes

$$\frac{D}{c^m}\cdot\frac{\omega a^{m+1}}{m+1}.$$

[Or, assuming Art. 16, Ex. 6,

$$\text{Mass}=\int_0^a D\frac{x^m}{c^m}\omega\,dx=\frac{D}{c^m}\frac{\omega a^{m+1}}{m+1} \text{ at once.]}$$

3. Find the position of the centroid of the rod in Question 2.

[For the centroid $\bar{x}=\frac{\Sigma mx}{\Sigma m}$, when m is the mass of an element.]

4. Find the moment of inertia of the same rod about the lighter end. [Moment of Inertia $=\Sigma mx^2$.]

5. Find the area bounded by the parabola $4y=x^2$, the ordinates $x=2$ and $x=4$ and the x-axis,

(1) by means of inscribed rectangles,
(2) „ circumscribed rectangles,

taking ordinates at distances ·1, and compare the results with that obtained by integration.

The sum of the inscribed rectangles is

$$\tfrac{1}{4}\times\tfrac{1}{10}(2^2+2{\cdot}1^2+2{\cdot}2^2+\ldots+3{\cdot}9^2).$$

The sum of the circumscribed rectangles is

$$\tfrac{1}{4}\times\tfrac{1}{10}(2{\cdot}1^2\times 2{\cdot}2^2+\ldots+3{\cdot}9^2+4^2).$$

The values of these expressions are respectively (taking the squares from Bottomley's tables or summing otherwise),

4·5175, which is a little too small,

and 4·8175, which is a little too large.

Their mean is 4·6675.

The true value is

$$\int_2^4\frac{x^2}{4}dx=\left[\frac{x^3}{12}\right]_2^4=\frac{4^3-2^3}{12}=\frac{14}{3}=4{\cdot}666\ldots.$$

6. Plot the graph of $y=\frac{1}{1+x^2}$ and mark on your figure the area represented by the definite integral $\int_0^1\frac{dx}{1+x^2}$.

Evaluate this integral by mensuration, and hence obtain an approximation for π. Note that $\frac{1}{1+x^2}=\frac{d}{dx}\tan^{-1}x$.

26. Mechanical Integration.

In a sense, any mechanical contrivance which performs additions and registers the results is an Integrating machine for the particular class of function to which it may be

adapted. Cash registers which record the day's takings, gas meters, water meters, electric-light meters, all record the amount passing into them. A slide rule adds up logarithms, and thereby performs multiplications. Various forms of planimeters add up the elements of area within a closed curve when a pointer is made to trace the perimeter. The indicator of a steam engine draws a work diagram and adds up work elements, representing them by elements of area, from which the Horse-Power of the engine may be deduced.

Such apparatus, however, though giving numerical results satisfactory for practical purposes, but subject to various errors both instrumental and observational, fails to produce an exact algebraical result, and therefore fails to satisfy the mathematician, however useful to the practical engineer.

We shall have occasion later to return to the theory of some apparatus of this kind. For the present it is sufficient to mention its existence.

To sum up then; we have discussed Four Methods of Integration, *i.e.* of finding

$$\int_a^b \phi(x)\,dx:$$

I. By obtaining

$$Lt_{h=0}\,h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b)].$$

II. By finding a function $\psi(x)$ such that $\frac{d\psi}{dx}=\phi(x)$, from which we obtain

$$\int_a^b \phi(x)\,dx=\psi(b)-\psi(a).$$

III. By drawing the graph of $y=\phi(x)$ and by some means or other obtaining its area, by the Trapezoidal or Simpson's or some other approximative rule, as, for instance, by drawing on squared paper and counting all the squares within the contour with a "give and take" rule round the perimeter.

IV. By approximating to the area of the contour by mechanical means.

It is obvious that III. and IV. can only give approximate results, though such results may approach a very high degree of accuracy.

For exact results we have to apply Method I. or II. As has been seen, Method I. leads to very difficult algebraic or trigonometric summation, except in the very simplest cases.

Hence we are forced upon Method II. for exact general work. This method we therefore shall in future rely upon and begin to develop the explanation of it in the next chapter.

EXAMPLES.

1. If the acceleration of a moving point be $\phi''(t)$, the initial velocity be u and $\phi'(0) = \phi(0) = 0$, show that, t being the time from a given epoch,

$$v = u + \phi'(t), \quad s = ut + \phi(t),$$

where v and s are respectively the velocity at time t and the space described.

If the acceleration be $10 \cos \omega t$ and the initial velocity be zero, show that

$$v = \frac{10}{\omega} \sin \omega t,$$

$$s = C - \frac{10}{\omega^2} \cos \omega t,$$

where C is a constant. To what kind of motion does this refer?

Show that the "periodic time" is $\frac{2\pi}{\omega}$.

2. If A be the area bounded by a curve, the coordinate axes and the ordinate at a given abscissa x, show that $y = \frac{dA}{dx}$, and hence that $A = \int_0^x y\,dx$. What difference would it make if the measurement of A commences from a standard ordinate y_0 whose abscissa is x_0?

If V be the volume of water in a pond, and A the horizontal sectional area at a height x above the bottom of the pond, show that

$$= \int_0^h A\,dx,$$

where h is the depth of the pond.

3. A large number of circular discs of the same thickness $\frac{h}{n}$ and successive radii

$$\frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \frac{4a}{n}, \ldots, a,$$

are threaded through their centres upon a straight wire and lie with

their plane faces in contact. Show that their total volume differs from that of a cone of height h and with a for the radius of its base by the ultimately vanishing quantity

$$\frac{\pi a^2 h}{6}\frac{3n+1}{n^2}.$$

If $n=1000$, show that the error in taking this sum as the volume of the cone is ·1505 per cent. of the true volume.

4. Consider a sphere of diameter $2a$ to be divided into $2n$ thin laminae of equal thickness by a series of parallel planes; show that the volume of the sphere is

$$2\,Lt_{n=\infty}\sum_{r=0}^{r=n}\pi\left(1-\frac{r^2}{n^2}\right)\frac{a^3}{n},$$

and that this limit is $\frac{4}{3}\pi a^3$.

Obtain by a similar method the volume of the spheroid formed by the revolution of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ round the axis of length $2b$.

5. Show by the method of summation that the volume of a paraboloid of revolution bounded by a plane at right angles to the axis is one half of the circumscribing cylinder.

Verify by consideration of the integral

$$4a\int_0^h x\,dx.$$

6. Draw on squared paper (one inch squares divided into tenths is convenient) a quadrant of a circle of radius 5 inches. Divide one of the bounding radii into 10 half-inch divisions, and erect ordinates at each point. Complete the inscribed and escribed rectangles. Show that the sum of the inscribed rectangles is 18·15 square inches very nearly. Also show that the mean of the inscribed and escribed rectangles falls short of the true area of the quadrant by about $\frac{6}{25}$ of a square inch.

7. Rectangles of the same breadth and of areas

$$\left(\frac{c}{n}\right)^t,\quad \frac{1}{2}\left(\frac{2c}{n}\right)^t,\quad \frac{1}{3}\left(\frac{3c}{n}\right)^t,\ \ldots,\ \frac{1}{r}\left(\frac{rc}{n}\right)^t,\ \ldots,\ \frac{1}{n}c^t,$$

are set up side by side on bases in a straight line.

Shew that when n is very great, the sum of their areas differs little from that enclosed by $y=x^{t-1}$, $y=0$, $x=c$.

Assume t to be positive.

Evaluate $$Lt_{n=\infty}\sum_{r=1}^{r=n}\frac{1}{r}\left(\frac{rc}{n}\right)^t.$$

[I. C. S. EXAM. 1902.]

8. In the curve in which the abscissa varies as the logarithm of the ordinate, prove that the area bounded by the curve, the x-axis and any two ordinates varies as the difference of the ordinates.

9. Approximate to the integral $\int_2^3 \frac{10}{x}\,dx$, regarding it as a summation (1) of inscribed parallelograms as in Art. 9,

(2) of circumscribed parallelograms,

and compare with the result of integration.

[The results are 3·9724 and 4·1391, the reciprocals being taken from Bottomley's tables. Their mean is 4·0557. The result to three places of decimals as computed from $10 \log_e \frac{3}{2}$ is 4·055.]

10. Draw a sketch showing the curvilinear area which is represented by the definite integral

$$\int_1^{10} \frac{10}{x}\,dx,$$

and evaluate the area approximately from the figure.

Without plotting, indicate roughly by dotted lines on your sketch the relative positions of the curvilinear areas represented by the definite integrals

$$\int_1^{10} 10x^{-0·9}dx \quad \text{and} \quad \int_1^{10} 10x^{-1·1}dx,$$

and calculate the values of these integrals.

Calculate also

$$\int_1^{10} 10x^{-0·99}dx \quad \text{and} \quad \int_1^{10} 10x^{-1·01}dx.$$ [I. C. S., 1908.]

11. In any curve in which the ordinate $PN \propto$ the n^{th} power of the abscissa, show that if any two ordinates be taken, viz. P_1N_1 and P_2N_2, and two others, P_3N_3 and P_4N_4, which are twice as far from the y-axis as P_1N_1 and P_2N_2 respectively, then

$$\text{Area } P_3P_4N_4N_3 : \text{Area } P_1P_2N_2N_1 :: 2^{n+1} : 1.$$

12. Prove that the area of the diagram formed by

$$\begin{aligned} x &= 0 \quad \text{from } (0, 0) \text{ to } (0, 4), \\ y &= 4 \quad \text{from } (0, 4) \text{ to } (1, 4), \\ x^2 - 10x + 25 &= 4y \quad \text{from } (1, 4) \text{ to } (5, 0), \\ y &= 0 \quad \text{from } (5, 0) \text{ to } (0, 0), \end{aligned}$$

is $9\frac{1}{3}$ square units.

13. In the construction of reservoir walls of great height, Rankine adopted the following plan:

Taking a vertical x-axis on which depths and ordinates are measured

in feet, the ordinates to the outer and inner faces are shown in the following scheme :

Depth in feet.	0	10	20	30	40	50	60
Ordinate to outer face in feet.	17·40	19·72	22·35	25·29	28·69	32·53	36·86
Ordinate to inner face in feet.	1·34	1·52	1·72	1·94	2·21	2·50	2·83
Depth in feet.	70	80	90	100	110	120	130
Ordinate to outer face in feet.	41·75	47·31	53·61	60·75	68·84	78·00	88·39
Ordinate to inner face in feet.	3·21	3·64	4·12	4·67	5·29	6·00	6·80
Depth in feet.	140	150	160	170	180		
Ordinate to outer face in feet.	100·15	113·49	128·60	146·72	165·14		
Ordinate to inner face in feet.	7·70	8·73	9·90	11·21	12·70		

(The two sets of ordinates are measured in opposite directions from the vertical.)

[RANKINE, *Applied Mechanics*, p. 638, and *Engineer*, Jan. 5, 1872.]

Construct a diagram showing the wall in elevation, and estimate in cubic yards the volume of material necessary to construct 100 yards length of the wall.

14. Find the centre of gravity of a rod whose density varies

(1) as the distance from one end;

(2) as the square of the distance from one end.

Find also the moment of inertia of the rod about the light end in each case.

15. Find the mass of a circular disc in which the density varies as the n^{th} power of the distance from the centre. $(n > -2.)$

Also find the moment of inertia of this disc about an axis through the centre at right angles to the plane of the disc.

16. If the graphs of $a \sin \frac{\pi x}{2b}$ and $a \sin \frac{\pi x}{b}$ be drawn, show that the areas bounded by the x-axis, the curves and the ordinate $x = b$ are equal.

17. A single wave on the sea is in the form defined by the curve of sines, $y = a \sin \frac{\pi x}{b}$. Show that the quantity of water raised above the mean sea level contained in a length c of the wave measured on the surface at right angles to the direction of progression, is $2abc/\pi$, the raised portion extending from $x = 0$ to $x = b$.

If c be 100 yards, $b = 20$ feet, $a = 2$ feet, and a cubic foot of water weighs $62\frac{1}{2}$ lbs. weight, find the number of tons weight in the portion of the wave higher than the mean sea level.

18. Show that when n becomes infinitely large,

$$Lt \frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1)}{n^3}$$

is the same as
$$Lt \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}.$$

Illustrate geometrically.

19. Show that the limit when $n = \infty$ of the ratio of the sum of all possible products, two and two together, of the first n natural numbers, to n^4, is $\frac{1}{8}$; and that the limit of the ratio of the sum of all products, three and three together, to n^6, is $\frac{1}{48}$.

20. If there be gas of volume v and pressure p below a piston in a cylinder of sectional area A and occupying a length x of the cylinder, show that in its expansion, so as to occupy a length $x + dx$ of the cylinder, the work done by the gas upon the piston is

$$pA\,dx \text{ or } p\,dv,$$

and that if the expansion continues so that the piston moves through a finite distance—say from $x = x_1$ to $x = x_2$, the work done on expansion is

$$\int_{x_1}^{x_2} p\,dv.$$

Remembering that
$$\frac{d}{dx} \log x = \frac{1}{x},$$

find the value of this integral in the two cases:

(1) Isothermal expansion, $pv = c$;

(2) Adiabatic expansion, $pv^\gamma = c'$.

Find in foot-lbs. the work done in the expansion of 10 cubic feet of gas, initially at a pressure of 1000 lbs. per square foot, to 40 cubic feet;

(1) According to the law, $pv = c$;

(2) According to the law, $pv^{1 \cdot 41} = c'$.

21. If the graph of $e^{\frac{x}{a}}$ be drawn, prove that the areas bounded by the curve, the x-axis and a set of equidistant ordinates are in geometrical progression, whose common ratio is the same as the common ratio of the tangents of the angles which the tangents at the ends of the successive ordinates make with the x-axis.

22. Show that the area bounded by a parabola, the axis and an ordinate is two-thirds of the circumscribing rectangle.

23. The circle $x^2 + y^2 = 5a^2$ and the parabola $y^2 = 4ax$ revolve about their common axis. Show that the smaller lens-shaped solid formed has for its volume

$$\tfrac{2}{3}\pi a^3(5\sqrt{5} - 4).$$

24. If $x_1, x_2, x_3, \ldots, x_{n-1}$ be a series of quantities taken between $a(=x_0)$ and $b(=x_n)$, prove that when n is made infinite, and the difference between any two consecutive terms of the series becomes indefinitely small, the limit of

$$\sum_{r=1}^{r=n}(x_r - x_{r-1})f(x_{r-1})$$

is $\phi(b) - \phi(a)$, where $\dfrac{d}{dx}\phi(x) = f(x)$.

Verify this in the case where $f(x) = \log x$, and the series $a, x_1, x_2, \ldots,$ is geometrical. [OXFORD, *2nd Public Examination*, 1900.]

25. Plot the value of $\cos^2 x$ for 10° intervals from 0° to 90°, and thus find as close an approximation as you can to

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

without integration. $\left[\text{The true value is } \dfrac{\pi}{4}.\right]$

26. If a cylindrical hole be drilled through a solid sphere, the axis of the cylinder passing through the centre of the sphere, show that the volume of the portion of the sphere left is equal to the volume of a sphere whose diameter is the length of the hole.

27. If the curve $y = a + bx + cx^2 + dx^3$ pass through the extremities of four equidistant ordinates y_1, y_2, y_3, y_4, the distance apart being h, show that the area bounded by the extreme ordinates, the curve and the x-axis is

$$\frac{3h}{8}(y_1 + 3y_2 + 3y_3 + y_4).$$

[SIMPSON'S "Three-eighths' Rule."]

28. If the curve $y = a + bx + cx^2 + dx^3 + ex^4$ pass through the extremities of 5 equidistant ordinates y_1, y_2, y_3, y_4, y_5, at mutual

distances h, show that the area bounded by the extreme ordinates, the curve and the x-axis is

$$h\frac{14(y_1+y_5)+64(y_2+y_4)+24y_3}{45}.$$

[BOOLE, *Finite Differences.*]

29. If a parabola whose axis is parallel to the y-axis pass through the points (a, y_1), (b, y_2), (c, y_3), show that its equation is

$$y=y_1\frac{(x-b)(x-c)}{(a-b)(a-c)}+y_2\frac{(x-c)(x-a)}{(b-c)(b-a)}+y_3\frac{(x-a)(x-b)}{(c-a)(c-b)},$$

and find the area bounded by the curve, the x-axis and the extreme ordinates y_1 and y_3.

30. In the cycloid $\left.\begin{aligned} x&=10(\theta+\sin\theta)\\ y&=10(1-\cos\theta)\end{aligned}\right\}$

tabulate the values of x and y for intervals of $\frac{\pi}{36}$ from $\theta=0$ to $\theta=\frac{\pi}{2}$. Hence obtain approximate results for

$$(1)\ \int y\,dx, \qquad (2)\ \int x\,dy$$

corresponding to the above limits for θ.

31. If $f(x)>\phi(x)$ where $a<x<b$, and if both functions are finite for this range of the variable, including both limits, prove that

$$\int_a^b f(x)\,dx>\int_a^b \phi(x)\,dx.$$

Explain why these conditions must be postulated. Must the functions also be continuous? [I. C. S., 1905.]

32. Prove that the integral

$$\int_0^{0\cdot5}\frac{dx}{\sqrt{1-x^n}}$$

is for all values of n greater than 2, nearly equal to 0·5. [I. C. S., 1905.]

33. A claret glass is 6 cm. deep and its rim is 5 cm. in diameter. Its vertical section is nearly parabolic. Calculate its capacity in c.c. to the nearest integer. [I. C. S., 1905.]

34. Trace the curve $y=x^m(1-x)^{2m}$ from $x=0$ to $x=1$ for the values $m=0{\cdot}5$ and 2. Show the two curves on one diagram. Show that the area enclosed by the curve and the x-axis diminishes as m increases. [I. C. S., 1902.]

35. A cask has a head diameter of a inches, a bung diameter of b inches, and length c. Find an expression for its volume, supposing

that a section along a stave is an arc of a curve of sines, the curvature vanishing at the ends of the stave.

Evaluate the result when $a = 13$, $b = 17$, $c = 18$. [I. C. S., 1902.]

36. Find the value of $\int_0^{0\cdot 3} x^{0\cdot 4}(1-x)^{0\cdot 6}\,dx$

to two significant figures,

(1) graphically,

(2) by calculation. [I. C. S., 1903.]

37. Show, without integration, that

$$I = \int_0^{0\cdot 644} \frac{64\,d\theta}{(5+3\cos\theta)^2}$$

lies between ·644 and ·753.

[Peterhouse and Sidney Sussex Scholarship Exam., 1917.]

Differentiate $\qquad 5\tan^{-1}\left(\frac{1}{2}\tan\frac{\theta}{2}\right) - \frac{6\sin\theta}{5+3\cos\theta}$,

and hence prove that the true value of I is about ·68.

(Take $\tan^{-1}\frac{1}{3} = \cdot 322$ and $\tan^{-1}\frac{1}{6} = \cdot 165$.)

38. In a diagram of the work done by the expansion of steam in a cylinder, given by Watt in 1782, there are 20 ordinates at equal (unit) distances. The respective lengths of the ordinates, of which the first is one unit distance from the beginning of the diagram, are 1, 1, 1, 1, 1, ·830, ·711, ·625, ·555, ·500, ·454, ·417, ·385, ·357, ·333, ·312, ·294, ·277, ·262, ·250, representing the steam pressure in pounds weight per square inch as the piston arrives at a position corresponding to the several ordinates. The initial ordinate is also of unit length. The steam pressure is supposed to be constant (14 lbs. weight per square inch), whilst the piston travels over the first five divisions, and then the steam being cut off suddenly, the pressure is assumed to fall according to Boyle's Law ($pv =$ constant). Show that the area of this diagram is very little more than 11·562 square units, and that the mean pressure is ·578 lb. weight per square inch.

Justify Watt's statement "whereby it appears that only $\frac{1}{4}$ of the steam necessary to fill the whole cylinder is employed, and that the effect is more than half of the effect which would have been produced by one whole cylinder full of steam, if it had been allowed to enter freely above the piston during the whole length of its descent." [Goodeve, *On the Steam Engine.*]

39. If steam at pressure p lbs. weight per square inch be admitted into a cylinder of length a feet, and be cut off when the piston has completed $\frac{1}{n}$ of its stroke, and the steam pressure then fall according to Boyle's Law for the rest of the stroke, prove by the Integral Calculus that if the piston area be A square inches, and there be no back pressure, the work done in one stroke is

$$\frac{Aap}{n}\log_e en \text{ foot-pounds.}$$

Show also that the approximate result found by the method of dividing the Indicator diagram as in the preceding question, and assuming the cut-off to be at half-stroke, differs from the true result by about 1·5 per cent. of the estimated work.

$$\left[\text{Assume } \int_a^b \frac{dx}{x} = \log\frac{b}{a},\ \log_e 2 = \cdot 69314718.\right]$$

40. Steam is admitted into a cylinder at double the atmospheric pressure (atmosph. pres. = 15 lbs. wt. per sq. inch), and on the opposite side of the piston the pressure is atmospheric continually. The steam is cut off at half stroke. Divide the stroke into 20 equal parts. Suppose the pressure at the beginning of each of these portions to remain uniform until the piston reaches the next in order, and assume the fall of pressure after cut-off to be that of Boyle's Law. Show that with these assumptions the work done in one stroke is nearly 8466 foot-lbs.; the area of the piston being 200 square inches and the length of the stroke 40 inches. [Draw the work diagram as accurately as possible on squared paper.]

41. An ellipse, whose major axis is 10 cm. and eccentricity 0·4, has a perimeter $20\int_0^{\frac{\pi}{2}}\sqrt{1-0\cdot 16\sin^2\phi}\,d\phi$ cm. in length. Draw on a large scale the graph of $\sqrt{1-0\cdot 16\sin^2\phi}$ as a function of ϕ from the following values, the angle ϕ being in radians:

ϕ	0·0	0·175	0·350	0·524	0·785	1·047	1·257	1·571
$\sin\phi$	0·0	0·030	0·117	0·250	0·500	0·750	0·904	1·000

Hence find graphically the value of the integral and the perimeter of the ellipse.

Check your result by drawing the ellipse, and stepping along the perimeter with your dividers opened to 1 cm. [I. C. S., 1907.]

42. Draw in one figure the graphs of $\frac{1}{x}$, $-\frac{1}{x}$, $\frac{\sin x}{x}$, showing how they are related; the angle x being taken in radians.

From the graph of $\frac{\sin x}{x}$ deduce the general shape of the graph of $\int_0^x \frac{\sin x}{x}\,dx$, finding its proportions roughly. What is the approximate value of the integral when x is large? [Use a large scale of representation and draw the graph, say, from $x=0$ to $x=15$. It is sufficient to describe the shape when x is negative.] [I.C.S., 1907.]

43. In an electric circuit of resistance R ohms, L is the self-inductance. The voltage suddenly rises to a value V. The current in ampères is I. The law of growth of the current is

$$I=\frac{V}{R}\left(1-e^{-\frac{R}{L}t}\right) \quad \text{and} \quad I=\frac{dQ}{dt}.$$

Taking Q to vanish initially, prove that $Q=\frac{V}{R}t-\frac{VL}{R^2}\left(1-e^{-\frac{R}{L}t}\right)$, and illustrate the growth of I graphically.

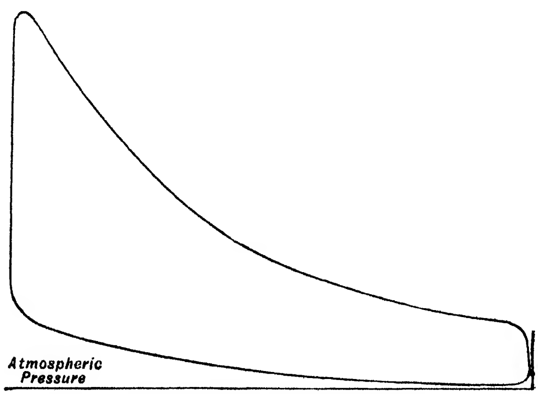
44. A current is changing according to the law

$$\frac{dQ}{dt}=I=a+bt-ct^2,$$

where t is measured in seconds and Q vanishes with t.

Also the voltage is given by $V=RI+L\frac{dI}{dt}$ where $R=$ resistance and $L=$ self-inductance. Express Q and V in terms of t.

45. The figure shows the indicator-diagram of a gas engine



Scale, 80 lbs per square inch per inch

Fig. 10.

which works on the Otto cycle. Estimate the horse-power of the engine from the diagram and from the following data:

Diameter of cylinder $9\frac{1}{2}''$,
Length of stroke $16''$,
Revolutions per minute 180. [MECH. SC. TRIP.]

46. Apply Weddle's Rule for the approximate evaluation of a definite integral, viz.

$$\int_0^{6h} u_x\,dx = \frac{3h}{10}\left[u_0 + u_{2h} + u_{4h} + u_{6h} + 5(u_h + u_{5h}) + 6u_{3h}\right],$$

to evaluate $\int_0^{\frac{\pi}{2}} \log \sin\theta\, d\theta$ to four places of decimals, and compare your result with the known value $\frac{\pi}{2}\log\frac{1}{2}$. [BOOLE, *Fin. Diff.*, p. 49.]

47. Prove from first principles that if

$$x_1,\quad x_2,\quad x_3,\ \ldots,\ x_n$$

be finite real quantities such that, as n tends to infinity and x_n to x, the limit of

$$(x_2 - x_1)^2 + (x_3 - x_2)^2 + \ldots + (x_n - x_{n-1})^2$$

is zero, then the limit of the sum

$$x_1^4(x_2 - x_1) + x_2^4(x_3 - x_2) + \ldots + x^4_{n-1}(x_n - x_{n-1})$$

is

$$\frac{x^5 - x_1^5}{5}.$$

[OXF. FIRST P., 1913.]

48. The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in miles per hour:

2 min.	10 m./h.	12 min.	20 m./h.
4 ,,	18 ,,	14 ,,	11 ,,
6 ,,	25 ,,	16 ,,	5 ,,
8 ,,	29 ,,	18 ,,	2 ,,
10 ,,	32 ,,	20 ,,	At rest.

Estimate approximately the total distance run in the 20 minutes. [MATH. TRIP. PT. I., 1913.]

49. Show that if $\phi(x)$ be any polynomial of the fifth degree,

$$\int_0^1 \phi(x)\,dx = \tfrac{1}{18}\{5\phi(\alpha) + 8\phi(\tfrac{1}{2}) + 5\phi(\beta)\},$$

where α and β are the roots of $x^2 - x + \frac{1}{10} = 0$.

[MATH. TRIP. PT. I., 1909.]

50. The specific heat of a substance at temperature $t°$ is $\frac{dQ}{dt}$, where Q is the quantity of heat required to raise 1 gram of the substance from some fixed temperature to $t°$.

The specific heat of water (s, in joules) at a temperature of $t°$ being given by the following table:

$t° =$	0°	10°	20°	30°	40°	50°
$s =$	4·219	4·195	4·181	4·174	4·173	4·174
$t° =$	60°	70°	80°	90°	100°	
$s =$	4·178	4·184	4·190	4·197	4·205	

show that to raise 1 gram of water from 0° to 100° requires 418·5 joules approximately. [MATH. TRIP. PT. I. 1910.]

51. Prove that $$\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx > \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{x} dx.$$

[MATH. TRIP. PT. I., 1912.]

52. A uniform solid is bounded by the surface obtained by revolving the area
$$y^2 = ax^2 + 2bx + c$$
about the axis of x. A slice is cut from the solid by two plane sections perpendicular to the axis at a distance h apart; prove that the volume of the slice is V, where
$$V = \tfrac{1}{2}(A_1 + A_2)h - \tfrac{1}{6}\pi a h^3,$$
A_1 and A_2 being the areas of the two plane faces of the slice.

Show also that the distance of the centroid of V from the face A_1 is equal to
$$\frac{h}{2} + \frac{(A_2 - A_1)}{12V}h^2.$$
[MATH. TRIP. PT. I., 1914.]

53. Apply Simpson's "Three-eighths' Rule" (see Ex. 27) to approximate to the value of the integral
$$\int_0^{\frac{\pi}{6}} (1 + 6 \sin \theta)^{\frac{1}{2}} d\theta.$$
[MATH. TRIP. PT. I., 1917.]

CHAPTER II.

STANDARD FORMS.

27. Reversal of Differentiation.

We now proceed to consider Integration as the purely analytical problem of reversal of the operation of Differentiation.

In the *Differential Calculus* the student has learnt how to differentiate a function of any assigned character with regard to the independent variable contained. In other words, having given $y=\psi(x)$, methods have been there explained of obtaining the form of the function $\psi'(x)$ in the equation

$$\frac{dy}{dx}=\psi'(x)=\phi(x), \text{ say.}$$

If we can reverse this operation and obtain the value of $\psi(x)$ when $\psi'(x)$ is the given function of x, we shall be able to perform the operation which has been indicated by the symbol

$$\int_a^b \phi(x)\,dx, \quad \textit{i.e.} \int_a^b \psi'(x)\,dx,$$

by merely (1) taking the function $\psi(x)$, (2) substituting b and a alternately for x in this function, and (3) subtracting the latter result from the former; thus obtaining

$$\psi(b)-\psi(a).$$

28. We shall therefore confine our attention for the next few chapters to the problem of this reversal of the operation of the Differential Calculus.

The quantity b has been assumed to have any real value whatever, provided it be finite; we may therefore replace it by x and write the result as

$$\int_a^x \phi(x)\,dx=\psi(x)-\psi(a).$$

When the lower limit is not specified and we are merely enquiring the *form* of the function $\psi(x)$, at present unknown, whose differential coefficient is the known function $\phi(x)$, the notation is

$$\int \phi(x)\,dx = \psi(x),$$

the limits being omitted.

29. Nomenclature.

The nomenclature of these expressions is as follows:

The function $\phi(x)$ whose integral is sought is termed the "**integrand,**" and the result $\psi(x)$ is termed the "**integral.**"

$$\int_a^b \phi(x)\,dx \quad \text{or} \quad \psi(b)-\psi(a)$$

is called the "**definite**" integral of $\phi(x)$ between the assigned limits a and b.

$$\int_a^x \phi(x)\,dx \quad \text{or} \quad \psi(x)-\psi(a),$$

where the lower limit is assigned and the upper limit is left undetermined, is called a "**corrected**" integral.

$$\int \phi(x)\,dx \quad \text{or} \quad \psi(x),$$

without any specified limits and regarded merely as the reversal of an operation of the differential calculus, is called an "**indefinite**" or "**uncorrected**" integral.

It is customary to read the expression $\int \phi(x)\,dx$ as "**the integral of $\phi(x)$ with respect to x,**" or as "**the integral of** $\phi(x)\,dx$." And the process of obtaining $\psi(x)$ is called **Integration**.

30. Addition of a Constant.

It will be observed that if $\phi(x)$ be the differential coefficient of $\psi(x)$, it is also the differential coefficient of $\psi(x)+C$, where C is any constant whatever, that is to say, a quantity which does not depend upon the variable x; for the differential coefficient of such a quantity with regard to x is zero. (See Art. 3.)

Accordingly, we might write

$$\int \phi(x)\,dx = \psi(x)+C.$$

This arbitrary constant is, however, not usually expressly written down, but will be understood to be existent in all cases where the lower limit of the integral is not expressed.

31. Different processes of indefinite integration will frequently give results of different form; for instance,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \quad \text{is} \quad \sin^{-1}x \quad \text{or} \quad -\cos^{-1}x,$$

for the expression $\dfrac{1}{\sqrt{1-x^2}}$ is the differential coefficient of either of these expressions. We cannot infer that $\sin^{-1}x$ and $-\cos^{-1}x$ are equal. What is really true is that $\sin^{-1}x$ and $-\cos^{-1}x$ differ by a constant, for

$$\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x.$$

So that
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C,$$

or
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = -\cos^{-1}x + C',$$

the arbitrary constants C and C' being necessarily different.

32. **Inverse Differential Notation.**

In agreement with the accepted notation for the Inverse Trigonometrical and Inverse Hyperbolic functions, we might express the equation

$$\int \phi(x)\,dx = \psi(x)$$

as
$$\left(\frac{d}{dx}\right)^{-1} \phi(x) = \psi(x),$$

or
$$D^{-1}\phi(x) = \psi(x)$$

or
$$\frac{1}{D}\phi(x) = \psi(x),$$

and it is not infrequently useful to employ this notation, which very well expresses the interrogative character of the operation we are conducting.

33. General Laws satisfied by the Integrating Symbol

$$\int dx \quad \text{or} \quad \frac{1}{D}.$$

I. It is plain from the meaning of the symbols that

$$\frac{d}{dx}\int \phi(x)\,dx \quad \text{is} \quad \phi(x) \qquad \text{or} \quad D\left[\frac{1}{D}\phi(x)\right]=\phi(x).$$

But $\displaystyle\int\left[\frac{d}{dx}\phi(x)\right]dx=\phi(x)+C$ or $\displaystyle\frac{1}{D}[D\phi(x)]=\phi(x)+C$,

C being any arbitrary constant.

II. The operation of integration is **distributive** for a finite number of terms.

For if u_1, u_2, u_3 be any functions of x,

$$\frac{d}{dx}\left\{\int u_1\,dx+\int u_2\,dx+\int u_3\,dx\right\}$$

$$=\frac{d}{dx}\left[\int u_1\,dx\right]+\frac{d}{dx}\left[\int u_2\,dx\right]+\frac{d}{dx}\left[\int u_3\,dx\right]$$

$$=u_1+u_2+u_3,$$

and therefore, omitting additive constants, *i.e.* supposing the lower limit to have been assigned and to be the same in each case,

$$\int u_1\,dx+\int u_2\,dx+\int u_3\,dx=\int (u_1+u_2+u_3)\,dx.$$

Similarly,

$$\int u_1\,dx+\int u_2\,dx-\int u_3\,dx=\int (u_1+u_2-u_3)\,dx.$$

If the lower limits in these several integrations are not the same, the left-hand member of the equation may differ from the right-hand by a constant. It is in this sense that the equality sign is used.

III. The operation of integration is **commutative** with regard to constants.

For if $\dfrac{du}{dx}=v$, and a be any constant,

$$\frac{d}{dx}(au)=a\frac{du}{dx}=av.$$

So that, omitting additive constants of integration,

$$au = \int av\,dx,$$

or

$$a\int v\,dx = \int av\,dx,$$

which establishes the theorem.

34. Case of an Infinite Series.

In the case of an infinite series of real quantities,

$$U = u_1 + u_2 + u_3 + \dots + u_n + \dots \text{ to } \infty,$$

of which the terms are connected by a definite law, we shall still have

$$\int_{x_1}^{x_2} U\,dx = \int_{x_1}^{x_2} u_1\,dx + \int_{x_1}^{x_2} u_2\,dx + \int_{x_1}^{x_2} u_3\,dx + \dots \text{ to } \infty = V, \text{ say},$$

provided the series U itself, and the series V formed by the integrations of the separate terms, are both *uniformly and unconditionally** *convergent* within a range of values of x, viz. $x = b$ to $x = a$, say, where $a > b$, between which quantities both limits of integration x_1 and x_2 lie, that is

$$a > x_2 > x_1 > b.$$

For let R and S be the remainders after n terms of the series U and V, *i.e.*

$$U = u_1 + u_2 + \dots + u_n + R,$$

$$V = \int_{x_1}^{x_2} u_1\,dx + \int_{x_1}^{x_2} u_2\,dx + \dots + \int_{x_1}^{x_2} u_n\,dx + S.$$

Then, by supposition, both R and S vanish when n is indefinitely increased for all values of x between a and b, and therefore so also does $\int_{x_1}^{x_2} R\,dx$, for it lies between $R'(x_2 - x_1)$ and $R''(x_2 - x_1)$, where R' and R'' are the greatest and least values of R as x changes continuously from a to b, and which are quantities vanishing in the limit.

Hence, $V - S = \int_{x_1}^{x_2} (U - R)\,dx = \int_{x_1}^{x_2} U\,dx - \int_{x_1}^{x_2} R\,dx$ (Art. 33, II.), and when n is indefinitely increased,

$$\int_{x_1}^{x_2} U\,dx = V.$$

* See Art. 1900, Vol. II.

If then a function $\phi(x)$ can be expanded in a power-series as $\phi(x)=\sum_{r=0}^{r=\infty} A_r x^r$, the series being uniformly and unconditionally convergent from $x=b$ to $x=a$, we can write

$$\int_{x_1}^{x_2}\phi(x)dx=\sum_{r=0}^{r=\infty}A_r\int_{x_1}^{x_2}x^r\,dx=\sum_{r=0}^{r=\infty}A_r\frac{x_2^{r+1}-x_1^{r+1}}{r+1}\text{ [Art. 16, Ex. 6],}$$

where $$a>x_2>x_1>b;$$

for if $\Sigma A_r x^r$ be uniformly and unconditionally convergent, so also will be

$$\Sigma A_r\frac{x^{r+1}}{r+1}\quad\text{and}\quad\Sigma A_r\frac{x_2^{r+1}-x_1^{r+1}}{r+1}.$$

Under such circumstances, therefore, we may expand before integrating.

35. Geometrical Illustrations.

We may illustrate these facts geometrically.

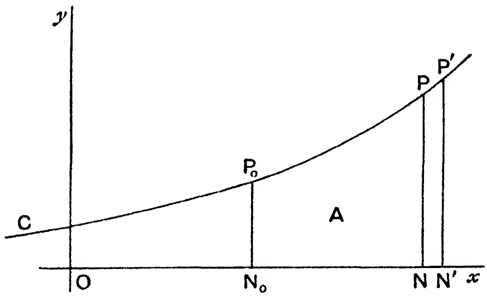


Fig. 11.

Let the graph of $y=\phi(x)$ be represented by the curve CP_0P. Let the coordinates of a fixed point P_0 on the curve be x_0, y_0, let x, y be the coordinates of a current point P on the curve, and let A be the area of the figure P_0N_0NP. Let x increase to $x+\delta x$, and in consequence let y become $y+\delta y$ and A become $A+\delta A$. Then δA is the area of the strip $PNN'P'$ between two contiguous ordinates NP and $N'P'$, and lies in magnitude between $y\,\delta x$ and $(y+\delta y)\delta x$, and therefore $\frac{\delta A}{\delta x}$ lies between y and $y+\delta y$.

Hence, in the limit, when δx is made indefinitely small we have

$$\frac{dA}{dx}=y.$$

Hence
$$A=\int y\,dx.$$

So long as the lower limit is unassigned the reckoning of the area may start from any arbitrary position of the ordinate N_0P_0, and the case is that of the "**indefinite**" integral.

When the lower limit is assigned, say $x=ON_0$, the area is reckoned from the ordinate N_0P_0 to any arbitrary ordinate NP, and is $\int_{ON_0}^{x}\phi(x)\,dx$, and is then "**corrected.**"

When both limits ON_0 and ON are numerically assigned the integral $\int_{ON_0}^{ON}\phi(x)\,dx$ is "**definite.**"

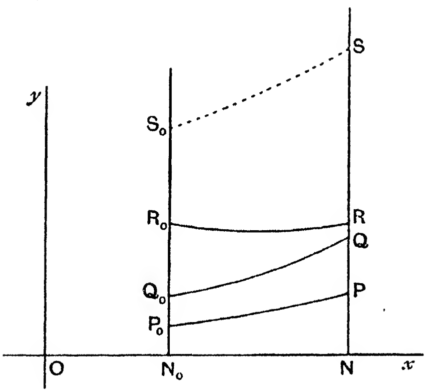


Fig. 12.

If there be several curves (a finite number of them, and all continuous, and none of the ordinates infinite within the limits of integration),

$y=F_1(x)$, $y=F_2(x)$, $y=F_3(x)$
$\equiv u_1$, $\equiv u_2$, $\equiv u_3$, viz. the curves P_0P, Q_0Q, R_0R,

and a curve be derived from them by the algebraic addition of ordinates so that

$$Y=F_1(x)+F_2(x)+F_3(x),\text{ viz. the curve } S_0S,$$

then the distributive property II. of the integration symbol asserts that

Area P_0N_0NP+area Q_0N_0NQ+area R_0N_0NR=area S_0N_0NS.

Again, if a curve be given by the equation

$$y=F(x),\ \textit{i.e.}\ \text{curve } P_0P,$$

and a new one be derived by increasing all the ordinates in the ratio $a:1$ so as to have an equation

$$y=aF(x),\ \textit{i.e.}\ \text{curve } S_0S, \text{ say,}$$

the commutative rule III. asserts that

$$\text{Area } S_0N_0NS=a\times\text{area } P_0N_0NP.$$

If the lower limit be not the same in each case, as assumed in the figure, the stated results would, instead of being equal, differ by constants which depend upon the positions of the initial ordinates in the several cases.

36. **Integration of x^n.**

By Differentation of $\dfrac{x^{n+1}}{n+1}$ we obtain $\dfrac{d}{dx}\dfrac{x^{n+1}}{n+1}=x^n$. Hence (as has already been seen, Art. 16, Ex. 6).

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+\text{an arbitrary constant.}$$

Thus the rule for the integration of any *constant* power of x may be stated in words;

Increase the index by unity, and divide by the new index.

E.g. $\displaystyle\int x^7\,dx=\frac{x^8}{8}$; $\displaystyle\int x^{\frac{3}{5}}\,dx=\frac{x^{\frac{8}{5}}}{\frac{8}{5}}=\tfrac{5}{8}x^{\frac{8}{5}}$;

$$\int x^{-\frac{11}{3}}\,dx=\frac{x^{-\frac{8}{3}}}{-\frac{8}{3}}=-\tfrac{3}{8}x^{-\frac{8}{3}};\qquad \int x^{-0\cdot45}\,dx=\frac{x^{\cdot55}}{\cdot55};$$

$$\int\sqrt[p]{x^q}\,dx=\frac{x^{\frac{q}{p}+1}}{\frac{q}{p}+1}=\frac{p}{p+q}x^{\frac{p+q}{p}}.$$

$$\int dx,\quad \textit{i.e.}\int 1\,dx\ \text{or}\int x^0\,dx=x.$$

37. **The case of x^{-1}.**

It will be remembered that x^{-1} or $\frac{1}{x}$ is the differential coefficient of $\log_e x$. Thus,

$$\int x^{-1}\,dx \quad \text{or} \quad \int \frac{1}{x}\,dx = \log_e x.$$

This therefore forms an apparent exception to the general rule,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

It is, however, only *apparent*. For we may deduce the logarithmic form as a limiting case. Supplying the arbitrary constant C, we have

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C = \frac{x^{n+1}-1}{n+1} + A,$$

where $A = C + \frac{1}{n+1}$ and still is an arbitrary constant, *i.e.* does not contain x. Taking the limit when $n+1=0$, $\frac{x^{n+1}-1}{n+1}$ takes the form $\log_e x$ (*Diff. Cal.*, Art. 21). And as C is an *arbitrary* constant, we may suppose that it contains a negatively infinite portion $-\frac{1}{n+1}$, together with another *arbitrary* portion A.

Then $$Lt_{n=-1}\int x^n\,dx = \log x + A.$$

This has also been seen in Art. 16, Ex. 6.

38. In the same way as in the integrations of x^n and x^{-1} we have

$$\frac{d}{dx}(ax+b)^{n+1} = (n+1)a(ax+b)^n$$

and $$\frac{d}{dx}\log(ax+b) = \frac{a}{ax+b},$$

and therefore $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a}$$

and $$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b).$$

[Although $\int dx$ is really one symbol indicating integration with regard to x, we shall often find $\int \frac{1}{ax+b}\,dx$ printed for convenience as $\int \frac{dx}{ax+b}$, $\int \frac{1}{\sqrt{x^2+a^2}}\,dx$ printed as $\int \frac{dx}{\sqrt{x^2+a^2}}$, etc.]

39. We are now in a position to integrate any expression of the form

$$\frac{\phi(x)}{ax+b},$$

where $\phi(x)$ indicates any rational integral algebraic function of x.

This can be done in two ways:

(1) By ordinary division of $\phi(x)$ by $ax+b$ we can express

$$\frac{\phi(x)}{ax+b} \text{ in the form } Q+\frac{R}{ax+b},$$

where Q consists of a series of descending powers of x and R is independent of x.

Every term is then integrable by the foregoing rules, and the result will be partly algebraic and partly logarithmic, the last term being $\frac{R}{a}\log(ax+b)$. The condition that it should be entirely algebraic is obviously that R should vanish, *i.e.* $\phi\left(-\frac{b}{a}\right)=0$, or that $\phi(x)$ should contain $ax+b$ as a factor.

✓ *E.g.*

$$\int\frac{x^4+x^3}{x+2}\,dx=\int\left(x^3-x^2+2x-4+\frac{8}{x+2}\right)dx$$
$$=\frac{x^4}{4}-\frac{x^3}{3}+x^2-4x+8\log(x+2).$$

(2) A second process would be to put $ax+b=ay$, *i.e.* $x=y-\frac{b}{a}$ and then

$$\frac{\phi(x)}{ax+b}=\frac{\phi\left(y-\frac{b}{a}\right)}{ay}.$$

Then expand $\phi\left(y-\frac{b}{a}\right)$ in descending powers of y, thus expressing the fraction ultimately in the form $Q'+\frac{R'}{y}$, where Q is a series of powers of y and R' is independent of y.

Thus $\frac{\phi(x)}{ax+b}$ is expressed in a series of powers of $\left(x+\frac{b}{a}\right)$, together with a term $\frac{R'}{x+\frac{b}{a}}$, R' being independent of x and each term is again integrable.

Thus, in the foregoing case, $\int \frac{x^4+x^3}{x+2}\,dx$, putting $x+2=y$,

$$\frac{x^4+x^3}{x+2}=\frac{(y-2)^4+(y-2)^3}{y}=y^3-7y^2+18y-20+\frac{8}{y}$$
$$=(x+2)^3-7(x+2)^2+18(x+2)-20+\frac{8}{x+2}.$$

Hence

$$\int \frac{x^4+x^3}{x+2}\,dx=\frac{(x+2)^4}{4}-\frac{7}{3}(x+2)^3+9(x+2)^2-20(x+2)+8\log(x+2).$$

The results are of different form, but of course equivalent, except that they differ by a constant.

40. It is also to be observed that since the differential coefficients of $[\phi(x)]^{n+1}$ and $\log\phi(x)$ are respectively

$$(n+1)[\phi(x)]^n\phi'(x) \quad \text{and} \quad \frac{\phi'(x)}{\phi(x)},$$

we have
$$\int [\phi(x)]^n\phi'(x)\,dx=\frac{[\phi(x)]^{n+1}}{n+1}$$

and
$$\int \frac{\phi'(x)}{\phi(x)}\,dx=\log\phi(x).$$

The second of these results especially is of great use. It may be put into words thus:

The integral of any fraction of which the numerator is the differential coefficient of the denominator is **log (denominator)**.

41. For example:

$$\int (ax^2+bx+c)^n(2ax+b)\,dx=\frac{(ax^2+bx+c)^{n+1}}{n+1}.$$

$$\int \frac{2ax+b}{ax^2+bx+c}\,dx=\log(ax^2+bx+c).$$

$$\int \cot x\,dx=\int \frac{\cos x}{\sin x}\,dx=\log\sin x.$$

$$\int \tan x\,dx=-\int \frac{-\sin x}{\cos x}\,dx=-\log\cos x=\log\sec x.$$

$$\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx=\log(e^x+e^{-x}).$$

42. More generally, since the differential coefficient of

$$F[\phi(x)] \quad \text{is} \quad F'[\phi(x)]\phi'(x),$$

we clearly have

$$\int F'[\phi(x)]\phi'(x)\,dx=F[\phi(x)].$$

Thus, for example, $\int \frac{1}{1+\sin^2 x}\cos x\,dx=\tan^{-1}\sin x.$

EXAMPLES.

Write down the indefinite integrals of :

1. x^{10}, x^{-10}, 1, 0, $x^{\frac{7}{5}}$, $x^{-\frac{5}{7}}$, $\sqrt[3]{x^{-2}}$, $\frac{1}{\sqrt{x}}$, $x^7 \times x^{-\frac{3}{2}}$,

2. $a\sqrt{x}+\frac{b}{\sqrt{x}}$, $a\sqrt[p]{x}+\frac{b}{\sqrt[p]{x}}$, $(ax^{\frac{1}{p}}+b)(cx^{\frac{1}{q}}+d)$, $\frac{ax^2+bx+c}{x^2}$.

3. $\frac{(ax^2+bx+c)(ax^{-2}+bx^{-1}+c)}{x}$, $\frac{1}{a-x}$, $\frac{1}{(a-x)^2}$, $\frac{1}{(a-x)^p}$.

4. $\frac{1}{a-x}+\frac{1}{a+x}$, $\frac{x}{a+x}$, $\frac{1}{(a+x)^2}+\frac{1}{(a-x)^2}$, $\frac{(x-a)(x+a)(x^2+a^2)}{x^4}$.

5. Calculate $\int_0^2 x^{0\cdot 3}\,dx$; $\int_3^5 x^{-\frac{1}{3}}\,dx$; $\int_1^2 \frac{1}{2x+3}\,dx$.

6. Calculate $\int_2^3 (a+bx^3)^2\,3bx^2\,dx$ for the values $a=2$, $b=5$.

7. $\int_a^{2a}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)^2 dx$.

8. If the retardation of a particle be 2 foot-seconds per second, and its initial velocity 10 f.s., when and where will it come to a stop?

9. Given $pv=$ constant, and that $p=40$ when $v=10$, calculate $\int_{10}^{20} p\,dv$. What does this integration mean?

10. Calculate $\int (x-1)(x-2)(x-3)(x-4)\,dx$

between limits (*a*) 0 and 1, (*b*) 1 and 2, (*c*) 2 and 3, (*d*) 3 and 4, (*e*) 4 and 5.

Explain the signs which occur in the results. Illustrate by a graph.

11. Write down the indefinite integrals of:

(i) $(ae^x+b)^n e^x$, (ii) $\frac{ce^x}{ae^x+b}$, (iii) $\left(ax+\frac{b}{x}+c\right)^n\left(\frac{ax^2-b}{x^2}\right)$,

(iv) $(ax^p+be^x)^n(pax^{p-1}+be^x)$.

12. Integrate (i) $\int \frac{ae^{ax}+be^{bx}}{e^{ax}+e^{bx}}\,dx$, (ii) $\int \cot 2x\,dx$, (iii) $\int \tanh x\,dx$,

(iv) $\int \frac{2ax^n+b}{(ax^{2n}+bx^n+c)^p}\,x^{n-1}\,dx$.

13. Integrate

(i) $\int \frac{dx}{(1+x^2)\tan^{-1}x}$, (ii) $\int \frac{dx}{(1+x^2)(\tan^{-1}x)^n}$, (iii) $\int \frac{(\sin^{-1}x)^n}{\sqrt{1-x^2}}\,dx$,

(iv) $\int \frac{1}{\sin^{-1}x}\frac{dx}{\sqrt{1-x^2}}$, (v) $\int \frac{\text{vers}^{-1}x}{\sqrt{2x-x^2}}\,dx$.

14. Integrate (i) $\int \frac{dx}{x \log x}$, (ii) $\int \frac{1}{x \log x \log \log x} dx$,

(iii) $\int \frac{1}{x \log x \log \log x (\log \log \log x)^n} dx$,

(iv) $\int \frac{dx}{x\, l(x) l^2(x) l^3(x) \dots l^r(x)[l^{r+1}(x)]^n}$,

where $l^r x$ represent $\log \log \log \dots \log x$, the log being repeated r times.

43. It will now be perceived that, the operations of the Integral Calculus being of a tentative nature, success in Integration will depend in the first place on a knowledge of the results of differentiating the ordinary simple functions which occur in Algebra and Trigonometry. It is therefore necessary to learn the table of Standard Forms which is now appended. It is practically the same list as that already learnt for Differentiation, and the proofs of the facts stated lie in differentiating the right-hand members of the several results. The list was printed on page 46 of the Author's *Differential Calculus*. There are a few additions, as we are now specifically considering Integration. The list will be gradually extended, and a supplementary list will be given when the results have been established.

44. Preliminary Table of Results to be committed to Memory.

(1) $\int x^n \, dx = \frac{x^{n+1}}{n+1}$. (2) $\int \frac{1}{x} \, dx = \log_e x$.

(3) $\int e^x \, dx = e^x$. (4) $\int a^x \, dx = \frac{a^x}{\log_e a}$.

(5) $\int \cos x \, dx = \sin x$. (6) $\int \sin x \, dx = -\cos x$.

(7) $\int \sec^2 x \, dx = \tan x$. (8) $\int \operatorname{cosec}^2 x \, dx = -\cot x$.

(9) $\int \sec x \tan x \, dx = \int \frac{\sin x}{\cos^2 x} dx = \sec x$.

(10) $\int \operatorname{cosec} x \cot x \, dx = \int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x$.

$$\left.\begin{aligned} &(11)\ \int \tan x \, dx = \log_e \sec x. \\ &(12)\ \int \cot x \, dx = \log_e \sin x. \end{aligned}\right\} \text{(Art. 41.)}$$

(13) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$ or $-\cos^{-1}\frac{x}{a}$.*

(14) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ or $-\frac{1}{a}\cot^{-1}\frac{x}{a}$.

(15) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}$ or* $-\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}$.

(16) $\int \frac{dx}{\sqrt{2ax-x^2}} = \operatorname{vers}^{-1}\frac{x}{a}$ or $-\operatorname{covers}^{-1}\frac{x}{a}$.

45. It is a help to the memory to observe the dimensions of each side. For instance, x and a being supposed linear, $\int \frac{dx}{\sqrt{a^2-x^2}}$ is of zero dimensions. There could, therefore, be no $\frac{1}{a}$ prefixed to the integral. On the other hand, $\int \frac{dx}{a^2+x^2}$ is of dimensions -1. Hence the result of integration must be of dimensions -1. Thus the integral could not be $\tan^{-1}\frac{x}{a}$, which is of zero dimensions. There should, therefore, be no difficulty in remembering in which cases the factor $\frac{1}{a}$ appears, and when it does not.

Also, so long as we are dealing with the trigonometrical functions, whenever the result begins with the letters "co," it must be with a negative sign. The reason is obvious; the cosine, cosecant, coversine and their inverses are all decreasing functions as x increases through the first quadrant, and their differential coefficients are negative.

The rule of the "co" does not apply to the hyperbolic functions.

Examples.

Write down the indefinite integrals of the following functions:

1. $\frac{1}{x+1}$, $\frac{x-a}{x+a}$, $\frac{x}{x^2+a^2}$, $\frac{x+a}{x^2+a^2}$, $\frac{x^2}{x^3+a^3}$, $\frac{x^{n-1}}{x^n+a^n}$.

2. 2^x, $2x$, $\frac{2}{x}$, x^2, x^3+3^x, $a+b^x+c^{2x}+d^{3x}$.

3. $\cos^2\frac{x}{2}$, $\sin^2\frac{x}{2}$, $\cot x+\tan x$, $\cos x\left(\frac{1}{\sin x}+\frac{1}{\sin^2 x}\right)$.

4. $\frac{1}{\sqrt{9-x^2}}$, $\frac{1}{\sqrt{9+x^2}}$, $\frac{1}{\sqrt{x^2-9}}$, $\frac{1}{9+x^2}$, $\frac{1}{9-x^2}$, $\frac{1}{x^2-9}$.

5. $\frac{1}{x\sqrt{x^2-4}}$, $\frac{x+1}{x\sqrt{x^2-4}}$, $\frac{ax+b}{\sqrt{c^2-x^2}}$, $\frac{ax+b}{\sqrt{x^2-c^2}}$, $\frac{ax+b}{\sqrt{x^2+c^2}}$.

6. $\frac{1}{\sqrt{x-x^2}}$, $\frac{1}{x\sqrt{3x^2-27}}$, $\frac{1}{\sqrt{27-3x^2}}$, $\frac{x^2-4}{x^2+4}$, $\frac{x^2+4}{x^2-4}$.

* See also Art. 1890, Vol. II.

7. Write down the indefinite integrals of:

$$\text{(i)} \int \sin^{-3} x \cos x \, dx, \quad \text{(ii)} \int \cot x \sec^2 x \, dx, \quad \text{(iii)} \int (e^x + a)^n e^x \, dx,$$

$$\text{(iv)} \int (x^3 + a^3)^n x^2 \, dx, \quad \text{(v)} \int (ax^3 + bx + c)^n (3ax^2 + b) \, dx.$$

8. Write down the indefinite integrals of:

$$\text{(i)} \int \frac{\frac{1}{1+x^2}}{\tan^{-1} x} dx, \quad \text{(ii)} \int \frac{dx}{\sqrt{1-x^2}(\sin^{-1} x)^2}, \quad \text{(iii)} \int \frac{dx}{x(\log x)^3}.$$

9. Evaluate $\text{(i)} \int_1^2 \frac{2x+1}{(x^2+x+1)} dx, \quad \text{(ii)} \int_1^2 \frac{2x+1}{(x^2+x+1)^2} dx.$

10. Draw the graph of $64(x-2)(x-3)(2x-5)$, and show that the area between $x=2$ and $x=2{\cdot}5$, bounded by the curve and the x-axis, is

$$32 \Big[(x-2)^2 (x-3)^2 \Big]_2^{2.5}, \quad i.e. = 2 \text{ square units.}$$

Verify by multiplying out and integrating each term.

11. Write down the values of:

$$\text{(i)} \int_0^x \frac{e^{3x} + e^{5x}}{e^x + e^{-x}} dx, \quad \text{(ii)} \int_0^x \frac{e^{(n+1)x} - e^{(n-1)x}}{\sinh x} dx,$$

$$\text{(iii)} \int_0^1 \frac{e^{2x} + e^{4x}}{e^{3x}} dx, \quad \text{(iv)} \int_a^b \cosh(\log x) \, dx.$$

12. Evaluate

$$\text{(i)} \int_0^{\frac{\pi}{2}} \cos x \, dx, \qquad \text{(ii)} \int_0^{\frac{\pi}{2}} \cos^2 x \, dx,$$

$$\text{(iii)} \int_0^{\frac{\pi}{4}} \cos 2x \, dx, \qquad \text{(iv)} \int_0^x (\cosh x + \cos x) dx.$$

13. Evaluate

$$\text{(i)} \int_0^{\frac{\pi}{4n}} \sec^2 nx \, dx, \qquad \text{(ii)} \int_0^{\frac{\pi}{4}} \sec x \tan x \, dx,$$

$$\text{(iii)} \int_1^{\sqrt{3}} \frac{dx}{1+x^2}, \qquad \text{(iv)} \int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

14. Evaluate

$$\text{(i)} \int \frac{x^n - a^n}{x-a} dx, \quad \text{(ii)} \int \frac{x^n}{x-a} dx, \quad \text{(iii)} \int \frac{x^5 - 2x + 1}{x-1} dx,$$

$$\text{(iv)} \int \frac{x^5}{x-1} dx, \qquad \text{(v)} \int \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 3x + 2} dx.$$

46. The processes of Integration being necessarily of a tentative nature and founded upon a knowledge of the forms obtained by differentiating the known functions—algebraic, logarithmic, exponential, trigonometric or hyperbolic, or the inverse forms, it will be realized that many expressions may be written down which are not the differential coefficients of such known functions or of any combination of them. A little consideration will show that this is necessarily the case.

If the inverse sine had never received the consideration of mathematicians, the expression $\frac{1}{\sqrt{1-x^2}}$ would have been the differential coefficient of something so far uninvented. In the same way, if the invention of a logarithm had not preceded the necessity for the integration of x^{-1}, the integral of $\frac{1}{x}$ would have been lacking and have presented difficulty.

Hence it will be seen that it is only certain classes of algebraic, trigonometrical, exponential, logarithmic, or hyperbolic functions, or the corresponding inverse functions, that admit of integration in finite terms. Some functions there are which admit of integration in terms of an infinite series though such series may not be otherwise expressible as the expansion of any known function. For example,

$\frac{1}{\sqrt{1-x^{100}}}$ is not the differential coefficient of any known function. But supposing $x<1$,

$$\int_0^x \frac{dx}{(1-x^{100})^{\frac{1}{2}}} = \int_0^x \left(1+\tfrac{1}{2}x^{100}+\frac{1 \cdot 3}{2 \cdot 4}x^{200}+\ldots\right)dx$$
$$= x+\tfrac{1}{2}\frac{x^{101}}{101}+\frac{1 \cdot 3}{2 \cdot 4}\frac{x^{201}}{201}+\ldots,$$

an infinite series, not capable of summation, but nevertheless useful for approximative purposes, supposing x to be a positive proper fraction, if such arithmetical approximation be required.

And to go back to the case of x^{-1}, it is also clear that as by the failure to integrate it, by considering it a case of $\int x^n\,dx = \frac{x^{n+1}}{n+1}$, there would have been a gap in the list of integrals of powers of x, viz.,

$$\int x^2 dx = \frac{x^3}{3}, \quad \int x\,dx = \frac{x^2}{2}, \quad \int x^0\,dx = \frac{x^1}{1}, \quad \int \mathbf{x^{-1}\,dx = ?} \quad \int x^{-2}dx = \frac{x^{-1}}{-1},$$

the properties of a function which had x^{-1} for its differential coefficient could not long have remained undiscovered.

For if $F(x)$ stand for $\int \frac{1}{x}\,dx$, we must have

$$F(x)+F(y) = \int \frac{dx}{x} + \int \frac{dy}{y}$$
$$= \int \left(\frac{dx}{x}+\frac{dy}{y}\right) = \int \frac{x\,dy+y\,dx}{xy}$$
$$= \int \frac{d(xy)}{xy} = F(xy), \quad i.e.\ xy = F^{-1}[F(x)+F(y)],$$

which constitutes the fundamental theorem of logarithms and indicates how an addition may be used to perform a multiplication when tables of $F(x)$ have been constructed.

In a similar way, the expression $\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, where k is a constant < 1, presents itself in the consideration of many problems geometric and kinetic. Now $\frac{1}{\sqrt{1-k^2\sin^2\theta}}$ is not the differential coefficient of any combination of algebraic, exponential or circular functions. Hence, this is a case in point. This is an integral where necessity for discussion has arisen prior to a knowledge of the expression of which it is the differential coefficient. Calling it u,

$$u=\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

We call the upper limit ϕ the amplitude of u, and write $\phi = \operatorname{am} u$, and inversely $u = \operatorname{am}^{-1}\phi$. Thus u receives a name. It is a function whose leading properties we propose to discuss later.

EXAMPLES.

1. Write down the indefinite integrals of:

(1) $\frac{(x+a)(x+b)(x+c)}{x^4}$, (2) $\frac{x^3+a^3+b^3-3abx}{x+a+b}$,

(3) $\sqrt[p-q]{x^{\frac{q+r}{r-p}}}\,\sqrt[q-r]{x^{\frac{r+p}{p-q}}}\,\sqrt[r-p]{x^{\frac{p+q}{q-r}}}$, (4) $\frac{a\cos x-b\sin x}{a\sin x+b\cos x+c}$,

(5) $e^{a\log x}+e^{x\log a}$, (6) $\frac{\tan^{-1}\frac{x}{3}}{9+x^2}$,

(7) $\frac{1}{\sin x\cos x}$, (8) $\frac{\cos x(1+\sin x)}{\sin^2 x}$,

(9) $\frac{1}{1-\cos x}$, (10) $\sin\left(x+\frac{\pi}{4}\right)$,

(11) $\frac{x^2+\sin^2 x}{x^2+1}\sec^2 x$, (12) $(1+\sin x\cos x)\sec^2 x$,

(13) $\tan x(1+\sec x)$, (14) $\frac{a\sin^3 x+b\cos^3 x}{\sin^2 x\cos^2 x}$,

(15) $(\cos x-\sin x)(2+\sin 2x)\sec^2 x\operatorname{cosec}^2 x$,

(16) $(a+\tan x)(b+\tan x)\sec^2 x$,

(17) $\dfrac{1}{x[1+(\log x)^2]}$, (18) $\dfrac{1}{x}\cos(\log x)$,

(19) $\dfrac{x^5+1}{x-1}$, (20) $\dfrac{e^x}{a^2e^{2x}+1}$.

2. (*a*) If $f(x)=\dfrac{1}{1-x}$, prove that $\displaystyle\int fff(x)\,dx=\frac{x^2}{2}$.

(*b*) If $f(x)=a+bx$, prove that $\displaystyle\int f^r(x)\,dx=a\frac{b^r-1}{b-1}x+b^r\frac{x^2}{2}$, where $f^r(x)$ means $fff\ldots f(x)$, the functionality symbol f occurring r times.

3. Show by expansion that

(*a*) $\displaystyle\int\log(1+x)\,dx=\frac{x^2}{1\,.\,2}-\frac{x^3}{2\,.\,3}+\frac{x^4}{3\,.\,4}-\frac{x^5}{4\,.\,5}+\ldots=(1+x)\log(1+x)-x$,

(*b*) $\displaystyle\int\log\frac{1+x+x^2}{1-x+x^2}\,dx$

$$=2\left\{\frac{x^2}{1.2}-\frac{2x^4}{3.4}+\frac{x^6}{5.6}+\frac{x^8}{7.8}-\frac{2x^{10}}{9.10}+\frac{x^{12}}{11.12}+\frac{x^{14}}{13.14}-\frac{2x^{16}}{15.16}+\ldots\right\},$$

(*c*) $\displaystyle\int(1+x)^n\,dx=x+\binom{n}{1}\frac{x^2}{2}+\binom{n}{2}\frac{x^3}{3}+\binom{n}{3}\frac{x^4}{4}+\binom{n}{4}\frac{x^5}{5}+\ldots$,

where $$\binom{n}{r}\equiv\frac{n(n-1)\ldots(n-r+1)}{1\,.\,2\ldots r}.$$

4. Prove by Differentiation or Integration from the Binomial Expansion of $(1+x)^n$, where n is a positive integer,

(*a*) $1C_1+2C_2+3C_3+\ldots+nC_n=n2^{n-1}$,

(*b*) $1\,.\,2C_2+2\,.\,3C_3+3\,.\,4C_4+\ldots+(n-1)nC_n=n(n-1)2^{n-2}$,

(*c*) $1C_1+3C_3+5C_5+\ldots=n2^{n-2}$,

(*d*) $1^2C_1+2^2C_2+3^2C_3+\ldots+n^2C_n=n(n+1)2^{n-2}$,

(*e*) $1^3C_1+2^3C_2x+3^3C_3x^2+\ldots+n^3C_nx^{n-1}$

$$=x\{n^2x^2+(3n-1)x+1\}(1+x)^{n-3},$$

(*f*) $1C_0+2C_1+3C_2+\ldots+(n+1)C_n=2^{n-1}(n+2)$,

(*g*) $\dfrac{C_0}{1}+\dfrac{C_1}{2}+\dfrac{C_2}{3}+\ldots+\dfrac{C_n}{n+1}=\dfrac{2^{n+1}-1}{n+1}$,

(*h*) $\dfrac{C_0}{1\,.\,2}+\dfrac{C_1}{2\,.\,3}+\dfrac{C_2}{3\,.\,4}+\ldots+\dfrac{C_n}{(n+1)(n+2)}=\dfrac{2^{n+2}-n-3}{(n+1)(n+2)}$,

(*i*) $\dfrac{C_0}{1.2.3}-\dfrac{C_1}{2.3.4}+\dfrac{C_2}{3.4.5}-\ldots+\dfrac{(-1)^nC_n}{(n+1)(n+2)(n+3)}=\dfrac{1}{2(n+3)}$,

(*j*) $\dfrac{3\,.\,4}{1\,.\,2}C_0-\dfrac{4\,.\,5}{2\,.\,3}C_1+\dfrac{5\,.\,6}{3\,.\,4}C_2-\ldots+(-1)^n\dfrac{(n+3)(n+4)}{(n+1)(n+2)}C_n$

$$=2\frac{3n+5}{(n+1)(n+2)}.$$

5. Prove from the expansions of $\sin x$ and $\cos x$ in powers of x that $\int_0^x \sin x\,dx = 1 - \cos x$ and that $\int_0^x \cos x\,dx = \sin x$.

6. Prove from the expansion of $\exp x$ that

$$\int_{-\infty}^{x} \exp x\,dx = \exp x \quad [\exp x \equiv e^x].$$

7. Prove that

$$(a)\ \int_1^x \frac{(1+x)^n}{x}\,dx = \log x + \binom{n}{1}(x-1) + \binom{n}{2}\frac{x^2-1}{2} + \binom{n}{3}\frac{x^3-1}{3} + \dots,$$

$$(b)\ \int_2^x \frac{(x+1)^n}{x-1}\,dx = 2^n\log(x-1) + \binom{n}{1}2^{n-1}[(x-1)-1]$$
$$+\binom{n}{2}2^{n-2}\frac{(x-1)^2-1}{2} + \binom{n}{3}2^{n-3}\frac{(x-1)^3-1}{3} + \dots.$$

8. Show that

$$\int (a+bx)(x+c)^n dx = b\frac{(x+c)^{n+2}}{n+2} + (a-bc)\frac{(x+c)^{n+1}}{n+1}.$$

9. Show that

$$\int_a^b (1+x)^n dx = (b-a) + \binom{n}{1}\frac{b^2-a^2}{2} + \binom{n}{2}\frac{b^3-a^3}{3} + \dots + \binom{n}{n}\frac{b^{n+1}-a^{n+1}}{n+1}.$$

10. If $\phi(x)$ be a rational integral algebraic function of x, show that

$$\int_c^x \phi(x)(x-c)^n dx = (x-c)^{n+1}\left[\frac{\phi(c)}{n+1} + \frac{\phi'(c)}{n+2}\frac{x-c}{\lfloor 1} + \frac{\phi''(c)}{n+3}\frac{(x-c)^2}{\lfloor 2} + \dots\right].$$

11. By considering $\int (x-a)^p(x-b)^q\,dx$, show that the difference of the series (p and q being positive integers)

$$\frac{(x-b)^{p+q+1}}{p+q+1} + p(b-a)\frac{(x-b)^{p+q}}{p+q} + \frac{p(p-1)}{1\,.\,2}(b-a)^2\frac{(x-b)^{p+q-1}}{p+q-1} + \dots,$$

$$\frac{(x-a)^{p+q+1}}{p+q+1} - q(b-a)\frac{(x-a)^{p+q}}{p+q} + \frac{q(q-1)}{1\,.\,2}(b-a)^2\frac{(x-a)^{p+q-1}}{p+q-1} - \dots$$

is independent of x.

12. Verify by differentiation that

$$(1)\ \int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2},$$

$$(2)\ \int \frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}\,[1+x\sqrt{1-x^2}]}\,dx = 2\tan^{-1}(x+\sqrt{1-x^2}).$$

13. If $\phi(x)=A_0x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots+A_n$, prove that

$$\int\frac{\phi(x)}{x-h}dx=B_0\frac{x^n}{n}+B_1\frac{x^{n-1}}{n-1}+B_2\frac{x^{n-2}}{n-2}+\ldots+\phi(h)\log(x-h)$$

where $B_r=hB_{r-1}+A_r$. Write down the values of B_0, B_1, B_2.

14. Show that $\int\frac{\phi(x)}{x-h}dx$ for rational integral algebraic forms of $\phi(x)$ may also be expressed as

$$\phi(h)\log(x-h)+\phi'(h)\frac{(x-h)}{1\,\underline{|1}}+\phi''(x)\frac{(x-h)^2}{2\,\underline{|2}}+\phi'''(h)\frac{(x-h)^3}{3\,\underline{|3}}+\ldots.$$

Prove that

$$\int\frac{e^x-e^h}{x-h}dx=e^h\left[\frac{x-h}{1^2}+\frac{(x-h)^2}{2^2\,\underline{|1}}+\frac{(x-h)^3}{3^2\,\underline{|2}}+\frac{(x-h)^4}{4^2\,\underline{|3}}+\ldots\right].$$

15. Prove that $\displaystyle\int_a^b\frac{\log x}{x}dx=\tfrac{1}{2}\log(ab)\log\left(\frac{b}{a}\right).$

16. If

$$J_n(x)=\frac{x^n}{2^n n!}\left[1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2\,.\,4(2n+2)(2n+4)}-\ldots\right]$$

(*i.e.* Bessel's function), prove that

$$(1)\quad\int_0^x J_1(x)dx=1-J_0(x),$$

$$(2)\quad\int_0^x[J_{n-1}(x)-J_{n+1}(x)]\,dx=2J_n(x)\quad(n>0).$$

17. Prove that

$$\lambda\xi^{-\lambda}\int_0^\xi x^{\lambda-1}(1-x^\mu)^\nu\,dx=F\left(-\nu,\frac{\lambda}{\mu},\frac{\lambda}{\mu}+1,\xi^\mu\right)$$

when $F(\alpha,\beta,\gamma,x)$ denotes the hypergeometric series

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}x+\frac{\alpha\,\overline{\alpha+1}}{1\,.\,2}\frac{\beta\,\overline{\beta+1}}{\gamma\,\overline{\gamma+1}}x^2+\frac{\alpha\,\overline{\alpha+1}\,\overline{\alpha+2}}{1\,.\,2\,.\,3}\cdot\frac{\beta\,\overline{\beta+1}\,\overline{\beta+2}}{\gamma\,\overline{\gamma+1}\,\overline{\gamma+2}}x^3+\ldots.$$

[I. C. S., 1898.]

18. Assuming that the speed of the current in a river at a distance x from the bank follows the law

$$v=v_0+kx(a-x),$$

where a is the breadth of the river and v_0 and k are constants, find by integration how far down stream a man will be carried who rows 4 miles an hour, pointing the boat's head always straight at the opposite bank, so as to cross in the least time possible: the width of the river being half a mile, the banks being straight and parallel, and the speed of the current being 2 miles an hour near the banks, and 3 miles an hour in mid-stream. [I. C. S., 1905.]

19. Find the moment of inertia of a rectangle of sides $2a$, $2b$ about a line joining the mid-points of the opposite sides of length $2a$.

The section of a ship at the water line is 120 feet long. If the middle line be divided into six equal portions, the ordinates of the boundary of the area at the middle points of the segments are given by the following table:

Distances from end	10	30	50	70	90	110
Ordinates - -	10	20	20	18	14	8

Draw a figure showing the section, remembering that it is symmetrical on both sides of the middle line. By approximate methods of summation (treating the segments as rectangles) find the value of Ak^2, the moment of inertia of the section about the middle line, and the height of the metacentre above the centre of buoyancy if the ship displaces 36,000 cubic feet of water.

$$\left[\text{The height required} = \frac{\text{Moment of Inertia}}{\text{Displacement}}.\right]$$

[I. C. S., 1908.]

20. A substance A transforms into a substance B, the rate of transformation in grammes per second at the time t being equal to ax, where x denotes the number of grammes of A existing at that instant. In like manner B transforms into a third substance C, the rate of transformation being by, where y is the number of grammes of B existing at time t.

Write down the relations between x, y and their differential coefficients with respect to the time, and show that these equations are satisfied by putting

$$x = Pe^{-at}, \quad y = Qe^{-at} + Re^{-bt},$$

provided Q and R are properly determined in terms of P, y being zero when $t = 0$.

Also show that the quantity of B existing will be greatest at

$$\frac{\log \text{Nap}\, a - \log \text{Nap}\, b}{a - b} \text{ seconds after the zero of time.}$$

[I. C. S., 1908.]

21. Prove that the integral of the sum of an infinite series taken over a range within which the series is absolutely convergent, is equal to the sum of the integrals of the terms.

Employ this theorem to find an expansion of $\log(1 + z)$ in ascending powers of z, pointing out the range for which the expansion is valid.

Having given $\log_e 2 = 0{\cdot}69315$, prove that $\log_e 61 = 4{\cdot}1109$ to five significant figures. [I. C. S., 1904.]

22. OK is the diameter bounding a semicircle of radius r, P any point on OK, and PQ an ordinate to the diameter OK. If x denote the length OP and z the area which PQ cuts off the semicircle, interpret $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$.

Find a curve for which the area bounded by the curve, the axes of x and y and the ordinate at a distance x from the axis of y, is $a^2 \tan \frac{x}{a}$. [I. C. S., 1902.]

23. From the equation

$$\frac{1}{y}\left(ah + \int_h^x y\,dx\right) = h,$$

where a and h are constants, find y in terms of x.

The value of a being 2 feet, and of h 10 feet, evaluate y when x is 30 feet. [I. C. S., 1910.]

24. Denoting by A the area between the curve $y=f(x)$ and the axis of x, from the value zero to the value a of x, show that, when $f(x)$ is a rational integral algebraic function of the *third* degree,

$$A = \frac{a}{6}(y_0 + 4y_1 + y_2),$$

where $$y_0 = f(0), \quad y_1 = f\left(\frac{a}{2}\right), \quad y_2 = f(a).$$

Compare the result given by this rule with the true value, taken to three places of decimals, for the curve $y = \sin x$, between the values 0 and 0·5 of x reckoned in radians. [I. C. S., 1912.]

25. Verify that the area of the curve

$$y = A + Bx + Cx^2 + Dx^3$$

between the limits $x=h$ and $x=-h$ is equal to the product of h and the sum of the ordinates at

$$x = h/\sqrt{3} \quad \text{and} \quad x = -h/\sqrt{3}.$$

In the case of the curve

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \equiv f(x),$$

verify in like manner that the area between $x=h$ and $x=-h$ is equal to

$$\{5f(h\sqrt{3/5}) + 8f(0) + 5f(-h\sqrt{3/5})\}h/9.$$

[I. C. S., 1913.]

26. Find the differential coefficient of

$$e^{-x}(1 + x + x^2/2! + x^3/3! + x^4/4!);$$

and deduce that the sum of the first five terms of the exponential series is less than e^x by the quantity

$$e^x \int_0^x \frac{a^4 e^{-a}}{4\,!}\, da.$$

What would be the corresponding result if the series were taken to n terms instead of to five terms? [I. C. S., 1913.]

27. Weddle's Rule for finding, approximately, the area bounded by a curve, two ordinates, and a base forming part of the axis of x, is: "Divide the base into six equal parts and draw the ordinates at the points of division, making, with the extreme ordinates, seven in all. Of these ordinates add the first, third, fourth, fifth and seventh, and five times the second, fourth and sixth. Multiply the sum by one-twentieth of the base."

Prove that the rule gives the true result when the limits are 0 and 1 and the curve has any of the forms

$$y = a, \quad y = ax^2, \quad y = ax^4, \quad y = a(x - 1/2)^n,$$

where a is a constant and n is an odd positive integer.

Find, by the rule, the value of $6 \times \int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1 + x^2}$ to seven places of decimals. Check the result by integration. [I. C. S., 1911.]

28. Show that the work done by a gas in altering its volume from v_1 to v_2 according to the Adiabatic Law

$$pv^\gamma = p_0 v_0^\gamma$$

is

$$\frac{p_0 v_0^\gamma}{\gamma - 1}\left[\frac{1}{v_1^{\gamma-1}} - \frac{1}{v_2^{\gamma-1}}\right].$$

If the law be Isothermal ($\gamma = 1$), show that this becomes

$$p_0 v_0 \log \frac{v_2}{v_1}.$$

If a gas expands isothermally from state p_1, v_1 to state p_2, v_2 (Operation I.),

then expands adiabatically from state p_2, v_2 to state p_3, v_3 (Operation II.),

then contracts isothermally from state p_3, v_3 to state p_4, v_4 (Operation III.),

then contracts adiabatically from state p_4, v_4 to state p_1, v_1 (Operation IV.),

(1) find the amounts of work done by or upon the gas during each of these four operations, drawing a graph of the whole cycle of changes;

(2) show that the work done in the whole cycle of operations is measured by

$$(p_1v_1 - p_3v_3)\log\frac{v_2}{v_1} \text{ (the adiabatic portions cancelling)};$$

(3) that $$v_1v_3 = v_2v_4;$$

(4) that, writing

$$p_1v_1 = p_2v_2 = \alpha_1, \qquad p_3v_3 = p_4v_4 = \alpha_2,$$
$$p_2v_2^\gamma = p_3v_3^\gamma = \beta_1, \qquad p_4v_4^\gamma = p_1v_1^\gamma = \beta_2,$$

the above expression for the work may be written

$$\frac{\alpha_1 - \alpha_2}{\gamma - 1}\log\frac{\beta_1}{\beta_2}.$$

[This cycle of operations is known as a Carnot's cycle for a perfect heat engine.]

29. If dQ be the whole heat absorbed by a body of uniform temperature whilst its temperature changes continuously from θ to $\theta + d\theta$, and if ϕ be a function of the independent variables which define the state of the body and such that

$$d\phi = \frac{dQ}{\theta},$$

ϕ is called the Entropy of the body (Clausius).

Show that if a graph be drawn to represent θ as a function of ϕ, the area between the graph, the ϕ-axis and the ordinates corresponding to the initial and final states represents on some scale the heat absorbed.

In the case of a perfect gas satisfying the law $\frac{pv}{\theta} = \text{const.} = R$, assume the Thermodynamic Equation

$$dQ = C_v d\theta + p\,dv,$$

where C_v is the specific heat at constant volume, and show that in changing from state θ_1, v_1, ϕ_1 to state θ_2, v_2, ϕ_2,

$$\left(\frac{\theta_2}{\theta_1}\right)^{C_v}\left(\frac{v_2}{v_1}\right)^{R} = \frac{e^{\phi_2}}{e^{\phi_1}}.$$

Taking the temperature as a function of the entropy and simultaneous values of ϕ and θ as given in the following table:

ϕ	1·60	1·70	1·75	1·85
θ	450	400	370	340

and assuming that one unit of area indicates one unit of heat, what is the total heat received during these changes?

[There is a brief sketch of the fundamental formulae of Thermodynamics on pages 56 and 57 of *Solutions of Senate House Problems for* 1878 which may be found useful. Students may also read Tait's *Thermodynamics* or Parker's *Thermodynamics* for detailed accounts of the theory; other useful books are Zeuner, *Théorie Mécanique de la Chaleur*; Briot, *Théorie Mécanique de la Chaleur*.]

30. In the case of a saturated vapour, if C' be the specific heat of the vapour, *i.e.* the heat imparted to one gramme of the saturated vapour to keep it constantly in the saturated state when slowly compressed till the temperature rises one degree Fahrenheit; C that of the liquid from which it is derived at the same pressure and temperature, L the latent heat, then it can be shown that

$$C' = C + \frac{dL}{d\theta} - \frac{L}{\theta},$$

where θ is the absolute temperature.

Let C_p be the specific heat of the liquid at constant pressure, which, as liquids are practically incompressible, is so nearly equal to C that no appreciable error results from regarding them as identical.

Then Regnault has shown experimentally that the sum of the free and latent heat, viz. $L + \int_{273}^{\theta} C_p d\theta$, is not a constant as had been supposed by Watt in his earlier experiments, but is a function of the temperature θ, viz. putting $\theta = 273 + \theta'$ and J being the number of ergs in one calorie (41,539,739·8 ergs or about 3 foot-lbs.), he obtained the equations

$$\left.\begin{aligned} &(1)\quad L + \int_{273}^{\theta} C_p d\theta = J(\alpha + \beta\theta' + \gamma\theta'^2), \\ &(2)\qquad\qquad\qquad C_p = J(\alpha' + \beta'\theta' + \gamma'\theta'^2), \end{aligned}\right\}$$

experimentally, determining the constants α, β, γ; α', β', γ' for several vapours.

Using these data, prove that

$$(1)\quad \frac{dL}{d\theta} + C_p = J(\beta + 2\gamma\theta'),$$

$$(2)\quad L = J\left[\alpha + (\beta - \alpha')\theta' + (2\gamma - \beta')\frac{\theta'^2}{2} - \gamma'\frac{\theta'^3}{3}\right],$$

$$(3)\quad c' \equiv \frac{C'}{J} = \beta + 2\gamma\theta' - \frac{\alpha + (\beta - \alpha')\theta' + (2\gamma - \beta')\dfrac{\theta'^2}{2} - \gamma'\dfrac{\theta'^3}{3}}{273 + \theta'}.$$

31. Show that the integral equivalent of the equation

$$\int_0^x \frac{dx}{1+x+x^2} + \int_0^y \frac{dy}{1+y+y^2} + \int_0^z \frac{dz}{1+z+z^2} = 0$$

is of the form

$$xyz + a(yz+zx+xy) + b(x+y+z) + c = 0,$$

where a, b, c are certain constants. [OXF. I. P., 1913.]

32. If the variables x, y, z be so related that

$$yz = F(x), \quad zx = F(y), \quad xy = F(z),$$

show that $$\int_{x_1}^x F(x)\,dx + \int_{y_1}^y F(y)\,dy + \int_{z_1}^z F(z)\,dz = xyz - x_1y_1z_1.$$

For example, if $$x+y+z=0,$$
and $$yz+zx+xy = -\tfrac{1}{4}x^2y^2z^2,$$
show that

$$\int_{x_1}^x \frac{\sqrt{1+x^4}}{x^2}\,dx + \int_{y_1}^y \frac{\sqrt{1+y^4}}{y^2}\,dy + \int_{z_1}^z \frac{\sqrt{1+z^4}}{z^2}\,dz = \tfrac{3}{4}(xyz - x_1y_1z_1).$$

[BERTRAND, *Calc. Int.*, p. 383.]

33. If $y = \int_0^x e^{\sin x}\,dx$, expand y in powers of x as far as x^5.

[OXF. I. P., 1911.]

34. Prove that $$\int_0^\infty \frac{x^2+3x+3}{(x+1)^3} e^{-x} \sin x\,dx = \tfrac{1}{2}.$$ [OXF. I. P., 1915.]

35. Integrate $$\int x^{-2n-2}(1-x)^n(1-cx)^n\,dx,$$

where n is a positive integer. [OXF. I. P., 1917.]

36. Prove that the fifth differential coefficient of

$$x^4\int \phi(x)\,dx - 4x^3\int x\phi(x)\,dx + 6x^2\int x^2\phi(x)\,dx - 4x\int x^3\phi(x)\,dx + \int x^4\phi(x)\,dx$$

is $$24\phi(x).$$ [OXF. I. P., 1917.]

37. Integrate $$\int \frac{\cos 8\theta - \cos 7\theta}{1+2\cos 5\theta}\,d\theta.$$

38. If $f(x)$ and $F(x)$ be two functions continuous and finite between 0 and x, such that

$$F(x) \equiv \int_0^x f(t)\,dt,$$

$$f(x) \equiv 1 - \int_0^x F(t)\,dt,$$

obtain their expansions in ascending powers of x. [OXF. I. P., 1915.]

39. Prove that $\int_0^1 \frac{dx}{(1+x)(2+x)} = 0{\cdot}288$ nearly.

[MATH. TRIP. I., 1916.]

40. If $y^2 = a^2x^2 + c$, express in terms of y the differential coefficients of the functions

$$\log(ax+y), \quad xy$$

with regard to x.

Hence evaluate $\int \frac{dx}{y}$ and $\int y\,dx$, and prove that

$$\int_2^{\frac{5}{2}} \sqrt{x^2-4}\,dx = \tfrac{15}{8} - 2\log 2.$$

[MATH. TRIP. I., 1914.]

41. Prove that

$$\{\log(a+\theta h) - \log a\} - \theta\{\log(a+h) - \log a\}$$

can be expressed in the form

$$\theta(1-\theta)\int_0^h \frac{x\,dx}{(a+x)(a+\theta x)}.$$

Deduce that in calculating a logarithm to base 10 by the method of proportional parts from tables which give the logarithms of all integers from 10^4 to 10^5, the error is one of defect and cannot amount to $\frac{1}{8}10^{-8}\mu$, where $\mu = \log_{10}e = {\cdot}43429$. Is this negligible in seven-figure tables?

[MATH TRIP. Pt. II., 1919.]

42. Integrate $\int \sin\theta \sqrt{\frac{1+\cos 9\theta}{1+\cos\theta}}\,d\theta$, and show that

$$\int \sqrt{\frac{1+\cos(4n+1)\theta}{1+\cos\theta}}\,d\theta = \theta - 2\left\{\frac{\sin\theta}{1} - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \dots - \frac{\sin 2n\theta}{2n}\right\},$$

n being a positive integer.

CHAPTER III.

CHANGE OF THE INDEPENDENT VARIABLE.

47. It will frequently facilitate integration if we change the independent variable x to a new variable z by a suitable choice of relation connecting the two.

Let $x=F(z)$ be the relation chosen, and let

$$\int V\,dx \quad \text{or} \quad \int f(x)\,dx$$

be the integral to be transformed.

Let $$u=\int V\,dx.$$

Then $$\frac{du}{dx}=V.$$

But $$\frac{du}{dz}=\frac{du}{dx}\frac{dx}{dz}=V\frac{dx}{dz}\equiv VF'(z), \quad i.e.\ f[F(z)]F'(z).$$

$$\therefore\ u=\int V\frac{dx}{dz}\,dz \quad \text{or} \quad \int f[F(z)]F'(z)\,dz.$$

48. Thus, to integrate $\int\frac{e^{\tan^{-1}x}}{1+x^2}\,dx$, let $\tan^{-1}x=z$ or $x=\tan z$.

$$\frac{dx}{dz}=\sec^2 z\,;$$

$$\therefore\ \int\frac{e^{\tan^{-1}x}}{1+x^2}\,dx=\int\frac{e^z}{1+\tan^2 z}\frac{dx}{dz}\,dz=\int\frac{e^z}{1+\tan^2 z}\sec^2 z\,dz$$

$$=\int e^z\,dz=e^z=e^{\tan^{-1}x}.$$

Instead of writing $x=\tan z$, it would be a little shorter to take $\tan^{-1}x=z$, and then $\frac{1}{1+x^2}\frac{dx}{dz}=1$. And

$$\text{the integral}=\int e^z\,dz, \text{ at once}$$

$$=e^z=e^{\tan^{-1}x}.$$

49. In the practical use of the formula

$$\int f(x)\,dx=\int f[F(z)]\,F'(z)\,dz,$$

after having made choice of the transformation $x=F(z)$, it is usual to make use of *differentials*, and instead of writing

$$\frac{dx}{dz}=F'(z),$$

we shall write the same equation as

$$dx=F'(z)\,dz,$$

and the formula will thus be reproduced by replacing dx in the integral $\int f(x)\,dx$ by $F'(z)\,dz$, and the x by $F(z)$. (See *Diff. Cal.*, Art. 156.)

Thus, in the example above, viz. $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$, after putting $\tan^{-1}x=z$, we may write

$$\frac{dx}{1+x^2}=dz,$$

and the integral becomes $\int e^z dz=e^z=e^{\tan^{-1}x}$.

50. When the integration is a definite one between specified limits, the limits for z will not in general be the same as those for x. But supposing a and b to be the inferior and superior limits for x, those for z must be such that whilst x ranges once over its values from a to b, z passes once and once only through the corresponding range of values for z, viz. from $F^{-1}(a)$ to $F^{-1}(b)$, where $x=F(z)$ is the connecting formula.

51. The transformation of the indefinite integral is

$$\int f(x)\,dx=\int f[F(z)]\,F'(z)\,dz.$$

Let $$f(x)=\psi'(x).$$

Then, if the limits for x be a and b,

$$\int_a^b f(x)=\int_a^b \psi'(x)\,dx=\psi(b)-\psi(a).$$

Now, when $x=a$, $\quad z=F^{-1}(a)$;

and, when $x=b$, $\quad z=F^{-1}(b)$.

Also $$f[F(z)]=\frac{d}{dx}\psi[F(z)],$$

and $$f[F(z)]F'(z)=\frac{d}{dx}\psi[F(z)]\frac{dx}{dz}=\frac{d}{dz}\psi[F(z)];$$

whence $$\int_{F^{-1}(a)}^{F^{-1}(b)} f\{F(z)\}F'(z)\,dz=\int_{F^{-1}(a)}^{F^{-1}(b)}\frac{d}{dz}\{\psi[F(z)]\}\,dz$$
$$=\psi[F\{F^{-1}b\}]-\psi[F\{F^{-1}(a)\}]$$
$$=\psi(b)-\psi(a).$$

So that the result of integrating $f[F(z)]F'(z)$ with regard to z between limits $F^{-1}(a)$ and $F^{-1}(b)$ is identical with that of integrating $f(x)$ with regard to x between the limits a and b.

52. Case of a Multiple-Valued Function.

It must be noted that $F^{-1}(a)$ and $F^{-1}(b)$ may be multiple-valued functions of a and b. Thus, for instance,

$$\sin^{-1}\tfrac{1}{2} \text{ being the same thing as } n\pi+(-1)^n\frac{\pi}{6},$$

where n is any integer whatever, is a multiple-valued function. The question will thus frequently arise as to which of a variety of values of $F^{-1}(a)$ and $F^{-1}(b)$ it is proper to take as the limits in the transformed integral.

If, however, we remember the connecting formula $x=F(z)$ and imagine x continuing its march in a continuous manner, always increasing from the value of a to the value of b, then, starting with any of the values of $F^{-1}(a)$, say α, $F^{-1}(x)$ is to change in a continuous manner from α to the *first occasion* on which it takes up the value $F^{-1}(b)$, or β say, increasing along the whole march from α to β, if x and z increase together, *i.e.* if $F'(z)$ be positive from $x=a$ to $x=b$, or decreasing along the whole march from α to β if x and z are such that z decreases as x increases, *i.e.* if $F'(z)$ be negative from $x=a$ to $x=b$. Then α and β are the limits for z which correspond to a and b respectively for x.

53. For instance, let it be required to find the value of $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, where we assign the positive sign to the radical $\sqrt{1-x^2}$. By the transformation $x=\sin\theta$, we have $\frac{dx}{d\theta}=\cos\theta$. And the indefinite integral is $\int(\pm 1)d\theta$ or $\pm\theta$, according as $+\sqrt{1-x^2}=+\cos\theta$ or $-\cos\theta$.

$$\left.\begin{array}{l}\text{When } x=0,\ \theta=\sin^{-1}0=n\pi.\\ \text{When } x=1,\ \theta=\sin^{-1}1=m\pi+(-1)^m\frac{\pi}{2},\end{array}\right\}\ n \text{ and } m \text{ being any integers.}$$

In the march of x from 0 to 1, $\sin\theta$ passes from 0 to 1 and is always positive. If the radius terminating θ lie in the first quadrant,

$$\theta \text{ increases from } 0 \text{ to } \frac{\pi}{2}, \text{ and } \frac{dx}{d\theta} \text{ is positive.}$$

If the terminating radius of θ lie in the second quadrant,

$$\theta \text{ decreases from } \pi \text{ to } \frac{\pi}{2}, \text{ and } \frac{dx}{d\theta} \text{ is negative.}$$

Generally, if θ starts from $2m\pi$, the next occasion on which $\sin\theta$ is 1 is at $\theta=2m\pi+\frac{\pi}{2}$ and $\sin\theta$ is *increasing* from 0 to 1.

If θ start at $(2m+1)\pi$, θ must decrease, as x increases, and therefore must pass from $(2m+1)\pi$, where $\sin\theta$ is zero, to $(2m+1)\pi-\frac{\pi}{2}$, where $\sin\theta$ is 1. Therefore it is proper to take our limits, either

$$0 \text{ to } \frac{\pi}{2}, \quad \sin\theta \text{ increasing}, \ \theta \text{ increasing};$$

or

$$\pi \text{ to } \frac{\pi}{2}, \quad \sin\theta \text{ increasing}, \ \theta \text{ diminishing};$$

or

$$2\pi \text{ to } \frac{5\pi}{2}, \quad \sin\theta \text{ increasing}, \ \theta \text{ increasing};$$

or

$$3\pi \text{ to } \frac{5\pi}{2}, \quad \sin\theta \text{ increasing}, \ \theta \text{ diminishing};$$

etc.

But we have noted that $+\sqrt{1-x^2}=\pm\cos\theta$, the $+$ sign to be taken if $\cos\theta$ be positive, the $-$ sign if $\cos\theta$ be negative. Accordingly,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}=\Big[\theta\Big]_0^{\frac{\pi}{2}}=\frac{\pi}{2}-0=\frac{\pi}{2},$$

or

$$=\Big[-\theta\Big]_\pi^{\frac{\pi}{2}}=-\frac{\pi}{2}+\pi=\frac{\pi}{2},$$

or

$$=\Big[\theta\Big]_{2\pi}^{\frac{5\pi}{2}}=\frac{5\pi}{2}-2\pi=\frac{\pi}{2},$$

or

$$=\Big[-\theta\Big]_{3\pi}^{\frac{5\pi}{2}}=\frac{-5\pi}{2}+3\pi=\frac{\pi}{2},$$

etc.

54. It will perhaps make the matter clearer if a graph of the transformation formula be drawn in such cases.

In the present case, $x=\sin\theta$ referred to θ, x axes is a curve of sines

whose axis is the θ-axis cutting it at O, L, M, N... ; x increases from 0 to 1 along any of the arcs, viz.,

$$O \text{ to } A, \quad L \text{ to } A,$$
$$M \text{ to } B, \quad N \text{ to } B,$$
etc.,

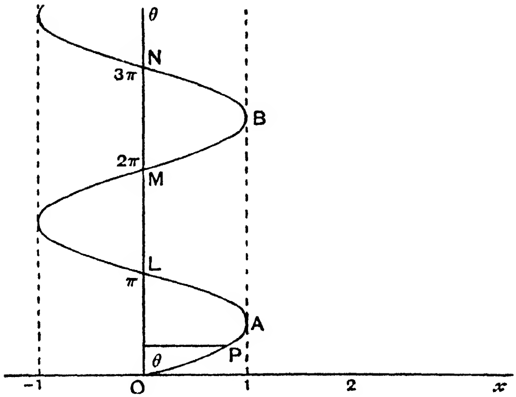


Fig. 13.

and the limits are as stated :

$$\text{along } OA \left[\theta\right]_0^{\frac{\pi}{2}}, \quad \text{along } LA \left[-\theta\right]_\pi^{\frac{\pi}{2}}, \quad \text{along } MB \left[\theta\right]_{2\pi}^{\frac{5\pi}{2}}, \text{ etc.}$$
$$\frac{dx}{d\theta} + ^{\text{ve}} \qquad \frac{dx}{d\theta} - ^{\text{ve}} \qquad \frac{dx}{d\theta} + ^{\text{ve}}$$

55. Purpose of a Substitution.

The purpose of a substitution is two-fold.

(1) Given an elementary known integral to construct a more complex one, and thus extend one's knowledge of integrable forms.

(2) Given an integral which does not fall under the list of fundamental forms, to reduce it to such form if possible.

And it must be noted that it often happens that though one substitution may reduce to a simpler form, that a further substitution, or further substitutions, may be necessary before the integration can be effected.

56. As an illustration of the first.

Beginning with the known result $\int \frac{dx}{1+x^2} = \tan^{-1} x$, let us put

$$x = y^4 + \frac{1}{y^4}, \quad \text{say.}$$

Then
$$dx=\left(4y^3-\frac{4}{y^5}\right)dy$$

whence
$$4\int\frac{\left(y^3-\frac{1}{y^5}\right)dy}{y^8+3+\frac{1}{y^8}}=\tan^{-1}\left(y^4+\frac{1}{y^4}\right),$$

or
$$\int\frac{y^3(y^8-1)\,dy}{y^{16}+3y^8+1}=\frac{1}{4}\tan^{-1}\left(y^4+\frac{1}{y^4}\right).$$

As an illustration of the second, let us try to get back from
$$I=\int\frac{y^3(y^8-1)\,dy}{y^{16}+3y^8+1}.$$

The presence of $y^3\,dy$ combined with the fact that all remaining powers of y are powers of y^4 suggests that we should put
$$y^4=z,\quad\text{and}\ \therefore\ 4y^3\,dy=dz.$$

Then
$$I=\frac{1}{4}\int\frac{z^2-1}{z^4+3z^2+1}\,dz.$$

The fact that the denominator is a reciprocal function (*i.e.* coefficients equidistant from the ends equal) suggests a division by z^2.

I is then written as
$$\frac{1}{4}\int\frac{\left(1-\frac{1}{z^2}\right)dz}{z^2+3+\frac{1}{z^2}},$$

which is seen to be
$$\frac{1}{4}\int\frac{\left(1-\frac{1}{z^2}\right)dz}{1+\left(z+\frac{1}{z}\right)^2}.$$

The form of this suggests further that we should now put
$$z+\frac{1}{z}=u,$$

for then
$$\left(1-\frac{1}{z^2}\right)dz=du.$$

I now becomes
$$\frac{1}{4}\int\frac{du}{1+u^2},$$

i.e.
$$=\frac{1}{4}\tan^{-1}u=\frac{1}{4}\tan^{-1}\left(z+\frac{1}{z}\right)=\frac{1}{4}\tan^{-1}\left(y^4+\frac{1}{y^4}\right).$$

57. Choice of Substitution.

It will be obvious that a proper choice of substitution can only be the result of experience. No general rules can be given, but the student may learn something as to the proper course to be taken from observation of the worked-out cases which follow and from the accompanying remarks.

✓Ex. 1. Evaluate $$\int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx.$$

Let $x = z^2$; then $$dx = 2z\, dz.$$

$$\therefore \int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx = \int \frac{1}{z} \cos z \,.\, 2z\, dz = 2\int \cos z\, dz = 2 \sin z = 2 \sin \sqrt{x}.$$

Here it was desirable to *get rid of the irrational form* of the angle.

✓ Ex. 2. Evaluate $$\int_0^1 \frac{x}{\sqrt{1+x^2}}\, dx.$$

Put $x = \tan\theta$; then $$dx = \sec^2\theta\, d\theta.$$

When $x = 0$ we have $$\theta = 0,$$

,, $x = 1$ we have $$\theta = \frac{\pi}{4};$$

$$\therefore \int_0^1 \frac{x}{\sqrt{1+x^2}}\, dx = \int_0^{\frac{\pi}{4}} \frac{\tan\theta}{\sec\theta} \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \sec\theta \tan\theta\, d\theta$$

$$= \Big[\sec\theta\Big]_0^{\frac{\pi}{4}} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1.$$

It is to be noted that when $\sqrt{a^2+x^2}$ occurs in the integrand, $x = a\tan\theta$ or $x = a\cot\theta$ or $x = a\sinh z$ *are likely substitutions, for they rationalize the radical.*

When $\sqrt{x^2-a^2}$ occurs, $x = a\sec\theta$, $x = a\operatorname{cosec}\theta$ or $x = a\coth z$, are good substitutions.

✓ Ex. 3. Evaluate $$I = \int \frac{dx}{x\sqrt{x^{2n}-a^{2n}}}.$$

Let $x^n = a^n z^{-1}$; then $$n\frac{dx}{x} = -\frac{dz}{z},$$

and $$I = -\frac{1}{n}\int \frac{dz}{z\sqrt{\frac{a^{2n}}{z^2} - a^{2n}}}$$

$$= -\frac{1}{na^n}\int \frac{dz}{\sqrt{1-z^2}}$$

$$= -\frac{1}{na^n}\sin^{-1} z = -\frac{1}{na^n}\sin^{-1}\left(\frac{a^n}{x^n}\right).$$

Note that $x^n = a^n z^{-1}$ *is generally a proper substitution in cases when* $\sqrt{a^{2n} \pm x^{2n}}$ *occurs.*

Also, $x^n = a^n\tan\theta$ or $a^n\cot\theta$ or $a^n\sinh z$ for $\sqrt{a^{2n}+x^{2n}}$,

or $x^n = a^n\sin\theta$ or $a^n\cos\theta$ or $a^n\operatorname{sech} z$ for $\sqrt{a^{2n}-x^{2n}}$,

might be used.

When $\sqrt{x^{2n}-a^{2n}}$ occurs, $x^n = a^n\sec\theta$ or $a^n\operatorname{cosec}\theta$, or $x^n = a^n\coth z$, would be useful.

Ex. 4. *When $\sqrt{2ax-x^2}$ occurs, a useful trial is*

$$x=a(1-\cos\theta), \quad i.e. \ x=a \operatorname{vers}\theta.$$

Thus, to evaluate $$I=\int\frac{x}{\sqrt{2ax-x^2}}\,dx,$$

$$dx=a\sin\theta\,d\theta, \qquad \sqrt{2ax-x^2}=a\sin\theta;$$

$$\therefore\ I=\int\frac{a(1-\cos\theta)\,a\sin\theta\,d\theta}{a\sin\theta}$$

$$=a\int(1-\cos\theta)\,d\theta$$

$$=a(\theta-\sin\theta)$$

$$=a\operatorname{vers}^{-1}\frac{x}{a}-\sqrt{2ax-x^2}.$$

Ex. 5. *When $\sqrt{a-x}$ or $\sqrt{\frac{a-x}{a+x}}$ occurs in the integrand the substitution $x=a\cos\theta$ will often be found useful, or perhaps better, $x=a\cos 2\theta$.*

Evaluate $$I=\int x\sqrt{\frac{a-x}{a+x}}\,dx.$$

Let $x=a\cos 2\theta$; then

$$dx=-2a\sin 2\theta\,d\theta.$$

$$I=-2a^2\int\cos 2\theta\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\sin 2\theta\,d\theta$$

$$=-2a^2\int(\cos 2\theta\tan\theta\, 2\sin\theta\cos\theta)\,d\theta$$

$$=-2a^2\int\cos 2\theta\,(1-\cos 2\theta)\,d\theta$$

$$=-2a^2\int\left(\cos 2\theta-\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$=-2a^2\left(\frac{\sin 2\theta}{2}-\frac{\theta}{2}-\frac{\sin 4\theta}{8}\right)$$

$$=\frac{a^2}{4}(4\theta-4\sin 2\theta+\sin 4\theta)$$

$$=\frac{a^2}{4}\left[2\cos^{-1}\frac{x}{a}-\frac{4}{a}\sqrt{a^2-x^2}+\frac{2x}{a^2}\sqrt{a^2-x^2}\right]$$

$$=\frac{a^2}{2}\cos^{-1}\frac{x}{a}+\frac{(x-2a)}{2}\sqrt{a^2-x^2}.$$

58. When an inverse function occurs in the integrand such as $\sin^{-1}\frac{x}{a}$, $\cos^{-1}\frac{x}{a}$, $\tan^{-1}\frac{x}{a}$, $\operatorname{vers}^{-1}\frac{x}{a}$, it is usually helpful to put $x=a\sin\theta$, $a\cos\theta$, $a\tan\theta$, or $a\operatorname{vers}\theta$, as the case may be, and work with the direct functions.

Many other forms of substitution will occur in due course, but what has been said will suffice for present purposes.

EXAMPLES.

1. Evaluate (i) $\int \frac{3x^2}{1+x^3}dx$. Put $x^3=z$.

(ii) $\int_0^1 \frac{dx}{x^2+4x+5}$. Put $x+2=z$.

(iii) $\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}}$. Put $x-1=z$; $\therefore \frac{dx}{x-1}=\frac{dz}{z}$, etc.

(iv) $\int_0 \frac{dx}{e^x+e^{-x}}$. Put $e^x=z$.

(v) $\int \frac{e^x\,dx}{2e^{2x}+2e^x+1}$. Put $e^x=\frac{z}{1-z}$.

(vi) $\int \tan^2 x \sec^2 x\,dx$. (vii) $\int \frac{dx}{\cosh^2 mx}$.

2. Evaluate (i) $\int_0^a \sqrt{a^2-x^2}\,dx$. Put $x=a\sin\theta$.

(ii) $\int_0^{2a} \sqrt{2ax-x^2}\,dx$. Put $x=a(1-\cos\theta)$.

Draw graphs to illustrate these two integrations.

3. Find the values of

(i) $\int_0^a x\sqrt{a^2-x^2}\,dx$. (ii) $\int_0^a x^2\sqrt{a^2-x^2}\,dx$.

Interpret the meaning of these integrations.

4. Integrate $\int \frac{(ax^2-b)\,dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$. Put $ax+\frac{b}{x}=z$.

5. Integrate $\int \frac{x^{2n}\,dx}{(a^2+x^2)^{n+\frac{3}{2}}}$. Put $x=a\tan\theta$. [ST. JOHN'S, 1883.]

6. Integrate $\int \sec^{\frac{6}{5}}\theta \operatorname{cosec}^{\frac{4}{5}}\theta\,d\theta$. Put $\tan\theta=z$. [ST. JOHN'S, 1883.]

7. Integrate $\int \sqrt{\frac{\sin x}{\cos^5 x}}\,dx$. Put $\tan x=z$. [TRINITY, 1883.]

8. Integrate

(i) $\int \frac{dx}{x\sqrt{x^4-1}}$. (ii) $\int \frac{dx}{x\sqrt{1-x^4}}$. (iii) $\int \frac{dx}{x\sqrt{1+x^4}}$.

9. Integrate

(i) $\int \left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,dx$. (ii) $\int \frac{ax^2-b}{x^2+(ax^2+b)^2}\,dx$.

(iii) $\int \frac{ax^2-b}{x^{n+2}}(ax^2+b)^n\,dx$. (iv) $\int \frac{1}{(c+ex)^2}\cos\frac{a+bx}{c+ex}\,dx$.

(v) $\int \frac{e^{a\tan^{-1}x}}{1+x^2}\,dx$. (vi) $\int \frac{e^{a\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$. (vii) $\int \frac{\sin x\cos x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$.

10. Integrate

(i) $\int \{\phi(x)\psi'(x)+\phi'(x)\psi(x)\}\,dx.$ (ii) $\int \frac{\phi(x)\psi'(x)-\phi'(x)\psi(x)}{[\phi(x)]^2}\,dx.$

(iii) $\int \frac{\phi'(x)\,dx}{1+[\phi(x)]^2}.$ (iv) $\int e^{\phi(x)}\phi'(x)\,dx.$

(v) $\int e^{-\psi(x)}\frac{\phi'(x)-\phi(x)\psi'(x)\log\phi(x)}{\phi(x)}\,dx.$

11. Show that

$$\int (x-a)\sqrt{\frac{x-4a}{x}}\,dx = a^2(\sinh 2u - 2\sinh u)$$

where $$x = 4a\cosh^2\frac{u}{2}$$ [Ox. I. Pub., 1899.]

59. The Hyperbolic Functions.

To avoid complexity of form in many integrations and to secure symmetry in the results of integrations of expressions of similar algebraic form, it is customary to make full use of the hyperbolic functions and their inverses. (*Diff. Calc.*, Art. 23.)

By analogy with the exponential values of the sine, cosine, tangent, etc., the exponential functions

$$\frac{e^x-e^{-x}}{2},\quad \frac{e^x+e^{-x}}{2},\quad \frac{e^x-e^{-x}}{e^x+e^{-x}},\ \text{etc.},$$

are respectively written

$$\sinh x,\quad \cosh x,\quad \tanh x,\ \text{etc.},$$

or sometimes more shortly as $\operatorname{sh} x$, $\operatorname{ch} x$, $\operatorname{th} x$, etc.

By further analogy with the inverse circular functions,

$$\text{if}\quad u=\sinh x \quad\text{or}\quad \cosh x \quad\text{or}\quad \tanh x,\ \text{etc.},$$

we write the inverse hyperbolic functions

$$x=\sinh^{-1}u \quad\text{or}\quad \cosh^{-1}u \quad\text{or}\quad \tanh^{-1}u,\ \text{etc., respectively,}$$

or sometimes as $\operatorname{sh}^{-1}x,\quad \operatorname{ch}^{-1}x,\quad \operatorname{th}^{-1}x.$

This notation is now commonly adopted by modern writers. Professor Sir George Greenhill (*Chapter on the Integral Calculus*, 1888) indicates it as being common amongst American writers, and as being frequently employed by writers on Applied Mathematics. The earlier notation used by Bertrand, viz.

$$\text{sect sin hyp } x,\quad \text{sect cos hyp } x,\quad \text{sect tan hyp } x,\ \text{etc.},$$

is far too cumbrous for free use.

The properties of these functions are now usually discussed at some length in books on Trigonometry [see Dr. Hobson's *Trigonometry*, pages 303-316]. It is therefore unnecessary to repeat them here fully. But for the convenience of students who have not already sufficient familiarity with their use, we reproduce those of the elementary properties which we shall require for the immediate purpose in hand.

60. **Elementary Properties.**

We clearly have

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1,$$

analogous to $\cos^2\theta + \sin^2\theta = 1$;

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x,$$

analogous to $\cos^2\theta - \sin^2\theta = \cos 2\theta$;

whence $\quad 2\cosh^2 x = 1 + \cosh 2x,$

analogous to $2\cos^2\theta = 1 + \cos 2\theta$;

$$2\sinh^2 x = \cosh 2x - 1,$$

analogous to $2\sin^2\theta = 1 - \cos 2\theta$;

$$\operatorname{sech}^2 x + \tanh^2 x = \left(\frac{2}{e^x + e^{-x}}\right)^2 + \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = 1,$$

analogous to $\sec^2\theta - \tan^2\theta = 1$;

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}, \quad \text{analogous to } \tan\theta = \frac{\sin\theta}{\cos\theta};$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\cosh x}{\sinh x}, \quad \text{analogous to } \cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$= \frac{1}{\tanh x},$$

etc.

It is unnecessary to point out methods of proof or analogies further, and the following results may be proved by the

student as exercises, and will form a convenient list for reference:

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y,$$

$$\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y,$$

$$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y,$$

$$\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y,$$

$$\sinh 2x=2\sinh x\cosh x,$$

$$\begin{aligned}\cosh 2x&=\cosh^2 x+\sinh^2 x.\\&=2\cosh^2 x-1\\&=1+2\sinh^2 x,\end{aligned}$$

$$\tanh 2x=\frac{2\tanh x}{1+\tanh^2 x},$$

$$\left.\begin{aligned}\sinh x+\sinh y=2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2},\\ \text{etc.,}\end{aligned}\right\}$$

$$\cosh x=1+\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}+\dots,$$

$$\sinh x=x+\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}+\dots,$$

$$\tanh^{-1}x=x+\frac{x^3}{3}+\frac{x^5}{5}+\dots.$$

61. It should be remarked that such expressions as $\sin\theta$, $\cos\theta$, etc., where θ is complex, *i.e.* of the form $u+\iota v$, do not come under the heading of the sines and cosines defined geometrically in the early parts of trigonometry. They are *re-defined now by the exponential values*

$\sin\theta$, standing for $\dfrac{e^{\iota\theta}-e^{-\iota\theta}}{2\iota}$; $\cos\theta$, standing for $\dfrac{e^{\iota\theta}+e^{-\iota\theta}}{2}$;

etc., for any value of θ real or complex.

Then writing $\theta=\iota x$, where $\iota=\sqrt{-1}$,

$$\sin\iota x=\iota\sinh x,$$

$$\cos\iota x=\cosh x,$$

$$\tan\iota x=\iota\tanh x,$$

$$\cot\iota x=-\iota\coth x,$$

Also the ordinary formulae of trigonometry can be proved *from these definitions*, viz., we have

$$\cos^2\theta+\sin^2\theta=1,$$

$$\cos^2\theta-\sin^2\theta=\cos 2\theta,$$

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi,$$

etc.,

and the restriction of the reality of θ and ϕ is removed.

Then, having proved the addition formulae for the sines and cosines from these definitions, we have

$$\sin(u+\iota v)=\sin u\cos\iota v+\cos u\sin\iota v$$

$$=\sin u\cosh v+\iota\cos u\sinh v,$$

etc.

62. Inverse Hyperbolic Functions.

We are, in the Integral Calculus, more particularly interested in the inverse forms.

Let us search for the meaning of the inverse function $\sinh^{-1}\dfrac{x}{a}$

Put $$\sinh^{-1}\frac{x}{a}=y.$$

Then $$\frac{x}{a}=\sinh y=\frac{e^y-e^{-y}}{2};$$

$$\therefore\ e^{2y}-2\frac{x}{a}e^y-1=0,$$

$$e^y=\frac{x\pm\sqrt{a^2+x^2}}{a},$$

and remembering that $e^{\pm\iota 2\lambda\pi}$, where λ is an integer,

$$=\cos 2\lambda\pi\pm\iota\sin 2\lambda\pi=1,$$

we may, to retain generality, write this as

$$e^{y-2\iota\lambda\pi}=\frac{x\pm\sqrt{a^2+x^2}}{a},$$

or $$y=2\iota\lambda\pi+\log\frac{x\pm\sqrt{a^2+x^2}}{a}.$$

Now $2\iota\lambda\pi+\log\dfrac{x-\sqrt{a^2+x^2}}{a}=2\iota\lambda\pi-\log\dfrac{a}{x-\sqrt{a^2+x^2}}$

$$=2\iota\lambda\pi-\log\left(-\frac{x+\sqrt{a^2+x^2}}{a}\right)$$

$$=2\iota\lambda\pi-\log(-1)-\log\frac{x+\sqrt{a^2+x^2}}{a}$$

$$=(2\lambda'-1)\iota\pi-\log\frac{x+\sqrt{a^2+x^2}}{a},$$

for $\log(-1)=\log e^{(2n+1)\iota\pi}=(2n+1)\iota\pi$,

n and λ' being integers.

Thus, $$y=\mu\iota\pi+(-1)^{\mu}\log\frac{x+\sqrt{a^2+x^2}}{a},$$

where μ is an integer

The "principal value" of y is then $\log\dfrac{x+\sqrt{a^2+x^2}}{a}$, and it is usual to take this as synonymous with $\sinh^{-1}\dfrac{x}{a}$, omitting the generality obtained by the addition of unreal constants.

63. Similarly putting

$$\cosh^{-1}\frac{x}{a}=y,$$

$$\frac{x}{a}=\cosh y=\frac{e^y+e^{-y}}{2},$$

$$e^{2y}-2\frac{x}{a}e^y+1=0$$

and $$e^y=\frac{x\pm\sqrt{x^2-a^2}}{a},$$

and omitting as before the generality derived from the unreal constants, we shall take the solution

$$y=\log\frac{x+\sqrt{x^2-a^2}}{a},$$

viz. the "principal value" of y with the positive sign as $\cosh^{-1}\dfrac{x}{a}$, and therefore $\cosh^{-1}\dfrac{x}{a}$ is to be understood as synonymous with

$$\log\frac{x+\sqrt{x^2-a^2}}{a}.$$

64. Again, putting $\tanh^{-1}\dfrac{x}{a}=y,$

$$\frac{x}{a}=\tanh y=\frac{e^y-e^{-y}}{e^y+e^{-y}},$$

and therefore $$e^{2y}=\frac{a+x}{a-x},$$

and omitting generalities as before,

$$\tanh^{-1}\frac{x}{a}=\frac{1}{2}\log\frac{a+x}{a-x}.$$

65. Similarly, $$\coth^{-1}\frac{x}{a}=\frac{1}{2}\log\frac{x+a}{x-a},$$

$$\operatorname{sech}^{-1}\frac{x}{a}=\log\frac{a+\sqrt{a^2-x^2}}{x},$$

$$\operatorname{cosech}^{-1}\frac{x}{a}=\log\frac{a+\sqrt{a^2+x^2}}{x}.$$

66. We shall therefore consider

$\sinh^{-1}\dfrac{x}{a}$ as meaning $\log\dfrac{x+\sqrt{x^2+a^2}}{a}$,

$\cosh^{-1}\dfrac{x}{a}$ as meaning $\log\dfrac{x+\sqrt{x^2-a^2}}{a}$,

$\tanh^{-1}\dfrac{x}{a}$ as meaning $\dfrac{1}{2}\log\dfrac{a+x}{a-x}$,

$\coth^{-1}\dfrac{x}{a}$ as meaning $\dfrac{1}{2}\log\dfrac{x+a}{x-a}$,

$\operatorname{sech}^{-1}\dfrac{x}{a}$ as meaning $\log\dfrac{a+\sqrt{a^2-x^2}}{x}$,

$\operatorname{cosech}^{-1}\dfrac{x}{a}$ as meaning $\log\dfrac{a+\sqrt{a^2+x^2}}{x}$.

67. **Periodicity of the Hyperbolic Functions.**

These hyperbolic functions are periodic. But the periodicity is imaginary.

For, since $e^{\pm\iota\lambda\pi}=\cos\lambda\pi\pm\iota\sin\lambda\pi=(-1)^\lambda$, ($\lambda\equiv$ an integer),

we have $$\cosh(x+\lambda\iota\pi)=\frac{e^{x+\lambda\iota\pi}+e^{-(x+\lambda\iota\pi)}}{2}=(-1)^\lambda\cosh x.$$

Similarly, $\sinh(x+\lambda\iota\pi)=(-1)^\lambda\sinh x,$

whence $\tanh(x+\lambda\iota\pi)=\tanh x.$

Thus, the periodicity of $\sinh x$ and $\cosh x$ is $2\pi\iota$,

that of $\tanh x$ and $\coth x$ is $\pi\iota$.

Also

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = 0, \quad \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1, \quad \tanh 0 = 0, \text{ etc.},$$

$$\sinh \iota\pi = \frac{e^{\iota\pi} - e^{-\iota\pi}}{2} = \iota \sin \pi = 0,$$

$$\cosh \iota\pi = \cos \pi = -1,$$

etc.

Again,

$$\begin{aligned}\cosh^{-1}(-z) &= \log(-z + \sqrt{z^2 - 1})\\ &= \log(z - \sqrt{z^2 - 1}) + \log(-1)\\ &= \log \frac{1}{z + \sqrt{z^2 - 1}} + \iota\pi\\ &= -\log(z + \sqrt{z^2 - 1}) + \iota\pi\\ &= -\cosh^{-1} z + \iota\pi,\end{aligned}$$

$$\begin{aligned}\sinh^{-1}(-z) &= \log(-z + \sqrt{z^2 + 1})\\ &= \log \frac{1}{z + \sqrt{z^2 + 1}}\\ &= -\log(z + \sqrt{z^2 + 1})\\ &= -\sinh^{-1} z,\end{aligned}$$

$$\begin{aligned}\tanh^{-1}(-z) &= \frac{1}{2} \log \frac{1 - z}{1 + z} = -\frac{1}{2} \log \frac{1 + z}{1 - z}\\ &= -\tanh^{-1} z,\end{aligned}$$

etc.,

analogous to the properties of the circular functions,

$$\cos^{-1}(-z) = -\cos^{-1} z + \pi, \quad \sin^{-1}(-z) = -\sin^{-1} z,$$

$$\tan^{-1}(-z) = -\tan^{-1} z,$$

etc.

68. Geometrical Interpretation.

Let a rectangular hyperbola $x^2 - y^2 = a^2$ and its auxiliary circle be drawn; then any point on the hyperbola may be represented by either of the parameters θ or u by putting

$$\left.\begin{aligned} x &= a \sec \theta, \\ y &= a \tan \theta, \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x &= a \cosh u, \\ y &= a \sinh u. \end{aligned}\right\}$$

Hence θ and u are connected by the equations

$$\sec\theta = \cosh u$$

or

$$\tan\theta = \sinh u.$$

Let P be the point θ (or u) on the hyperbolic arc AP; PN the ordinate, NT the tangent from N to the auxiliary circle.

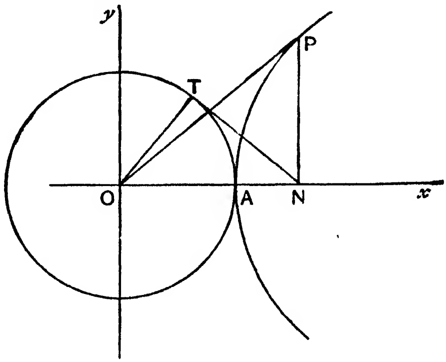


Fig. 14.

Then obviously the abscissa

$$ON = x = OT \sec NOT = a \sec NOT.$$

Hence, the angle NOT is the parameter θ.

Also, since $ON^2 - NT^2 = a^2$, it follows that $NT = y$, as is obvious, since $y = a\tan\theta$, as also $NT = a\tan\theta$.

The area of the portion NAP of the hyperbola

$$= \int_a^x y\,dx = \int_0^u a\sinh u \,.\, a\sinh u\,du,$$

$$= a^2\int_0^u \sinh^2 u\,du$$

$$= \frac{a^2}{2}\int_0^u (\cosh 2u - 1)\,du$$

$$= \frac{a^2}{2}\left(\frac{\sinh 2u}{2} - u\right) = \frac{a^2\sinh 2u}{4} - \frac{a^2 u}{2}.$$

Also, area of triangle $ONP = \frac{1}{2}xy$

$$= \frac{1}{2}a\cosh u \,.\, a\sinh u = \frac{a^2\sinh 2u}{4}.$$

Hence the area of the hyperbolic sector OAP

$$= \triangle ONP - \text{area } ANP$$

$= \dfrac{a^2 u}{2}$, analogous to $\dfrac{a^2\theta}{2}$ for the circular sector

This indicates the meaning of u, viz.

$$u = \frac{2 \text{ area of hyperbolic sector } CAP}{a^2}.$$

It is this connection with the hyperbola from which these transcendental functions are termed hyperbolic functions. For other properties in connection with this figure, see Greenhill's *Chapter on the Integral Calculus*, p. 27, or Hobson's *Trigonometry*, p. 309, and an "Essai sur les Fonct. Hyperboliques," *Mem. d. l. Soc. des Sciences Phys.*, Bordeaux, 1875, cited by Greenhill.

Since $\cosh u = \sec\theta$

and $\therefore\ \sinh u = \tan\theta$,

we have $\tanh u = \dfrac{\tan\theta}{\sec\theta} = \sin\theta.$

$$\coth u = \operatorname{cosec}\theta,$$

etc.,

which express functions of u in terms of θ. Again, expressing θ in terms of u, we obviously have

$$\sin\theta = \tanh u,$$
$$\cos\theta = \operatorname{sech} u,$$
$$\tan\theta = \sinh u,$$
$$\cot\theta = \operatorname{cosech} u, \text{ etc.}$$

69. **The Gudermannian.**

The angle θ, which may therefore be variously expressed as $\sin^{-1}(\tanh u)$, $\cos^{-1}(\operatorname{sech} u)$, $\tan^{-1}(\sinh u)$, $\cot^{-1}(\operatorname{cosech} u)$, $\sec^{-1}(\cosh u)$, or $\operatorname{cosec}^{-1}(\coth u)$, is called by Cayley the "Gudermannian" of u* (*Elliptic Functions*, p. 56), and denoted by him by the convenient notation

$$\theta = \operatorname{gd} u,$$

or inversely $u = \operatorname{gd}^{-1}\theta.$

Then $\sin\theta$, $\cos\theta$, $\tan\theta$ he denotes by $\operatorname{sg} u$, $\operatorname{cg} u$, $\operatorname{tg} u$.

Again, $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \log(\sec\theta+\tan\theta)$

$$= \log(\cosh u + \sinh u) = \log e^u = u.$$

*So named from Gudermann, who specially discussed this function (Cayley, p. 44).

Hence, $\operatorname{gd} u$ is such that

$$\log\tan\left(\frac{\pi}{4}+\frac{\operatorname{gd} u}{2}\right)=u,$$

or $$\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\operatorname{gd}^{-1}\theta,$$ which is the same thing.

Differentiating $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ or $\log(\sec\theta+\tan\theta)$, we get $\sec\theta$ as the differential coefficient. Hence,

$$\int\sec\theta\,d\theta=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\operatorname{gd}^{-1}\theta,$$

and $\int\sec\theta\,d\theta$ is a degenerate form of the more general integral $\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, where $k=1$, which is to be discussed later.

70. Tables of the values of $u=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, *i.e.* the inverse Gudermannian of θ, are given by Legendre, *Théorie des Fonctions Elliptiques*, vol. ii., to 12 places of decimals for angles in the first quadrant. They will be found to seven places at degree intervals in Hobson's *Trigonometry*, p. 316, and to five places at degree intervals in Greenhill's *Elliptic Functions*, p. 16, whence it is easy to extract the values of u corresponding to any angle θ, or the value of θ corresponding to any given value of u, and hence from the relations $\cosh u=\sec\theta$, $\sinh u=\tan\theta$, etc., we can find the values of the hyperbolic functions $\cosh u$, $\sinh u$, etc., for any values of u by the use of the intermediary angle θ by means of the ordinary tables of secants, tangents, etc. In the absence of direct tables of the hyperbolic functions this will be the proper mode of computation to follow in numerical calculations. See Lodge's *Report to Brit. Assoc.* 1888, and remarks by Greenhill on p. 15, *Elliptic Functions*.*

71. Unless extremely close approximations are required it will be sufficient to take the values of $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ from

*The Smithsonian Institute of the City of Washington publishes a set of *Mathematical Tables of the Hyperbolic Functions*, by G. F. Becker and C. E. van Orstand.

The Harvard University Press publishes *Tables of Complex Hyperbolic and Circular Functions*, by A. E. Kennelly.

the following graph, which indicates the march of the function from $\theta=0$ to $\theta=90°$. There is not much deviation from a straight line from $\theta=0$ to $\theta=45°$, but beyond that the function begins to increase more rapidly, passing from 4·7413 at 89° to ∞ at 90°. For the first part of the graph, obviously the ordinary rule of proportional parts will give a fair approxi-

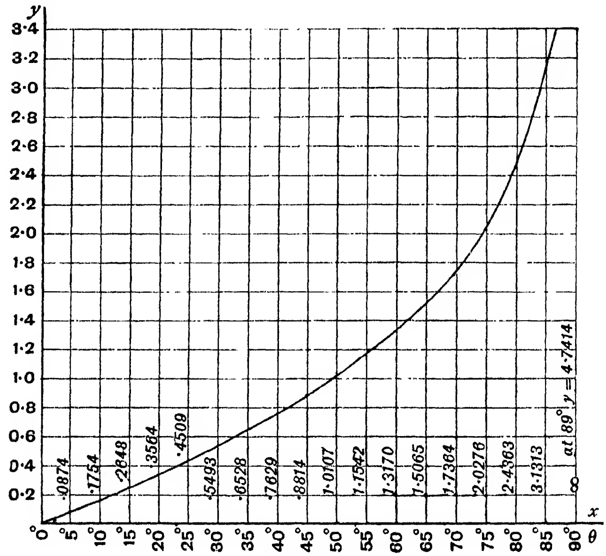


Fig. 15.

[Graph of $y=\log_e \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\mathrm{gd}^{-1}\theta$, the abscissae being the sexagesimal measures of θ, showing the march of the inverse Gudermannian function.]

mation. Within the last 10° or so of 90° it is desirable to adopt the special mode of approximation shown in Greenhill, *loc. cit.* The ordinates are given to four places of decimals at 5° intervals, which will be sufficient for the purposes of approximation in this book. The numerical data for the graph were taken from Hobson's *Trigonometry.*

This graph is that of

$$y=\mathrm{gd}^{-1}x,$$

the inverse Gudermannian, and equally serves to illustrate the graph of

$$x=\mathrm{gd}\,y,$$

i.e. the march of the direct Gudermannian.

72. Let us illustrate the use of the graph—or the values tabulated in Fig. 15.

If, for instance, we require the value of sinh 1 from the tables (which should of course be $\frac{e^1 - e^{-1}}{2}$, and therefore we can check it).

u lies between ·8814 and 1·0107,

i.e. $\tan^{-1}\sinh \cdot 8814 = 45°$,
$\tan^{-1}\sinh 1{\cdot}0107 = 50°$,
$\tan^{-1}\sinh 1{\cdot}0000 = \theta$.

By proportional parts,

$$\frac{\theta - 45°}{5°} = \frac{\cdot 1186}{\cdot 1293},$$

$$\theta = 4°\,35' + 45° = 49°\,35'.$$

$\therefore \sinh 1 = \tan 49°\,35' = 1{\cdot}1744$, from the tables of natural tangents.

To check this,

$$\sinh 1 = \frac{e - e^{-1}}{2} = \frac{2{\cdot}7183 - \cdot 3679}{2} = \frac{2{\cdot}3504}{2} = 1{\cdot}1752,$$

which shows an error of about ·0008.

73. There is also a useful table giving the values of various powers of e, viz. $e^{\pm 1}$, $e^{\pm 2}$, $e^{\pm 3}, \ldots e^{\pm 10}$; $e^{\pm\frac{1}{2}}$, $e^{\pm\frac{3}{2}}$, $e^{\pm\frac{5}{2}}$, $e^{\pm\frac{7}{2}}$; $e^{\pm\frac{1}{4}}$, $e^{\pm\frac{1}{8}}$, $e^{\pm\frac{1}{16}}$, $e^{\pm\frac{1}{32}}$;
$e^{\pm\frac{1}{2}}$, $e^{\pm\frac{1}{3}}$, $e^{\pm\frac{1}{4}}$, $e^{\pm\frac{1}{5}}$, $e^{\pm\frac{1}{6}}$, $e^{\pm\frac{1}{7}}$, $e^{\pm\frac{1}{8}}$, $e^{\pm\frac{1}{9}}$, $e^{\pm\frac{1}{10}}$,
in Bottomley's tables, p. 56, which will be convenient in some cases.

E.g. (extracting the values from these tables)

$$\cosh \cdot 1 = \frac{e^{\cdot 1} + e^{-\cdot 1}}{2} = \frac{1{\cdot}1052 + 0{\cdot}9048}{2} = \frac{2{\cdot}0100}{2} = 1{\cdot}0050.$$

If great accuracy be required it will be necessary to use the 7, or perhaps, in cases, the 12-figure tables, but such extreme accuracy would but seldom be required in practice.

Examples.

Establish the following results:

1. $\int \cosh x\,dx = \sinh x.$ 2. $\int \sinh x\,dx = \cosh x.$

3. $\int \operatorname{sech}^2 x\,dx = \tanh x.$ 4. $\int \operatorname{cosech}^2 x\,dx = -\coth x.$

5. $\int \frac{\sinh x}{\cosh^2 x}\,dx = \int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x.$

6. $\int \frac{\cosh x}{\sinh^2 x}\,dx = \int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x.$

7. $\int \operatorname{cg} x\,dx = \operatorname{gd} x.$ 8. $\int \operatorname{cg}^2 x\,dx = \operatorname{sg} x.$ 9. $\int \frac{dx}{\operatorname{cg} x} = \operatorname{tg} x.$

10. (*a*) $\operatorname{sg}(u+v) = \frac{\operatorname{sg} u + \operatorname{sg} v}{1 + \operatorname{sg} u \operatorname{sg} v}$, (*b*) $\operatorname{cg}(u+v) = \frac{\operatorname{cg} u \operatorname{cg} v}{1 + \operatorname{sg} u \operatorname{sg} v}$,

(*c*) $\operatorname{sg} 2u = \frac{2 \operatorname{sg} u}{1 + \operatorname{sg}^2 u}$. (*d*) $\operatorname{cg} 2u = \frac{\operatorname{cg}^2 u}{1 + \operatorname{sg}^2 u}$,

(*e*) $\operatorname{tg} 2u = 2\frac{\operatorname{sg} u}{\operatorname{cg}^2 u}$.

74. Integrals of cosec x and sec x.

Let $\tan\frac{x}{2}=z$; then, taking the logarithmic differential,

$$\frac{1}{2\tan\frac{x}{2}}\sec^2\frac{x}{2}\,dx=\frac{dz}{z},\quad i.e.\quad \frac{dx}{\sin x}=\frac{dz}{z}.$$

Thus $$\int\operatorname{cosec} x\,dx=\int\frac{dx}{\sin x}=\int\frac{dz}{z}=\log z=\log\tan\frac{x}{2}.$$

In this result put $x=\frac{\pi}{2}+y$; then $dx=dy$.

And $$\int\sec y\,dy=\log\tan\left(\frac{\pi}{4}+\frac{y}{2}\right).$$

That is, $$\int\sec x\,dx \text{ or } \int\frac{dx}{\cos x}=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\operatorname{gd}^{-1}x,$$

as we have seen before.

75. From this result we may infer the integral of

$$\int\frac{dx}{a\cos x+b\sin x}.$$

For $$a\cos x+b\sin x=R\sin(x+\alpha),$$

where $R=\sqrt{a^2+b^2}$ and $\tan\alpha=\frac{a}{b}$;

$$\int\frac{dx}{a\cos x+b\sin x}=\frac{1}{R}\int\frac{dx}{\sin(x+\alpha)}$$
$$=\frac{1}{R}\int\frac{d(x+\alpha)}{\sin(x+\alpha)}$$
$$=\frac{1}{R}\log\tan\frac{x+\alpha}{2}.$$

76. The **integrals of cosech x and sech x** give no trouble.

$$\int\operatorname{cosech} x\,dx=\int\frac{dx}{\sinh x}=2\int\frac{dx}{e^x-e^{-x}}=2\int\frac{e^x}{e^{2x}-1}\,dx$$
$$=\int\left(\frac{1}{e^x-1}-\frac{1}{e^x+1}\right)de^x$$
$$=\log\frac{e^x-1}{e^x+1}=\log\frac{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}$$
$$=\log\tanh\frac{x}{2},$$

$$\int \operatorname{sech} x\,dx=\int\frac{dx}{\cosh x}=2\int\frac{e^x}{1+e^{2x}}dx=2\int\frac{de^x}{1+e^{2x}}$$
$$=2\tan^{-1}e^x \quad\text{or}\quad =\cos^{-1}\frac{1-e^{2x}}{1+e^{2x}}$$
$$=\cos^{-1}(-\tanh x)$$
$$=-\cos^{-1}(\tanh x)+\text{const.}$$
$$=-\sin^{-1}(\operatorname{sech} x)+\text{const.}$$

77. Integrals of $\dfrac{1}{\sqrt{x^2+a^2}}$ and $\dfrac{1}{\sqrt{x^2-a^2}}$.

The differential coefficient of $\log_e\dfrac{x+\sqrt{x^2+a^2}}{a}$ is $\dfrac{1}{\sqrt{x^2+a^2}}$.

Thus, $$\int\frac{dx}{\sqrt{x^2+a^2}}=\log\frac{x+\sqrt{x^2+a^2}}{a}=\sinh^{-1}\frac{x}{a}.$$

Similarly, $$\int\frac{dx}{\sqrt{x^2-a^2}}=\log\frac{x+\sqrt{x^2-a^2}}{a}=\cosh^{-1}\frac{x}{a}.$$

In the inverse hyperbolic forms which it is now possible to use, these results resemble that for the integral

$$\int\frac{dx}{\sqrt{a^2-x^2}},\quad\text{viz.}\ =\sin^{-1}\frac{x}{a},$$

and the analogy is an aid to the memory.

The student will note the avoidance of complexity and the gain of symmetry referred to in Art. 59 as the result of using these forms.

We might have established these results thus:

To find $\displaystyle\int\frac{dx}{\sqrt{x^2+a^2}}$, put $x=a\sinh u$; then

$$dx=a\cosh u\,du \quad\text{and}\quad \sqrt{x^2+a^2}=a\cosh u.$$

Hence, $$\int\frac{dx}{\sqrt{x^2+a^2}}=\int du=u=\sinh^{-1}\frac{x}{a}.$$

Similarly, putting $x=a\cosh u$ we have

$$\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\sinh u\,du}{a\sinh u}=\int du=u=\cosh^{-1}\frac{x}{a}.$$

78. Integrals of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$.

I. To find $\int\sqrt{a^2-x^2}\,dx$, put $x=a\sin\theta$; then

$$dx=a\cos\theta\,d\theta;$$

$$\therefore\quad \int\sqrt{a^2-x^2}\,dx=a^2\int\cos^2\theta\,d\theta=\frac{a^2}{2}\int(1+\cos 2\theta)\,d\theta$$

$$=\frac{a^2}{4}\sin 2\theta+\frac{a^2}{2}\theta$$

$$=\frac{1}{2}a\sin\theta\,.\,a\cos\theta+\frac{a^2}{2}\theta$$

or
$$\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

II. To find $\int\sqrt{a^2+x^2}\,dx$, put $x=a\sinh z$; then

$$dx=a\cosh z\,dz.$$

Then, since $1+\sinh^2 z=\cosh^2 z$, we have

$$\int\sqrt{a^2+x^2}\,dx=a^2\int\cosh^2 z\,dz=\frac{a^2}{2}\int(\cosh 2z+1)\,dz$$

$$=\frac{a^2}{4}\sinh 2z+\frac{a^2 z}{2}=\frac{1}{2}a\sinh z\,.\,a\cosh z+\frac{a^2}{2}z,$$

i.e.
$$\int\sqrt{a^2+x^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}$$

or
$$=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log\frac{x+\sqrt{a^2+x^2}}{a};$$

and in the latter form, if the integral be indefinite, we may drop out the a in the denominator of the logarithm, as this will only add a constant to the whole.

III. To find $\int\sqrt{x^2-a^2}\,dx$, put $x=a\cosh z$; then

$$dx=a\sinh z\,dz.$$

Then, since $\cosh^2 z-1=\sinh^2 z$,

$$\int\sqrt{x^2-a^2}\,dx=a^2\int\sinh^2 z\,dz=\frac{a^2}{2}\int(\cosh 2z-1)\,dz$$

$$=\frac{a^2}{4}\sinh 2z-\frac{a^2 z}{2}=\frac{1}{2}a\sinh z\,.\,a\cosh z-\frac{a^2 z}{2},$$

i.e. $$\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

or $$=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\frac{x+\sqrt{x^2-a^2}}{a},$$

and the a in the denominator may be omitted, as before, if the integral be indefinite.

[This last integral has already appeared in Art. 68 in finding the area of a portion of space bounded by a rectangular hyperbola, an ordinate and the x-axis].

79. From Art. 78 we may deduce the integration of $\sec^3 x$.

For, putting $\tan x=t$, $\sec^2 x\,dx=dt$, we have

$$\int\sec^3 x\,dx=\int\sqrt{1+t^2}\,dt=\frac{t\sqrt{1+t^2}}{2}+\frac{1}{2}\sinh^{-1}t$$

or $$=\frac{t\sqrt{1+t^2}}{2}+\frac{1}{2}\log(t+\sqrt{1+t^2}),$$

i.e. $$\int\sec^3 x\,dx=\frac{1}{2}\tan x\sec x+\frac{1}{2}\log(\sec x+\tan x)$$

or $$=\frac{1}{2}\tan x\sec x+\frac{1}{4}\log\frac{1+\sin x}{1-\sin x}$$

or $$=\frac{\sin x}{2\cos^2 x}+\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

Just in the same way, putting $\cot x=c$, $\operatorname{cosec}^2 x\,dx=-dc$, we have

$$\int\operatorname{cosec}^3 x\,dx=-\int\sqrt{1+c^2}\,dc=-\frac{c\sqrt{1+c^2}}{2}-\frac{1}{2}\log(c+\sqrt{1+c^2})$$

$$=-\frac{1}{2}\cot x\operatorname{cosec} x-\frac{1}{2}\log(\operatorname{cosec} x+\cot x)$$

$$=-\frac{1}{2}\frac{\cos x}{\sin^2 x}-\frac{1}{4}\log\frac{1+\cos x}{1-\cos x}$$

$$=-\frac{1}{2}\frac{\cos x}{\sin^2 x}+\frac{1}{2}\log\tan\frac{x}{2}.$$

80. We may now deduce from Art. 77 the integration of

$$\int\frac{dx}{\sqrt{R}},$$

where R is a quadratic function of x, viz.

$$R=ax^2+2bx+c.$$

CASE I. a Positive.

When a is positive we may write this integral as

$$\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{x^2+2\frac{b}{a}x+\frac{c}{a}}},$$

which we may arrange as

$$\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{a}\right)^2-\frac{b^2-ac}{a^2}}}\quad\text{or}\quad\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{a}\right)^2+\frac{ac-b^2}{a^2}}},$$

according as b^2 is greater or less than ac, and the real form of the integral is therefore (Art. 77)

$$\frac{1}{\sqrt{a}}\cosh^{-1}\frac{ax+b}{\sqrt{b^2-ac}}\quad\text{or}\quad\frac{1}{\sqrt{a}}\sinh^{-1}\frac{ax+b}{\sqrt{ac-b^2}},$$

according as b^2 is $>$ or $<$ ac.

In either case the integral may be written in the logarithmic form

$$\frac{1}{\sqrt{a}}\log\left(ax+b+\sqrt{a}\sqrt{ax^2+2bx+c}\right),\ \text{i.e.}\ \frac{1}{\sqrt{a}}\log\left(ax+b+\sqrt{aR}\right),$$

the constant $\frac{1}{\sqrt{a}}\log\sqrt{b^2\sim ac}$ being omitted.

Also, since $\quad\cosh^{-1}z=\sinh^{-1}\sqrt{z^2-1}$

and $\quad\sinh^{-1}z=\cosh^{-1}\sqrt{z^2+1}$,

$$\frac{1}{\sqrt{a}}\cosh^{-1}\frac{ax+b}{\sqrt{b^2-ac}}=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{\sqrt{aR}}{\sqrt{b^2-ac}}\quad(b^2>ac)$$

and

$$\frac{1}{\sqrt{a}}\sinh^{-1}\frac{ax+b}{\sqrt{ac-b^2}}=\frac{1}{\sqrt{a}}\cosh^{-1}\frac{\sqrt{aR}}{\sqrt{ac-b^2}}\quad(b^2<ac),$$

which forms may therefore be taken when a is positive and $b^2>$ or $<ac$ respectively.

81. CASE II. a Negative.

If in the integral $\int\frac{dx}{\sqrt{ax^2+2bx+c}}$, a be negative, write $-a=A$.

Then our integral may be written

$$\frac{1}{\sqrt{A}}\int\frac{dx}{\sqrt{-x^2+\frac{2b}{A}x+\frac{c}{A}}}$$

or
$$\frac{1}{\sqrt{A}}\int\frac{dx}{\sqrt{\frac{Ac+b^2}{A^2}-\left(x-\frac{b}{A}\right)^2}}$$

or
$$\frac{1}{\sqrt{A}}\sin^{-1}\frac{Ax-b}{\sqrt{Ac+b^2}},\quad i.e.\quad \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-ax-b}{\sqrt{b^2-ac}}$$

or, omitting a constant,
$$\frac{1}{\sqrt{-a}}\cos^{-1}\frac{ax+b}{\sqrt{b^2-ac}}\quad\left[\text{for} -\sin^{-1}z=\cos^{-1}z-\frac{\pi}{2}\right].$$

Also, since $\cos^{-1}z=\sin^{-1}\sqrt{1-z^2}$, we have
$$\cos^{-1}\frac{ax+b}{\sqrt{b^2-ac}}=\sin^{-1}\frac{\sqrt{-aR}}{\sqrt{b^2-ac}}.$$

82. To sum up then; it appears that when $R=ax^2+2bx+c$ we have the results:

$$\int\frac{dx}{\sqrt{R}}=\begin{cases}\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{\sqrt{-aR}}{\sqrt{b^2-ac}}, & a \text{ negative},\\[2ex] \dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{\sqrt{aR}}{\sqrt{b^2-ac}} & b^2>ac,\quad a\text{ positive},\\[2ex] \text{or}\quad \dfrac{1}{\sqrt{a}}\cosh^{-1}\dfrac{\sqrt{aR}}{\sqrt{ac-b^2}} & b^2<ac,\end{cases}$$

and the real form is to be chosen in each case.

83. Ex. 1. Integrate $\displaystyle\int\frac{dx}{\sqrt{2x^2+3x+4}}$.

We may write this $\displaystyle\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(x+\frac{3}{4})^2+\frac{23}{16}}}=\frac{1}{\sqrt{2}}\sinh^{-1}\frac{4x+3}{\sqrt{23}}$,

or it may be written $\displaystyle\frac{1}{\sqrt{2}}\cosh^{-1}\frac{2\sqrt{2}}{\sqrt{23}}\sqrt{2x^2+3x+4}$

or
$$\frac{1}{\sqrt{2}}\log(4x+3+2\sqrt{2}\sqrt{2x^2+3x+4}),$$

rejecting the constant $\displaystyle\frac{1}{\sqrt{2}}\log\frac{1}{\sqrt{23}}$.

Ex. 2. Integrate $\displaystyle\int\frac{dx}{\sqrt{4+3x-2x^2}}$.

This may be written $\displaystyle\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{41}{16}-(x-\frac{3}{4})^2}}$,

and therefore is $\displaystyle\frac{1}{\sqrt{2}}\sin^{-1}\frac{4x-3}{\sqrt{41}}$,

which may be written as
$$\frac{1}{\sqrt{2}}\cos^{-1}\frac{2\sqrt{2}}{\sqrt{41}}\sqrt{4+3x-2x^2}.$$

84. In exactly the same way, $\int \sqrt{ax^2+2bx+c}\,dx$, when a is positive or when a is negative, can be deduced from the results of Art. 78.

It appears then that the general rule in all cases of

$$\int\frac{dx}{\sqrt{R}} \quad \text{or} \quad \int\sqrt{R}\,dx,$$

where R is quadratic, will be, "Divide out the coefficient of x^2 and then complete the square, and then make use of the suitable standard form."

85. Functions of the form $\dfrac{Ax+B}{\sqrt{ax^2+2bx+c}}$ may be integrated by first putting $Ax+B$ into the form $\lambda(ax+b)+\mu$, which may be done either by inspection or by equating the coefficients, and we obtain

$$Ax+B=\frac{A}{a}(ax+b)+\left(B-\frac{Ab}{a}\right).$$

Thus, $$\frac{Ax+B}{\sqrt{R}}=\frac{A}{a}\cdot\frac{ax+b}{\sqrt{R}}+\frac{aB-Ab}{a}\cdot\frac{1}{\sqrt{R}}.$$

The integral of the first fraction is $\dfrac{A}{a}\sqrt{R}$, and that of the second has been discussed in Arts. 80, 81.

More general forms, such as

$$\frac{f(x)}{\sqrt{ax^2+2bx+c}} \quad \text{or} \quad \frac{f(x)}{\phi(x)}\frac{1}{\sqrt{ax^2+2bx+c}},$$

where f and ϕ are rational integral algebraic polynomials in x, are to be discussed later.

86. Before leaving the integration of $\displaystyle\int\frac{dx}{\sqrt{R}}$, the student should observe other forms into which the results may be thrown.

For some purposes a 'double angle' result is preferable to that given, *e.g.*

(1) $$\int\frac{dx}{\sqrt{x(a-x)}}=\int\frac{dx}{\sqrt{\frac{a^2}{4}-\left(\frac{a}{2}-x\right)^2}}=\cos^{-1}\frac{\frac{a}{2}-x}{\frac{a}{2}}=\cos^{-1}\left(1-\frac{2x}{a}\right).$$

But we may throw this into the form $2\sin^{-1}z$ by making $z^2=\dfrac{x}{a}$ and using $2\sin^{-1}z=\cos^{-1}(1-2z^2)$.

Then $\int \frac{dx}{\sqrt{x(a-x)}} = 2\sin^{-1}\sqrt{\frac{x}{a}} = 2\cos^{-1}\sqrt{\frac{a-x}{a}} = 2\tan^{-1}\sqrt{\frac{x}{a-x}}.$

(2) $$\int \frac{dx}{\sqrt{(x+b)(a-x)}} = \int \frac{d(x+b)}{\sqrt{\{(x+b)(a+b-x+b)\}}}$$
$$= 2\sin^{-1}\sqrt{\frac{b+x}{a+b}} = 2\cos^{-1}\sqrt{\frac{a-x}{a+b}} = 2\tan^{-1}\sqrt{\frac{b+x}{a-x}}$$
$(a>x)$.

(3) $$\int \frac{dx}{\sqrt{(x-b)(a-x)}} = 2\sin^{-1}\sqrt{\frac{x-b}{a-b}} = 2\cos^{-1}\sqrt{\frac{a-x}{a-b}} = 2\tan^{-1}\sqrt{\frac{x-b}{a-x}}$$
$(a>x>b)$.

(4) $$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = \int \frac{dx}{\sqrt{\left(x+\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}} = \cosh^{-1}\frac{2x+a+b}{a-b},$$

the ordinary form; but writing this $=2\sinh^{-1}z$, *i.e.* $\cosh^{-1}(2z^2+1)$,

$$z^2 = \tfrac{1}{2}\left(\frac{2x+a+b}{a-b} - 1\right) = \frac{x+b}{a-b};$$

$$\int \frac{dx}{\sqrt{(x+a)(x+b)}} = 2\sinh^{-1}\sqrt{\frac{x+b}{a-b}} = 2\cosh^{-1}\sqrt{\frac{x+a}{a-b}} = 2\tanh^{-1}\sqrt{\frac{x+b}{x+a}},$$
if $a>b$,

or $$2\sinh^{-1}\sqrt{\frac{x+a}{b-a}} = 2\cosh^{-1}\sqrt{\frac{x+b}{b-a}} = 2\tanh^{-1}\sqrt{\frac{x+a}{x+b}},$$ if $a<b$,

and so for other cases.

87. Of such the following forms are particularly useful:

$$\left.\begin{aligned} \int \frac{dx}{\sqrt{x(a-x)}} &= 2\sin^{-1}\sqrt{\frac{x}{a}}, \\ \int \frac{dx}{\sqrt{x(a+x)}} &= 2\sinh^{-1}\sqrt{\frac{x}{a}}, \\ \int \frac{dx}{\sqrt{x(x-a)}} &= 2\cosh^{-1}\sqrt{\frac{x}{a}}, \end{aligned}\right\}$$
and the others can be derived from these forms as shown above.

88. It will be noticed also in many cases, as, for instance, in the integral of Art. 81, viz.

$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{-a}}\sin^{-1}\frac{\sqrt{-aR}}{\sqrt{b^2-ac}} \quad (R=ax^2+2bx+c),$$

that the $\sqrt{R}$ of the integrand reappears in the integral. It did not do so when the result was arrived at as

$$\frac{1}{\sqrt{-a}}\cos^{-1}\frac{ax+b}{\sqrt{b^2-ac}},$$

but was made to do so by the subsequent transformation $\cos^{-1} z = \sin^{-1}\sqrt{1-z^2}$. Examine the earlier integral (Art. 44)

$$\int \frac{dx}{\sqrt{a^2-x^2}} \text{ given as } \sin^{-1}\frac{x}{a}.$$

This could be written $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \cos^{-1}\frac{\sqrt{a^2-x^2}}{a}$,

i.e. $$\int \frac{dx}{\sqrt{R}} = \cos^{-1}\frac{\sqrt{R}}{a} \quad (R = a^2 - x^2).$$

So also $$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a}$$

could be written as $$\int \frac{dx}{\sqrt{x^2+a^2}} = \cosh^{-1}\frac{\sqrt{x^2+a^2}}{a},$$

i.e. $$\int \frac{dx}{\sqrt{R}} = \cosh^{-1}\frac{\sqrt{R}}{a} \quad (R = x^2 + a^2).$$

Similarly $$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} = \sinh^{-1}\frac{\sqrt{x^2-a^2}}{a}$$

could be written as $$\int \frac{dx}{\sqrt{R}} = \sinh^{-1}\frac{\sqrt{R}}{a} \quad (R = x^2 - a^2).$$

And though these forms are obviously not the *simplest forms* of the various integrals, it is frequently desirable to adopt them, as they exhibit a visible relation between the integrand and the result of integration. The *simplest* forms are those tabulated to be remembered in the two lists of standard forms, Arts 44 and 89.

89. We are now in a position to make our list of

ADDITIONAL STANDARD FORMS.

1. $\int \cosh x\, dx = \sinh x$ and $\int \sinh x\, dx = \cosh x$.

2. $\int \operatorname{sech}^2 x\, dx = \tanh x$ and $\int \operatorname{cosech}^2 x\, dx = -\coth x$.

3. $\int \frac{\sinh x}{\cosh^2 x}\, dx = \int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x$.

4. $\int \frac{\cosh x}{\sinh^2 x}\, dx = \int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x$.

5. $\displaystyle\int\frac{dx}{\sqrt{x^2+a^2}}=\log\frac{x+\sqrt{x^2+a^2}}{a}=\sinh^{-1}\frac{x}{a}.$*

6. $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}=\log\frac{x+\sqrt{x^2-a^2}}{a}=\cosh^{-1}\frac{x}{a}.$

7. $\displaystyle\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

8. $\displaystyle\int\sqrt{a^2+x^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}.$

9. $\displaystyle\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}.$

10. $\displaystyle\int\frac{dx}{\sqrt{x(a-x)}}=2\sin^{-1}\sqrt{\frac{x}{a}}.$

11. $\displaystyle\int\frac{dx}{\sqrt{x(a+x)}}=2\sinh^{-1}\sqrt{\frac{x}{a}}.$

12. $\displaystyle\int\frac{dx}{\sqrt{x(x-a)}}=2\cosh^{-1}\sqrt{\frac{x}{a}}.$

13. $\displaystyle\int\operatorname{cosec}x\,dx=\int\frac{dx}{\sin x}=\log\tan\frac{x}{2}.$

14. $\displaystyle\int\sec x\,dx\quad=\int\frac{dx}{\cos x}=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\log(\sec x+\tan x)$

$$=\frac{1}{2}\log\frac{1+\sin x}{1-\sin x}=\text{gd}^{-1}x.$$

15. $\displaystyle\int\frac{dx}{x\sqrt{a^2-x^2}}=-\frac{1}{a}\operatorname{sech}^{-1}\frac{x}{a}.$

16. $\displaystyle\int\frac{dx}{x\sqrt{a^2+x^2}}=-\frac{1}{a}\operatorname{cosech}^{-1}\frac{x}{a}.$

17. $\displaystyle\int\frac{dx}{x^2-a^2}(x>a)=\frac{1}{2a}\log\frac{x-a}{x+a}=-\frac{1}{a}\coth^{-1}\frac{x}{a}.$

18. $\displaystyle\int\frac{dx}{a^2-x^2}(x<a)=\frac{1}{2a}\log\frac{a+x}{a-x}=\frac{1}{a}\tanh^{-1}\frac{x}{a}.$

It is customary to obtain 17 and 18 when wanted, rather than to commit them to memory. They will be discussed later (Art. 127).

* See also Art. 1890, Vol. II.

EXAMPLES.

Write down the integrals of

1. $\dfrac{1}{9-x^2}$, $\dfrac{1}{9-4x^2}$, $\dfrac{1}{x^2-4}$, $\dfrac{1}{9x^2-4}$, $\sqrt{16-9x^2}$, $\sqrt{3x^2-5}$, $\sqrt{2+3x^2}$.

2. $\dfrac{1}{\sqrt{x(x-4)}}$, $\dfrac{1}{\sqrt{x(4-x)}}$, $\dfrac{1}{\sqrt{x(4+x)}}$, $\dfrac{1}{\sqrt{2+2x-x^2}}$, $\dfrac{1}{\sqrt{x^2-2x+2}}$, $\sqrt{x^2+2ax}$.

3. $\dfrac{x}{\sqrt{9-x^2}}$, $\dfrac{x}{\sqrt{x^2-9}}$, $\dfrac{x}{\sqrt{9-4x^2}}$, $\dfrac{x^2}{\sqrt{1-x^2}}$, $\dfrac{x^2}{\sqrt{x^2+1}}$.

4. $x\sqrt{x^2+a^2}$, $(x+b)\sqrt{x^2+a^2}$, $\dfrac{ax+b}{\sqrt{x^2+c^2}}$.

5. $x(x^2+a^2)^{\frac{n}{2}}$, $(x+a)(x^2+2ax+b)^{\frac{n}{2}}$, $(ax-b)(ax^2-2bx+c)^{\frac{n}{2}}$.

6. $\dfrac{x^2+2x+3}{\sqrt{1-x^2}}$, $\dfrac{x^2+2x+3}{\sqrt{x^2+1}}$, $\dfrac{x^2+2x+3}{\sqrt{x^2+x+1}}$, $\dfrac{x^2+ax+b}{\sqrt{x^2+cx+d}}$.

7. $\sqrt{x^2+4x+5}$, $\sqrt{-x^2+4x+5}$, $\sqrt{4x^2+4x+5}$, $\sqrt{-4x^2+4x+5}$.

8. $\sqrt{\dfrac{x+a}{x-a}}$, $\sqrt{\dfrac{a+x}{a-x}}$, $x\sqrt{\dfrac{a+x}{a-x}}$, $(x+a)\sqrt{\dfrac{x+b}{x-b}}$, $\dfrac{(x+a)^{\frac{3}{2}}}{(x-a)^{\frac{1}{2}}}$.

9. $\operatorname{cosec} nx$, $\operatorname{cosec}(2x+b)$, $\dfrac{1}{4\cos^3 x-3\cos x}$, $\dfrac{1+\tan^2 x}{1-\tan^2 x}$, $\frac{1}{2}(\cot x+\tan x)$.

10. $\dfrac{1}{a\sin x+b\cos x}$, $\dfrac{1}{\sin 2x+\cos 2x}$, $\dfrac{a\sin x+b\cos x}{c\sin x+d\cos x}$.

11. Deduce $\int \operatorname{cosec} x\,dx=\log\tan\dfrac{x}{2}$ by expressing $\operatorname{cosec} x$ as

$$\frac{1}{2}\left(\cot\frac{x}{2}+\tan\frac{x}{2}\right)$$

12. Deduce $\int \sec x\,dx=\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ by

(i) putting $\sin x=z$,

(ii) putting $\sec x+\tan x=z$.

Show that
$$\int \sec x\,dx=\cosh^{-1}(\sec x).$$

13. Integrate $\displaystyle\int\frac{\cos\theta\,d\theta}{\sin\theta\sqrt{1-\sin^{2n}\theta}}$.

[redacted] MPLES.

1. If APB be a [redacted], and PN an ordinate to the diameter AB, and P'[redacted]dinate, show that

$$Lt \sum \frac{NN'}{NP} = \text{c}\ [\text{redacted}]\ \text{the angle } OPN,$$

the summation being [redacted] and B, and NN' being [redacted]

2. Find the area in [redacted] ordinates, the ordinate $x=x_1$ an[redacted]

If the range $x=0$ to $x=a$ on the x-axis [redacted] portions of length h and rectangles be inscribed in the Newtonian manner, examine the limit of the area of the last of these rectangles when h is indefinitely diminished. Find the whole area from $x=0$ to $x=a$.

3. Find the value of $\int \sqrt{e^{2x}+ae^x}\,dx$ [R. P.]

4. Evaluate (i) $\displaystyle\int \frac{dx}{\sqrt{1-3x-x^2}}$. [I. C. S., 1884.]

(ii) $\displaystyle\int \frac{dx}{x\sqrt{x^2+x-6}}$. $\left(\text{Put } x=\frac{1}{y}.\right)$

[Oxford Second Public Ex., 1880.]

(iii) $\displaystyle\int \frac{(1+x)\,dx}{(2x^2+3x+4)^{\frac{1}{2}}}$ [Colleges β, 1891.]

(iv) $\displaystyle\int \frac{x-1}{(x^2+2x-1)^{\frac{1}{2}}}\,dx$. [Trinity, 1892.]

(v) $\displaystyle\int \frac{3x+4}{(x^2+2x+5)^{\frac{1}{2}}}\,dx$. [Math. Trip., 1887.]

5. Show that the result of integrating $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}$ may be exhibited as

(i) $2\cos^{-1}\dfrac{\sqrt{a+x}-\sqrt{a-x}}{2\sqrt{a}}$ or as (ii) $2\sin^{-1}\dfrac{\sqrt{a+x}+\sqrt{a-x}}{2\sqrt{a}}$

or as (iii) $2\tan^{-1}\sqrt{\dfrac{a+R}{a-R}}$, where $R=\sqrt{a^2-x^2}$.

6. If $R = ax^2 + 2bx + c$ and [illegible] that

$$\int \frac{dx}{\sqrt{R}} = \text{[illegible]}$$

or [illegible] $= \text{[illegible]} \frac{R}{+K}$,

[illegible] $\tan\theta$.) [Oxf. I., 1888.]

7. [illegible] $\int \frac{dx}{(x+1)\sqrt{x^2-1}}$. (Put $x = \sec\theta$.)

[Oxf. I, 1888.]

8. Show that (i) $\displaystyle\int_0^1 \frac{dx}{(1+x^2)(1-x^2)^{\frac{1}{2}}} = \frac{\pi}{2\sqrt{2}}$. [Trinity, 1888.]

(ii) $\displaystyle\int \frac{dx}{x^4\sqrt{1+x^2}} = -\frac{2x^2-1}{3x^3}\sqrt{1+x^2}$. [Trinity, 1882.]

9. Integrate $\displaystyle\int \sqrt{1+e^x+e^{2x}}\, dx$.

10. A sphere of given radius a consists of an infinite number of concentric shells of very small thickness, the density at the surface of any shell varying as the n^{th} power of its radius. Find the mass of the sphere. [Ox. I. Pub., 1903.]

If any diameter AB cut one of the shells at P and the density of the shell varies inversely as (i) $OP\sqrt{AP.PB}$, (ii) $OP^2\sqrt{AP.PB}$, find the mass of the sphere in each case, O being the centre.

11. A triangle ABC is divided into strips by lines parallel to BC; a point is taken in each strip, and the square of the perpendicular from this point to BC is multiplied by the area of the strip; the same is done with all the strips, and the sum of the products is formed. Express by a definite integral the limiting value of this sum when the breadths of all the strips are diminished indefinitely, and evaluate the integral in terms of the base BC and the distance of A therefrom. [Ox. I. Pub., 1901.]

12. Prove that if $u^2 = x^2 + 2px + q$, an integral of the form

$$\int f(x, u)\, dx$$

can always be rationalized (provided f is a rational algebraic function) by one of the substitutions

$$\frac{u}{\sqrt{p^2-q}}=\frac{2y}{1-y^2} \quad \text{or} \quad \frac{u}{\sqrt{q-p^2}}=\frac{1+y^2}{1-y^2}. \qquad \text{[Coll. } a\text{, 1890.]}$$

13. Find the relation connecting x and y, being given

$$x^2y^4\left(\frac{dy}{dx}\right)^2=a^2+b^2y^3. \qquad \text{[I. C. S., 1889.]}$$

14. Show that $\displaystyle\int_{\sqrt{3}}^{2\sqrt{3}} z^3(z^2-3)^{\frac{3}{2}}\,dz=\frac{16038}{35}.$ [Ox. I., 1888.]

15. Integrate

$$\text{(i)} \int\frac{\sin\theta\,d\theta}{\sqrt{a\cos^2\theta+2b\cos\theta+c}}, \quad \text{(ii)} \int\frac{\cos\theta\,d\theta}{\sqrt{a\sin^2\theta+2b\sin\theta+c}},$$

$$\text{(iii)} \int\frac{d\theta}{\cos\theta\sqrt{a\cos^2\theta+2b\sin\theta\cos\theta+c\sin^2\theta}},$$

$$\text{(iv)} \int\frac{d\theta}{\sin\theta\sqrt{a\cos^2\theta+2b\sin\theta\cos\theta+c\sin^2\theta}},$$

$$\text{(v)} \int\frac{d\theta}{\sin\theta\sqrt{a\cos^2\theta+b\sin^2\theta+c}}. \qquad \text{[Trin., 1888.]}$$

16. Integrate

$$\text{(i)} \int\frac{x^4}{(a^2-x^2)^{\frac{1}{2}}}\,dx, \quad \text{(ii)} \int\frac{x^2\,dx}{(a+bx^2)\sqrt{c^2-x^2}}. \qquad \text{[Trin., 1888.]}$$

17. (*a*) Evaluate $\displaystyle\int_1^7 (x^2-6x+13)\,dx$, first directly, second by putting $x^2-6x+13=y$. (Draw a graph and explain fully.)

(*b*) Evaluate $\displaystyle\int_0^{\frac{2b}{a}} (ax^2-2bx+c)\,dx,$

and explain by a graph the result when $2b^2=3ac$.

Obtain the same result by substituting

$$ax^2-2bx+c=y, \quad \text{taking } b^2<ac.$$

Also obtain $\displaystyle\int_0^{\frac{3b}{a}} (ax^2-2bx+c)\,dx$ by this substitution, explaining your limits for y by means of the graph.

18. Point out the fallacy in the following argument:

$$\int_{-1}^{1}\frac{dx}{1+x^2}=\Big[\tan^{-1}x\Big]_{-1}^{1}=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}.$$

But putting $x=\dfrac{1}{y}$, $\quad dx=-\dfrac{dy}{y^2}$. $\quad\left.\begin{array}{l}\text{When } x=-1,\ y=-1.\\ \text{When } x=\ \ 1,\ y=\ \ 1.\end{array}\right\}$

$$\therefore \int_{-1}^{1}\frac{dx}{1+x^2}=-\int_{-1}^{1}\frac{dy}{1+y^2}=-\int_{-1}^{1}\frac{dx}{1+x^2},$$

for, as the result is numerical, the letter used in integration cannot affect the result.

Hence $\quad 2\displaystyle\int_{-1}^{1}\frac{dx}{1+x^2}=0$; $\quad$ but $\displaystyle\int_{-1}^{1}\frac{dx}{1+x^2}=\frac{\pi}{2}$; $\ \therefore\ \pi=0$!

19. Point out the fallacy in the following reasoning:

We have, if we put $x=e^t$,

$$\left(x\frac{d}{dx}\right)^a x^k=\left(\frac{d}{dt}\right)^a e^{kt}=k^a e^{kt}=k^a x^k.$$

But when $a=-1$, we have

$$\left(x\frac{d}{dx}\right)^{-1} x^k=\frac{1}{x}\int x^k dx=\frac{x^k}{k+1},$$

and these two results do not agree. [R.P.]

20. Prove that $\qquad \iota\,\mathrm{gd}\left(\dfrac{1}{\iota}\mathrm{gd}\,u\right)=u,$ [CAYLEY, *E.F.*]

and show that if $\qquad \mathrm{gd}\,u=a_1u+a_3u^3+a_5u^5+\dots,$

then will $\qquad \mathrm{gd}^{-1}u=a_1u-a_3u^3+a_5u^5-\dots.$

21. If $\qquad \sec x+\tan x=1+S_1\dfrac{x}{1!}+S_2\dfrac{x^2}{2!}+S_3\dfrac{x^3}{3!}+\dots,$

show that $\qquad \mathrm{gd}^{-1}x=x+S_3\dfrac{x^3}{3!}+S_5\dfrac{x^5}{5!}+\dots,$

that $\qquad S_{p+1}=S_p+\dbinom{p}{2}S_{p-2}S_2+\dbinom{p}{4}S_{p-4}S_4+\dots$

and $\qquad S_n-\dbinom{n}{2}S_{n-2}+\dbinom{n}{4}S_{n-4}-\dots+\cos\dfrac{n\pi}{2}=\sin\dfrac{n\pi}{2},$

and that $\quad S_3=2,\quad S_5=16,\quad S_7=272,\quad S_9=7936$, etc.

[*Diff. Calc.*, Art. 573, etc.]

22. Integrate $\qquad \displaystyle\int\frac{dx}{(a+x)(c+x)^{\frac{1}{2}}}$

by putting $\qquad c+x=(a-c)z^2 \quad$ or $\quad (c-a)z^2,$

according as $\qquad a >$ or $< c$.

Taking the case $a>c$, consider the same integral with $a+da$ replacing a, subtract the original integral, divide by da, and take the limit when da is indefinitely diminished.

Hence obtain $$\int \frac{dx}{(a+x)^2 (c+x)^{\frac{1}{2}}}.$$

Deduce also $$\int \frac{dx}{(a+x)(c+x)^{\frac{3}{2}}}.$$

23. Evaluate $$\int \frac{dx}{(x^2-a^2)(x^2-c^2)^{\frac{1}{2}}}.$$

(i) if $a > c$,
(ii) if $a < c$. (Put $x = c \sec \phi$). [MATH. TRIP., 1878.]

24. Show that $$\int \frac{(x-p)^{2n+1}}{(ax^2+2bx+c)^{n+\frac{3}{2}}}\,dx = \frac{1}{(b^2-ac)^{n+1}} \int (y^2-q^2)^n\,dy,$$

where $$q^2 \equiv ap^2 + 2bp + c$$

and $$y\sqrt{ax^2+2bx+c} = (ap+b)x + bp + c.$$ [COLLEGES, 1901.]

25. If $$F(x) = a_1 f(x) + a_2 f(2x) + a_3 f(3x) + \dots,$$

prove that $$\frac{a_1}{1^x} + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \dots = \frac{\int_0^\infty t^{x-1} F(t)\,dt}{\int_0^\infty t^{x-1} f(t)\,dt}.$$

[OX. J. M. SCH., 1904.]

26. Integrate (i) $$\int \frac{1+x^2}{1-x^2} \frac{dx}{\sqrt{1+x^4}}.$$

(ii) $$\int \frac{1-x^2}{1+x^2} \frac{dx}{\sqrt{1+x^4}}.$$ [EULER.]

27. Show that if $F(x, y)$ be a rational function of x and y,

$$\int F\left(x, \sqrt[n]{\frac{\alpha x+\beta}{\gamma x+\delta}}\right) dx$$

can be thrown into rational form by the substitution

$$\frac{\alpha x+\beta}{\gamma x+\delta} = z^n.$$

Hence show that

$$\int \left(\frac{1-2x}{1+2x}\right)^{\frac{3}{2}} dx = 3\tan^{-1}\sqrt{\frac{1-2x}{1+2x}} - 2\sqrt{\frac{1-2x}{1+2x}} - \frac{\sqrt{1-4x^2}}{2}.$$

28. Show that if $F(x, y)$ be any rational integral function of x and y,

$$\int F(x, \sqrt{ax^2+2bx+c})\,dx$$

can be thrown into rational form by any of the substitutions

$$\begin{aligned}&(1)\ \sqrt{ax^2+2bx+c}=\sqrt{a}\,(x+z),\\ &(2)\ \sqrt{ax^2+2bx+c}=xz+\sqrt{c},\\ &(3)\qquad x-x_2=z^2\,(x-x_1),\end{aligned}$$

where x_1, x_2 are the roots of $ax^2+2bx+c=0$. [BERTRAND, *C.I.*, p. 39.]

Apply each of these methods to the integration of

$$\int\frac{x\,dx}{\sqrt{x^2-6x+8}},$$

showing that the result in each case reduces to

$$\sqrt{x^2-6x+8}+3\cosh^{-1}(x-3),$$

as derived by the method of Art. 85.

29. If $$x^3-3a^2x=a^2y,$$

show that $$\frac{dx}{\sqrt{x^2-4a^2}}=\frac{1}{3}\frac{dy}{\sqrt{y^2-4a^2}},$$

and hence obtain Cardan's formula for the solution of a cubic.

[J. M. SCH. OX.]

30. Evaluate $$\int_0^{\frac{\pi}{2}}\frac{\cos x\,dx}{1-\sin^2\alpha\cos^2 x},$$

and deduce the expansion

$$\frac{2\alpha}{\sin 2\alpha}=1+\frac{2}{3}\sin^2\alpha+\frac{2\,.\,4}{3\,.\,5}\sin^4\alpha+\ldots,\quad\text{where }\frac{\pi}{2}>\alpha>0.$$

[OXF. I. P., 1915.]

31. Show that

$$\int\frac{dx}{(1+x^4)\{(1+x^4)^{\frac{1}{2}}-x^2\}^{\frac{1}{2}}}=\tan^{-1}\frac{x}{\{(1+x^4)^{\frac{1}{2}}-x^2\}^{\frac{1}{2}}}.$$

[EULER, *C.I.*, iv.]

Integrate $$\int\frac{dx}{(1+x^{2n})\{(1+x^{2n})^{\frac{1}{n}}-x^2\}^{\frac{1}{2}}}.$$

32. Show that the integrals

$$\int\frac{dx}{(1-x^m)(2x^m-1)^{\frac{1}{2m}}}\quad\text{and}\quad\int\frac{x^{m-1}dx}{(1-x^m)(2x^m-1)^{\frac{1}{2m}}}$$

are reduced to the integration of rational fractions by the respective substitutions $2x^m-1=u^{2m}x^{2m}$ and $2x^m-1=u^{2m}$.

[LEXELL, *Actes de Pétersbourg*, 1781, ii.; LACROIX, *C.D.*, ii., p. 65.]

CHAPTER IV

INTEGRATION BY PARTS. POWERS OF SINES AND COSINES.

Integration by Parts.

90. Let u and w be functions of x, and let accents denote differentiations, and suffixes integrations, with respect to x.

Thus u'' stands for $\frac{d^2u}{dx^2}$ and w_2 for $\int\left[\int w\,dx\right]dx$, and so on with u''', w_3, etc.

Then $$\frac{d}{dx}(uw)=u\frac{dw}{dx}+w\frac{du}{dx}$$

which we may write as

$$(uw)'=uw'+wu'.$$

It follows that $$uw=\int uw'\,dx+\int wu'\,dx$$

or $$\int uw'\,dx=uw-\int wu'\,dx.$$

This may be put into another form.

Let $u=\phi(x)$ and w' $\left(i.e.\ \frac{dw}{dx}\right)=\psi(x)=v$, say; so that

$$w=\int\psi(x)\,dx=v_1.$$

Then the above rule may be written

$$\int\phi(x)\,\psi(x)\,dx=\phi(x)\left\{\int\psi(x)\,dx\right\}-\int\phi'(x)\left\{\int\psi(x)\,dx\right\}dx,$$

i.e. $$\int uv\,dx=uv_1-\int u'v_1\,dx,$$

or the two functions ϕ and ψ may be interchanged, and then

$$\int \phi(x)\,\psi(x)\,dx = \psi(x)\left\{\int \phi(x)\,dx\right\} - \int \psi'(x)\left\{\int \phi(x)\,dx\right\}dx;$$

i.e.
$$\int uv\,dx = vu_1 - \int v'u_1\,dx.$$

Thus, in integrating the product of two functions, if the integral be not at once obtainable, it is possible to connect the integral

$$\int \phi(x)\,\psi(x)\,dx$$

with either of two new integrals, viz. those of

$$\int \phi'(x)\left\{\int \psi(x)\,dx\right\}dx, \quad \int \psi'(x)\left\{\int \phi(x)\,dx\right\}dx,$$

and supposing that the integral of one of the two factors $\phi(x)$, $\psi(x)$ is known, one of these new integrals *may be more easily obtainable than that of the original product.*

91. The rule may be put into words thus:

Int. of Prod. $\phi \cdot \psi$ = 1st function × Integral of 2nd
− Integral of [Diff. Co. of 1st × Int. of 2nd].

92. Ex.
$$\int x \sin nx\,dx.$$

Here it is important to connect if possible $\int x \sin nx\,dx$ with another *in which the factor x has been removed.* There is a choice as to whether we put
$$u = x \quad \text{and} \quad v = \sin nx$$
or
$$u = \sin nx \quad \text{and} \quad v = x;$$
but it will be observed that in the connected integral $\int u'v_1\,dx$, u has been differentiated, v integrated. Hence the removal of x will be effected if we take the first alternative.

Then
$$u = x, \quad u' = 1, \quad v = \sin nx, \quad v_1 = -\frac{\cos nx}{n}.$$

Thus, by the rule,

$$\begin{aligned}\int x \sin nx\,dx &= x\left[-\frac{\cos nx}{n}\right] - \int 1\left[-\frac{\cos nx}{n}\right]dx \\ &= -\frac{x\cos nx}{n} + \frac{1}{n}\int \cos nx\,dx \\ &= -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}.\end{aligned}$$

93. It is to be noted that *unity* may be regarded as one of the factors to aid an integration.

Thus
$$\int \log x\,dx = \int 1 \log x\,dx$$
$$= x\log x - \int x\frac{d}{dx}(\log x)\,dx$$
$$= x\log x - \int x\frac{1}{x}\,dx$$
$$= x\log x - \int 1\,dx$$
$$= x\log x - x = x(\log_e x - 1),$$
or as it may be written $= x\log_e\left(\frac{x}{e}\right)$.

94. **Repetition of the Operation.**

The operation of integration by parts may be repeated as often as may be considered necessary for the evaluation of the original integral.

Thus
$$\int x^4\sin nx\,dx = (x^4)\left(-\frac{\cos nx}{n}\right) - \int (4x^3)\left(-\frac{\cos nx}{n}\right)dx,$$
$$\int (4x^3)\left(-\frac{\cos nx}{n}\right)dx = (4x^3)\left(-\frac{\sin nx}{n^2}\right) - \int (4\,.\,3x^2)\left(-\frac{\sin nx}{n^2}\right)dx,$$
$$\int (4\,.\,3x^2)\left(-\frac{\sin nx}{n^2}\right)dx = (4\,.\,3x^2)\left(+\frac{\cos nx}{n^3}\right) - \int (4\,.\,3\,.\,2\,.\,x)\left(+\frac{\cos nx}{n^3}\right)dx$$
$$\int (4.3.2.x)\left(+\frac{\cos nx}{n^3}\right)dx = (4.3.2.x)\left(\frac{\sin nx}{n^4}\right) - \int (4\,.\,3\,.\,2\,.\,1)\left(\frac{\sin nx}{n^4}\right)dx,$$
$$\int (4\,.\,3\,.\,2\,.\,1)\left(\frac{\sin nx}{n^4}\right)dx = (4\,.\,3\,.\,2\,.\,1)\left(-\frac{\cos nx}{n^5}\right).$$

Then adding and subtracting alternately,
$$\int x^4\sin nx\,dx = (x^4)\left(-\frac{\cos nx}{n}\right) - (4x^3)\left(-\frac{\sin nx}{n^2}\right)$$
$$+(4\,.\,3x^2)\left(+\frac{\cos nx}{n^3}\right) - (4\,.\,3\,.\,2\,x)\left(\frac{\sin nx}{n^4}\right)$$
$$+(4\,.\,3\,.\,2\,.\,1)\left(-\frac{\cos nx}{n^5}\right).$$

The student will note that no arithmetical simplification is attempted until the whole operation is complete. The total operation is much less liable to error if simplification be postponed to the end.

We now obviously have
$$\int x^4\sin nx\,dx = P\cos nx + Q\sin nx,$$
where
$$P = -\frac{x^4}{n} + 4\,.\,3\frac{x^2}{n^3} - \frac{4\,.\,3\,.\,2\,.\,1}{n^5},$$
$$Q = 4\frac{x^3}{n^2} - 4\,.\,3\,.\,2\frac{x}{n^4}.$$

95. The General Rule.

It is obviously possible to formulate a general rule for the repeated operation. And such a method is most serviceable in practice.

The rule is

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots + (-1)^{n-1}u^{(n-1)}v_n$$
$$+ (-1)^n \int u^{(n)}v_n\,dx,$$

where $u^{(n-1)}$ is written for u with $n-1$ accents, *i.e.* the $(n-1)^{\text{th}}$ differential coefficient of u.

For
$$\int uv\,dx = uv_1 - \int u'v_1\,dx,$$
$$\int u'v_1\,dx = u'v_2 - \int u''v_2\,dx,$$
$$\int u''v_2\,dx = u''v_3 - \int u'''v_3\,dx,$$
$$\int u'''v_3\,dx = u'''v_4 - \int u''''v_4\,dx,$$
$$\text{etc.} = \text{etc.},$$
$$\int u^{(n-2)}v_{n-2}\,dx = u^{(n-2)}v_{n-1} - \int u^{(n-1)}v_{n-1}\,dx,$$
$$\int u^{(n-1)}v_{n-1}\,dx = u^{(n-1)}v_n - \int u^{(n)}v_n\,dx.$$

Hence, adding and subtracting alternately,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots + (-1)^{n-1}u^{(n-1)}v_n$$
$$+ (-1)^n \int u^{(n)}v_n\,dx.$$

Ex. 1. Thus applying this to the last example (Art. 94),

$$\int x^4 \sin nx\,dx = (x^4)\left(-\frac{\cos nx}{n}\right) - (4x^3)\left(-\frac{\sin nx}{n^2}\right)$$
$$+ (4\,.\,3x^2)\left(+\frac{\cos nx}{n^3}\right) - (4\,.\,3\,.\,2x)\left(\frac{\sin nx}{n^4}\right)$$
$$+ (4\,.\,3\,.\,2\,.\,1)\left(-\frac{\cos nx}{n^5}\right),$$

each term being derived from the preceding by the simple rule of "**diff. 1st factor and integ. 2nd**" **and connecting by alternate signs.** When one of the factors is a rational integral algebraic polynomial, it is ultimately destroyed by the successive differentiations.

Ex. 2. $$\int x^m e^{ax}\,dx = x^m\frac{e^{ax}}{a} - mx^{m-1}\frac{e^{ax}}{a^2} + m(m-1)x^{m-2}\frac{e^{ax}}{a^3} - m(m-1)(m-2)x^{m-3}\frac{e^{ax}}{a^4} + \ldots + (-1)^m m!\frac{e^{ax}}{a^{m+1}}.$$

96. If one of the subsidiary integrals *returns to the original form*, this fact may be utilized to infer the result of the integration.

Ex. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx \quad\ldots\ldots\ldots\text{(i)}$$

and $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx. \quad\ldots\ldots\ldots\text{(ii)}$$

Hence, if $P = \int e^{ax}\sin bx\,dx$ and $Q = \int e^{ax}\cos bx\,dx$,

$$P = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\frac{e^{ax}}{a}\cos bx + \frac{b}{a}P\right]$$

and $$Q = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left[\frac{e^{ax}}{a}\sin bx - \frac{b}{a}Q\right],$$

whence $$P = e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2},$$

$$Q = e^{ax}\frac{b\sin bx + a\cos bx}{a^2+b^2}.$$

Or we might have written equations (i) and (ii) as

$$\left.\begin{aligned} aP + bQ &= e^{ax}\sin bx,\\ -bP + aQ &= e^{ax}\cos bx,\end{aligned}\right\}\text{ and then solve for } P \text{ and } Q.$$

We may write P and Q as follows:

$$\left.\begin{aligned} P &= (a^2+b^2)^{-\frac{1}{2}}e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right),\\ Q &= (a^2+b^2)^{-\frac{1}{2}}e^{ax}\cos\left(bx - \tan^{-1}\frac{b}{a}\right),\end{aligned}\right\}$$

forms which are frequently useful and which are *derivable at once from the formula for the n^{th} differential coefficient*, viz.

$$\frac{d^n}{dx^n}e^{ax}\frac{\sin}{\cos}bx = (a^2+b^2)^{\frac{n}{2}}e^{ax}\frac{\sin}{\cos}\left(bx + n\tan^{-1}\frac{b}{a}\right),$$

by putting $n = -1$. [*Diff. Calc.*, Art. 93.]

And this is what we should be led to expect. For if to differentiate $e^{ax}\frac{\sin}{\cos}(bx+c)$ is the same as to multiply it by $\sqrt{a^2+b^2}$ and to increase the angle by $\tan^{-1}\frac{b}{a}$, the effect of integration, which is the inverse operation, must be to divide out by the factor $\sqrt{a^2+b^2}$ and to diminish the angle by $\tan^{-1}\frac{b}{a}$.

And it is in this form, viz.

$$\frac{e^{ax}}{\sqrt{a^2+b^2}}\frac{\sin}{\cos}\left(bx+c-\tan^{-1}\frac{b}{a}\right),$$

that the integration of $\int e^{ax}\frac{\sin}{\cos}(bx+c)\,dx$ is most easily remembered.

97. In cases of the form

$$e^{ax}\sin bx\sin cx\sin dx,\quad e^{ax}\sin^p x\cos^q x,\quad e^{ax}\sin^p x\cos nx,\text{ etc.},$$

p and q being positive integers, the trigonometrical factor must first be expressed as the sum of a series of sines or cosines of multiples of x by trigonometrical means, and then each term being of form $e^{ax}\frac{\sin}{\cos}mx$ can be integrated.

98. Ex. 1. $$I=\int e^x\sin 2x\cos x\,dx.$$

Now $$\sin 2x\cos x=\tfrac{1}{2}(\sin 3x+\sin x);$$

$$\therefore\ I=\tfrac{1}{2}\int e^x(\sin 3x+\sin x)\,dx$$

$$=\tfrac{1}{2}e^x\left[\frac{1}{\sqrt{10}}\sin(3x-\tan^{-1}3)+\frac{1}{\sqrt{2}}\sin\left(x-\frac{\pi}{4}\right)\right].$$

Ex. 2. $$I=\int e^{3x}\sin^2 x\cos^3 x\,dx.$$

Now $$\sin^2 x\cos^3 x=\tfrac{1}{4}\sin^2 2x\cos x=\tfrac{1}{8}(1-\cos 4x)\cos x$$
$$=\tfrac{1}{16}(2\cos x-\cos 3x-\cos 5x);$$

$$\therefore\ \int e^{3x}\sin^2 x\cos^3 x\,dx=\tfrac{1}{16}\int e^{3x}(2\cos x-\cos 3x-\cos 5x)\,dx$$

$$=\frac{e^{3x}}{16}\left[\frac{2}{\sqrt{10}}\cos\left(x-\tan^{-1}\frac{1}{3}\right)-\frac{1}{3\sqrt{2}}\cos\left(3x-\frac{\pi}{4}\right)-\frac{1}{\sqrt{34}}\cos\left(5x-\tan^{-1}\frac{5}{3}\right)\right]$$

Ex. 3. Integrate $\int\sqrt{a^2-x^2}\,dx$ by "Parts."

$$\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}-\int x\frac{d}{dx}\sqrt{a^2-x^2}\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$$

[Note this step. Some such rearrangement is frequently necessary.]

$$=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}-\int\sqrt{a^2-x^2}\,dx,$$

whence, transposing and dividing by 2,

$$\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a},$$

which agrees with the result of Art. 78 obtained by the method of substitution of $a\sin\theta$ for x.

99. The method of Integration by "Parts" shows immediately that whenever a direct function $\phi(x)$ can be integrated, so also can the corresponding inverse function $\phi^{-1}(x)$, *i.e.* if

$\int\phi(x)\,dx$ can be found, so also can $\int\phi^{-1}(x)\,dx$ be found.

For, putting $\phi^{-1}(x)=z$,

$$x=\phi(z)\quad\text{and}\quad dx=\phi'(z)\,dz.$$

Hence
$$\int\phi^{-1}(x)\,dx=\int z\phi'(z)\,dz$$
$$=z\phi(z)-\int\phi(z)\,dz,$$

which establishes the rule.

100. **Geometrical Consideration.**

This is no more than might have been anticipated from geometrical considerations.

Let PQ be any arc of a curve referred to rectangular axes Ox, Oy, and let the coordinates of P be (x_0, y_0) and of Q (x_1, y_1). Let the equation of the curve be $y=\phi(x)$; or if x, y be expressed in terms of a single variable t, let the equations of the curve be

$$x=f_1(t)\equiv u,\ \text{say},$$
$$y=f_2(t)\equiv v,\ \text{say};$$

and let t_0 and t_1 be the values of t corresponding to the values x_0, y_0 and x_1, y_1, of x and y respectively.

Let PN, QM be the ordinates and PN_1, QM_1 the abscissae of the points P, Q. Then plainly

$$\text{area } PNMQ=\text{rect. } OQ-\text{rect. } OP-\text{area } PQM_1N_1.$$

But
$$\text{area } PNMQ=\int_{x_0}^{x_1}y\,dx=\int_{x_0}^{x_1}\phi(x)\,dx,$$

$$\text{area } PQM_1N_1=\int_{y_0}^{y_1}x\,dy=\int_{y_0}^{y_1}\phi^{-1}(y)\,dy.$$

Also
$$\text{rect. } OQ=x_1y_1\quad\text{and}\quad\text{rect. } OP=x_0y_0.$$

Thus $$\int_{x_0}^{x_1} y\,dx = (x_1y_1 - x_0y_0) - \int_{y_0}^{y_1} x\,dy, \quad \ldots\ldots\ldots\ldots\text{(1)}$$

$$i.e.\ \int_{x_0}^{x_1} \phi(x)\,dx = (x_1y_1 - x_0y_0) - \int_{y_0}^{y_1} \phi^{-1}(y)\,dy.$$

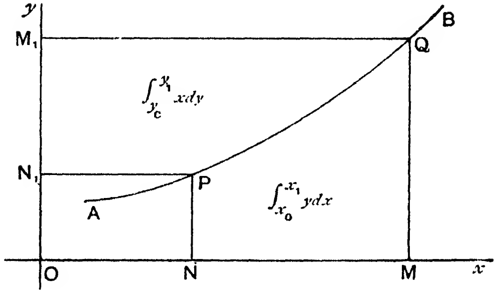


Fig. 16.

Hence the dependence of the one integral upon the other is obvious, and to establish the possibility of calculating the area $PNMQ$ is to establish incidentally the possibility of obtaining the area of PQM_1N_1.

Further, $$\int_{x_0}^{x_1} y\,dx = \int_{f_1(t_0)}^{f_1(t_1)} v\,du = \int_{t_0}^{t_1} v\frac{du}{dt}\,dt$$

and $$\int_{y_0}^{y_1} x\,dy = \int_{f_2(t_0)}^{f_2(t_1)} u\,dv = \int_{t_0}^{t_1} u\frac{dv}{dt}\,dt$$

and $$x_1y_1 - x_0y_0 = \Big[uv\Big]_{t_0}^{t_1}.$$

So that the equation (1) may be written

$$\int_{t_0}^{t_1} v\frac{du}{dt}\,dt = \Big[uv\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} u\frac{dv}{dt}\,dt,$$

and thus the general rule of integration by parts is established geometrically.

The meaning of the process is therefore this: In cases where there is a difficulty in finding the area $PNMQ$, we may find instead the area PQM_1N_1 and deduce the former result from the latter.

EXAMPLES.

Integrate by parts

1. xe^{3x}, x^2e^{ax}, x^5e^{-x}, $x\cosh x$, $x^2\sinh x$.
2. $x\cos x$, $x^5\cos 2x$, $x^2\cos^2 x$, $x^2\cos 3x\sin x$, $x\sin x\sin 2x\sin 3x$.
3. $e^x\sin 2x$, $e^x\sin^2 x$, $e^{3x}\sin^3 x\cos x$, $e^{-6x}\cos x\sin^2 x\cos 3x$.
4. $x^3\log x$, $x^n\log x$, $x^n(\log x)^2$, $x^n(\log x)^3$.
5. $e^{ax}\sin px\sin qx\sin rx$, $e^{ax}\sin px\sin qx\cos rx$.
6. $e^{ax}\sin px\sin qx\cos^2 rx$, $e^{ax}\cos px\cos qx\cos^2(p+q)x$.
7. Evaluate
$$\int_0^{\pi} x\sin x\,dx,\quad \int_0^{\frac{\pi}{2}} x^2\cos x\,dx,\quad \int_0^{\frac{\pi}{2}} x^2\cos 2x\,dx.$$
8. Integrate
$$\int\sin^{-1}x\,dx,\quad \int x\sin^{-1}x\,dx,\quad \int x^3\sin^{-1}x\,dx,\quad \int x\tan^{-1}x\,dx.$$

101. **Reduction Formulae.**

It not infrequently occurs that a function which it is desired to integrate is not immediately integrable or reducible by substitution to one or other of the standard forms whose integrals have been committed to memory. But it may happen in such a case that the integral may be connected in a linear manner with the integral of another function, or with the integrals of other functions, which are simpler or easier to integrate than the original function.

Such a connecting formula is called a Reduction Formula. Thus an integration by parts makes one integral depend upon a second integral, and is a Reduction Formula.

Many Formulae of this type will be found and used in subsequent chapters.

102. We have seen how a repetition of the process of integration by parts will enable us to calculate the integrals

$$S_m=\int x^m\sin nx\,dx,\quad C_m=\int x^m\cos nx\,dx.$$

We propose to construct "Reduction Formulae" for these integrals, giving S_m, C_m in terms of S_{m-2}, C_{m-2} respectively.

Integrating by parts, we have at once

$$S_m=-x^m\frac{\cos nx}{n}+\frac{m}{n}C_{m-1}$$

and
$$C_m=x^m\frac{\sin nx}{n}-\frac{m}{n}S_{m-1}.$$

Thus,

$$S_m = -x^m \frac{\cos nx}{n} + \frac{m}{n}\left[x^{m-1}\frac{\sin nx}{n} - \frac{m-1}{n} S_{m-2}\right],$$

and $$C_m = x^m \frac{\sin nx}{n} - \frac{m}{n}\left[-x^{m-1}\frac{\cos nx}{n} + \frac{m-1}{n} C_{m-2}\right],$$

i.e. $$S_m = -x^m \frac{\cos nx}{n} + mx^{m-1}\frac{\sin nx}{n^2} - \frac{m(m-1)}{n^2} S_{m-2},$$

and $$C_m = x^m \frac{\sin nx}{n} + mx^{m-1}\frac{\cos nx}{n^2} - \frac{m(m-1)}{n^2} C_{m-2}.$$

Thus, when the four integrals for the cases $m=0$ and $m=1$ are found, viz.

$$S_0 = \int \sin nx\, dx = -\frac{\cos nx}{n}, \quad C_0 = \int \cos nx\, dx = \frac{\sin nx}{n},$$

$$S_1 = \int x \sin nx\, dx = -x\frac{\cos nx}{n} + \frac{\sin nx}{n^2},$$

$$C_1 = \int x \cos nx\, dx = x\frac{\sin nx}{n} + \frac{\cos nx}{n^2},$$

all others can be deduced by *successive applications* of the above formulae.

This *illustrates the use of a reduction formula.* But for expressions like $x^m \sin nx$, $x^m \cos nx$ it is ordinarily better in practice to apply the method of Art. 95 at once and avoid the successive substitutions.

Examples.

Write down the integrals of

1. $\int x^6 e^x\, dx$, $\int x^5 \sinh x\, dx$, $\int x^5 \cosh^2 x\, dx$.

2. $\int_0^{\frac{\pi}{2}} x^3 \sin x\, dx$, $\int_0^{\frac{\pi}{2}} x^3 \sin^2 x\, dx$, $\int_0^{\frac{\pi}{2}} x^4 \sin x \cos x\, dx$.

3. $\int_0^{\pi} x^5 \sin x\, dx$, $\int_0^{\pi} x^6 \cos^2 x\, dx$, $\int_0^1 x^3 \cosh x\, dx$.

4. $\int_0^{\frac{\pi}{2}} x^3(a^2\cos^2 x + b^2 \sin^2 x)\, dx$, $\int_1^3 x^3 \log x\, dx$, $\int_0^1 x \tan^{-1} x\, dx$.

5. $\int_0^{\frac{\pi}{2}} e^x \sin x \cos^2 x\, dx$, $\int_0^{\frac{\pi}{2}} x \sin x \sin 2x \sin 3x\, dx$.

103. The Determination of the Integrals

$$\int x^n e^{ax} \sin bx\,dx, \quad \int x^n e^{ax} \cos bx\,dx,$$

may be at once effected.

For remembering

$$\int e^{ax} \begin{matrix}\sin\\ \cos\end{matrix} bx\,dx = \frac{e^{ax}}{r} \begin{matrix}\sin\\ \cos\end{matrix} (bx-\phi),$$

where $r=\sqrt{a^2+b^2}$ and $\tan\phi=\frac{b}{a}$, we have

$$\int x^n e^{ax} \sin bx\,dx = \frac{x^n}{r} e^{ax}\sin(bx-\phi) - \frac{nx^{n-1}}{r^2} e^{ax}\sin(bx-2\phi)$$

$$+\frac{n(n-1)}{r^3} x^{n-2}e^{ax}\sin(bx-3\phi) - \dots$$

$$+(-1)^n \frac{n!}{r^{n+1}} e^{ax}\sin(bx-\overline{n+1}\,\phi)$$

or $$= e^{ax}(P\sin bx - Q\cos bx),$$

where

$$P=\frac{x^n}{r}\cos\phi - n\frac{x^{n-1}}{r^2}\cos 2\phi + n(n-1)\frac{x^{n-2}}{r^3}\cos 3\phi - \dots,$$

$$Q=\frac{x^n}{r}\sin\phi - n\frac{x^{n-1}}{r^2}\sin 2\phi + n(n-1)\frac{x^{n-2}}{r^3}\sin 3\phi - \dots.$$

Similarly,

$$\int x^n e^{ax}\cos bx\,dx = e^{ax}\{P\cos bx + Q\sin bx\}.$$

104. Integration of

$$C_n = \int e^{ax}\cos^n bx\,dx, \quad S_n = \int e^{ax}\sin^n bx\,dx.$$

We may now express $\cos^n bx$ and $\sin^n bx$ in a series of cosines or sines of multiples of bx and then integrate each term by Art. 96; or, we may obtain formulae connecting C_n with C_{n-2} and S_n with S_{n-2}, thus:

$$C_n = \int e^{ax}\cos^n bx\,dx = \frac{e^{ax}}{a}\cos^n bx + \int \frac{e^{ax}}{a}\,.\,nb\cos^{n-1}bx\sin bx\,dx$$

$$= \frac{e^{ax}}{a}\cos^n bx + \frac{nb}{a}\Big[\frac{e^{ax}}{a}\cos^{n-1}bx\sin bx$$

$$- \int \frac{b}{a}e^{ax}\{\cos^n bx - (n-1)\cos^{n-2}bx\sin^2 bx\}\,dx\Big]$$

$$=\frac{e^{ax}}{a}\cos^n bx+\frac{nb}{a}\left[\frac{e^{ax}}{a}\cos^{n-1}bx\sin bx\right.$$

$$\left.-\int\frac{b}{a}e^{ax}\{n\cos^n bx-(n-1)\cos^{n-2}bx\}\,dx\right];$$

$$\therefore\ \left(1+\frac{n^2b^2}{a^2}\right)C_n=\frac{e^{ax}}{a}\cos^n bx+\frac{nb}{a^2}e^{ax}\cos^{n-1}bx\sin bx$$

$$+n(n-1)\frac{b^2}{a^2}C_{n-2}.$$

Hence

$$C_n=e^{ax}\cos^{n-1}bx\frac{a\cos bx+nb\sin bx}{a^2+n^2b^2}+\frac{n(n-1)b^2}{a^2+n^2b^2}C_{n-2}.$$

Similarly

$$S_n=e^{ax}\sin^{n-1}bx\frac{a\sin bx-nb\cos bx}{a^2+n^2b^2}+\frac{n(n-1)b^2}{a^2+n^2b^2}S_{n-2}.$$

And as $\int e^{ax}\,dx$, $\int e^{ax}\sin bx\,dx$, $\int e^{ax}\cos bx\,dx$ (that is, S_0, C_0, S_1 and C_1) can be written down (Art. 96), the integration of $\int e^{ax}\cos^n bx\,dx$ and $\int e^{ax}\sin^n bx\,dx$ can be completed, in any case where n is a positive integer, by successive reduction.

105. Ex. Integrate $\int e^x\sin^5 x\,dx$ (i) by the "multiple angle" method,
(ii) by "reduction."

(i) Let $\cos x+\iota\sin x=y$; then $2\iota\sin x=y-\frac{1}{y}$ (see Art. 112).

$$2^5\iota^5\sin^5 x=\left(y-\frac{1}{y}\right)^5=\left(y^5-\frac{1}{y^5}\right)-5\left(y^3-\frac{1}{y^3}\right)+10\left(y-\frac{1}{y}\right)$$

$$=2\iota\sin 5x-10\iota\sin 3x+20\iota\sin x;$$

$$\therefore\ \sin^5 x=\frac{1}{2^4}(\sin 5x-5\sin 3x+10\sin x).$$

$$\therefore\ \int e^x\sin^5 x\,dx$$

$$=\frac{1}{2^4}\int e^x(\sin 5x-5\sin 3x+10\sin x)\,dx$$

$$=\frac{e^x}{2^4}\left[\frac{1}{\sqrt{26}}\sin(5x-\tan^{-1}5)-\frac{5}{\sqrt{10}}\sin(3x-\tan^{-1}3)+\frac{10}{\sqrt{2}}\sin\left(x-\frac{\pi}{4}\right)\right].$$

(ii) Proceeding with the reduction formula, $a=1$, $b=1$, $n=5$,

$$S_5=e^x\sin^4 x\frac{\sin x-5\cos x}{1^2+5^2}+\frac{5\,.\,4}{1^2+5^2}S_3.$$

Similarly $$S_3 = e^x \sin^2 x \frac{\sin x - 3\cos x}{1^2+3^2} + \frac{3\,.\,2}{1^2+3^2} S_1$$

and $$S_1 = \int e^x \sin x \, dx = \frac{e^x \sin\left(x - \frac{\pi}{4}\right)}{\sqrt{1^2+1^2}};$$

$$\therefore\ S_5 = e^x \left[\frac{1}{26} \sin^4 x (\sin x - 5\cos x) + \frac{5\,.\,4}{26}\left\{ \sin^2 x \frac{\sin x - 3\cos x}{10} + \frac{3\,.\,2}{10\sqrt{2}} \sin\left(x - \frac{\pi}{4}\right)\right\}\right].$$

106. **Integrals of form $I_n = \int x^m (\log x)^n \, dx$, n being a positive integer and m not equal to -1.**

Integrating by parts, we have

$$I_n = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^{m+1} \frac{1}{x} (\log x)^{n-1} dx$$

$$= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx,$$

i.e. $$I_n = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} I_{n-1}. \qquad (1)$$

Writing l for $\log x$,

$$I_n = \frac{x^{m+1}}{m+1} l^n - \frac{n}{m+1} \left[\frac{x^{m+1}}{m+1} l^{n-1} - \frac{(n-1)}{m+1} I_{n-1} \right];$$

and proceeding in this way, we ultimately get down to I_1, which is

$$\int x^m \log x \, dx, \quad i.e.\ \frac{x^{m+1}}{m+1} l - \frac{x^{m+1}}{(m+1)^2}.$$

Hence

$$I_n = \frac{x^{m+1}}{m+1} \left[l^n - \frac{n}{m+1} l^{n-1} + \frac{n(n-1)}{(m+1)^2} l^{n-2} - \frac{n(n-1)(n-2)}{(m+1)^3} l^{n-3} + \dots + \frac{(-1)^{n-1} n!}{(m+1)^{n-1}} l + \frac{(-1)^n n!}{(m+1)^n} \right]. \quad \dots(2)$$

107. If the *definite* integral $\int_0^1 x^m (\log x)^n \, dx$ be required $(m > -1)$, note that

$$x^{m+1} (\log x)^r = 0 \text{ when } x = 1 \text{ and } r > 0,$$

and that $$Lt_{x=0} x^{m+1} (\log x)^r = 0.$$

[*Diff. Calc.*, Art. 474, Ex. 3.]

Hence

$$\left[I_n\right]_0^1=(-1)\frac{n}{m+1}\left[I_{n-1}\right]_0^1=(-1)^2\frac{n(n-1)}{(m+1)^2}\left[I_{n-2}\right]_0^1=\text{etc.},$$

and finally,
$$\left[I_1\right]_0^1=\frac{-1}{(m+1)^2}.$$

Hence
$$\left[I_n\right]_0^1=(-1)^n\frac{n!}{(m+1)^{n+1}},$$

$$i.e.\quad \int_0^1 x^m(\log x)^n\,dx=(-1)^n\frac{n!}{(m+1)^{n+1}},\quad\text{..............(3)}$$

which is also directly obvious from result (2).

When $m=-1$,

$$I_n=\int\frac{1}{x}(\log x)^n\,dx=\frac{(\log x)^{n+1}}{n+1}.$$

108. The reduction formula established by integration by parts was

$$I_n=\frac{x^{m+1}}{m+1}(\log x)^n-\frac{n}{m+1}I_{n-1}.$$

We may point out that this could be obtained by the rule of "the smaller index+1" of Art. 217 by putting $P=x^{m+1}(\log x)^n$ and differentiating, but in this case there is no advantage in using this method, as the same formula is immediately written down by "parts" as above.

109. We may add, in passing, that $\int\frac{x^m}{\log x}dx$ cannot be integrated in finite terms except when $m=-1$. In that case, we have

$$\int\frac{1}{x\log x}dx=\log(\log x).$$

In other cases put $x=e^y$.

Then
$$\int\frac{x^m}{\log x}dx=\int\frac{e^{my}}{y}e^y\,dy=\int\frac{e^{(m+1)y}}{y}dy,$$

and expanding the exponential, we have

$$=\int\left[\frac{1}{y}+(m+1)+\frac{(m+1)^2}{\lfloor 2}y+\frac{(m+1)^3}{\lfloor 3}y^2+\dots\right]dy$$

$$=\log y+(m+1)y+\frac{(m+1)^2}{\lfloor 2}\frac{y^2}{2}+\frac{(m+1)^3}{\lfloor 3}\frac{y^3}{3}+\dots$$

$$=\log(\log x)+(m+1)\log x+\frac{(m-1)^2}{\lfloor 2}\frac{(\log x)^2}{2}+\frac{(m+1)^3}{\lfloor 3}\frac{(\log x)^3}{3}+\dots,$$

and the integration is expressed as an infinite series.

110. Integrals of the form $\int x^m(\log x)^n dx$, where n is a negative integer, may be reduced to the above form by using the reduction formula in the reversed form, and writing n for $n-1$,

$$\int x^m(\log x)^n dx = \frac{x^{m+1}}{n+1}(\log x)^{n+1} - \frac{m+1}{n+1}\int x^m(\log x)^{n+1} dx.$$

Thus

$$\int \frac{x^2}{(\log x)^2}dx = -\frac{x^3}{\log x} + 3\int \frac{x^2}{\log x}dx$$

$$= -\frac{x^3}{\log x} + 3\left[\log(\log x) + 3\log x + \frac{3^2(\log x)^2}{2\,\underline{|2}} + \dots\right].$$

But as these expansions are not finite in expression, they are of but little practical importance.

111. Integrals, however, where m is negative and n is positive, can be expressed in finite terms by the reduction formulae, and present no difficulty.

Ex. $$I_3 = \int \frac{(\log x)^3}{x^{10}}dx.$$

$$I_3 = \frac{x^{-9}}{-9}(\log x)^3 - \frac{3}{-9}\int x^{-10}(\log x)^2 dx$$

$$= -\frac{1}{9}x^{-9}(\log x)^3 + \frac{3}{9}\left[\frac{x^{-9}}{-9}(\log x)^2 + \frac{2}{9}\int x^{-10}\log x\, dx\right]$$

$$= -\frac{1}{9}\frac{(\log x)^3}{x^9} - \frac{3}{9^2}\frac{(\log x)^2}{x^9} - \frac{3.2}{9^3}\frac{\log x}{x^9} - \frac{3.2.1}{9^4 x^9},$$

i.e. $$-\frac{1}{9x^9}\left[(\log x)^3 + \frac{3}{9}(\log x)^2 + \frac{3.2}{9^2}(\log x) + \frac{3.2.1}{9^3}\right].$$

Note on a Trigonometrical Process.

112. We return to the Method of Multiple Angles already introduced in Arts. 97, 105.

The process of expressing $\sin^p x\cos^q x$ in multiple angles is a matter of Trigonometry. But for the convenience of the student it is briefly indicated here, as it will be extensively required in what follows.

Remembering that

$$(\cos x + \iota\sin x)^n = \cos nx + \iota\sin nx \text{ (Demoivre)},$$

let $\quad \cos x + \iota \sin x = y$; then $\cos x - \iota \sin x = \dfrac{1}{y}$,

$$\cos nx + \iota \sin nx = y^n \quad \text{and} \quad \cos nx - \iota \sin nx = \frac{1}{y^n}.$$

Thus $\quad 2\cos x = y + \dfrac{1}{y},\quad 2\iota \sin x = y - \dfrac{1}{y},$

$$2\cos nx = y^n + \frac{1}{y^n},\quad 2\iota \sin nx = y^n - \frac{1}{y^n}.$$

Thus, if we require, say, $\sin^8 x$ in a series of sines or cosines of multiples of x, we proceed thus:

$$2^8\iota^8 \sin^8 x = \left(y - \frac{1}{y}\right)^8 = y^8 + \frac{1}{y^8} - 8\left(y^6 + \frac{1}{y^6}\right) + 28\left(y^4 + \frac{1}{y^4}\right) - 56\left(y^2 + \frac{1}{y^2}\right) + 70$$
$$= 2\cos 8x - 16\cos 6x + 56\cos 4x - 112\cos 2x + 70$$

and $\quad \sin^8 x = \dfrac{1}{2^7}(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35)$.

$\sin^8 x$ thus expressed is then ready either for finding *the n^{th} differential coefficient*, or *for integration*, or for *expansion* in powers of x, as may be required.

If we required $\sin^6 x \cos^2 x$, say, in a series of sines or cosines of multiples of x, then

$$2^6\iota^6 \sin^6 x \,.\, 2^2 \cos^2 x = \left(y - \frac{1}{y}\right)^6 \left(y + \frac{1}{y}\right)^2 \quad \text{(See the next article.)}$$
$$= y^8 + \frac{1}{y^8} - 4\left(y^6 + \frac{1}{y^6}\right) + 4\left(y^4 + \frac{1}{y^4}\right) + 4\left(y^2 + \frac{1}{y^2}\right) - 10$$
$$= 2\cos 8x - 8\cos 6x + 8\cos 4x + 8\cos 2x - 10,$$

and $\quad \sin^6 x \cos^2 x = \dfrac{1}{2^7}\left\{-\cos 8x + 4\cos 6x - 4\cos 4x - 4\cos 2x + 5\right\}$,

and is ready for integration, etc.

113. It is convenient for such examples to remember that the several sets of binomial coefficients may be quickly reproduced in the following scheme:

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

etc.,

each number being formed at once as the sum of the one immediately above it and the preceding one. Thus, in forming the seventh row,

$$0+1=1, \quad 1+5=6, \quad 5+10=15, \quad 10+10=20, \text{ etc.},$$

and in multiplying out such a product as the one in Art. 112, we only need the coefficients of $(1-t)^6(1+t)^2$, and all the work appearing will be

Coefficients of $(1-t)^6$ are $1-6+15-20+15-6+1$,

Coefficients of $(1-t)^6(1+t)$ are $1-5+9-5-5+9-5+1$,

Coefficients of $(1-t)^6(1+t)^2$ are $1-4+4+4-10+4+4-4+1$,

each row of figures being formed according to the same law as before.

The student will discover the reason of this by performing the actual multiplication of

$$a+bt+ct^2+dt^3+\dots \text{ by } 1+t,$$

in which the several coefficients in the result are

$$0+a, \quad a+b, \quad b+c, \quad c+d, \dots.$$

Similarly, if the coefficients in $(1+t)^4(1-t)^2$ were required, the work appearing would be

$$1+4+6+4+1$$

$$1+3+2-2-3-1$$

$$1+2-1-4-1+2+1,$$

and the last row gives the coefficients required.

The coefficients here are formed thus:

$$1-0=1, \quad 4-1=3, \quad 6-4=2, \quad 4-6=-2, \text{ etc.}$$

Powers and Products of Sines and Cosines.

114. Sine or Cosine with Positive Odd Integral Index.

Any *odd positive power* of a sine or cosine can be integrated immediately thus:

To integrate

$$\int \sin^{2n+1} x\,dx, \quad \text{let } \cos x = c; \quad \therefore \sin x\,dx = -dc.$$

Hence

$$\int \sin^{2n+1} x\,dx = -\int (1-c^2)^n\,dc$$

$$= -\int \left[1 - nc^2 + \frac{n(n-1)}{1.2}c^4 - \dots + (-1)^n c^{2n}\right] dc$$

$$= -c + \frac{nc^3}{3} - \frac{n(n-1)}{1.2}\frac{c^5}{5} + \ldots - (-1)^n \frac{c^{2n+1}}{2n+1}$$

$$= -\cos x + {}^nC_1 \frac{\cos^3 x}{3} - {}^nC_2 \frac{\cos^5 x}{5} + \ldots - (-1)^n\, {}^nC_n \frac{\cos^{2n+1} x}{2n+1}.$$

Similarly, putting $\sin x = s$, and therefore $\cos x\, dx = ds$, we have

$$\int \cos^{2n+1} x\, dx = \int (1-s^2)^n\, ds$$

$$= \sin x - {}^nC_1 \frac{\sin^3 x}{3} + {}^nC_2 \frac{\sin^5 x}{5} - \ldots + (-1)^n\, {}^nC_n \frac{\sin^{2n+1} x}{2n+1}.$$

115. **Products of form $\sin^p x . \cos^q x$, p or q being an odd positive integer.**

In the same way as before, any product of the form $\sin^p x \cos^q x$ admits of immediate integration by the same method whenever either p or q is a positive odd integer, whatever the other may be.

Thus, to integrate $\int \sin^p x \cos^{2n+1} x\, dx$. Let $\sin x = s$; then

$$\cos x\, dx = ds \quad \text{and} \quad \int \sin^p x \cos^{2n+1} x\, dx = \int s^p (1-s^2)^n\, ds,$$

and expanding as before,

$$= \frac{\sin^{p+1} x}{p+1} - {}^nC_1 \frac{\sin^{p+3} x}{p+3} + {}^nC_2 \frac{\sin^{p+5} x}{p+5} - \ldots + (-1)^n\, {}^nC_n \frac{\sin^{p+2n+1} x}{p+2n+1}$$

116. **When $p+q$ is a negative even integer,** the expression $\sin^p x \cos^q x$ admits of immediate integration in terms of $\tan x$ or $\cot x$.

For, put $\tan x = t$, and therefore $\sec^2 x\, dx = dt$, and let

$$p+q = -2n,$$

n being positive and integral.

Thus

$$\int \sin^p x \cos^q x\, dx = \int \tan^p x \cos^{p+q+2} x\, dt = \int t^p (1+t^2)^{n-1}\, dt$$

$$= \int (t^p + {}^{n-1}C_1\, t^{p+2} + {}^{n-1}C_2 t^{p+4} + \ldots + {}^{n-1}C_{n-1}\, t^{p+2n-2})\, dt$$

$$= \frac{\tan^{p+1} x}{p+1} + {}^{n-1}C_1 \frac{\tan^{p+3} x}{p+3} + {}^{n-1}C_2 \frac{\tan^{p+5} x}{p+5} + \ldots + {}^{n-1}C_{n-1} \frac{\tan^{p+2n-1} x}{p+2n-1}.$$

Similarly, if we put

$$\cot x = c, \quad \text{then} \quad -\operatorname{cosec}^2 x\,dx = dc,$$

and

$$\int \sin^p x \cos^q x\,dx = -\int \cot^q x \sin^{p+q+2} x\,dc = -\int c^q (1+c^2)^{n-1}\,dc$$

$$= -\frac{\cot^{q+1} x}{q+1} - {}^{n-1}C_1 \frac{\cot^{q+3} x}{q+3} - {}^{n-1}C_2 \frac{\cot^{q+5} x}{q+5} - \dots - {}^{n-1}C_{n-1} \frac{\cot^{q+2n-1} x}{q+2n-1}.$$

This result is the same as the former, arranged in the opposite order.

117. **Use of Multiple Angles.** $\sin^p x$, $\cos^q x$, $\sin^p x \,.\, \cos^q x$, where p and q are *positive integers, either odd or even.*

To sum up then, when in $\sin^p x$, p is odd, or in $\cos^q x$, q is odd, or in $\sin^p x \cos^q x$ *one* of the two p, q is odd, the best method of procedure is that of Arts. 114, 115.

But when both p and q are positive even indices, this method cannot be adopted, for the series used are not terminating series.

We then express the function to be integrated as the sum of a series of sines or cosines of multiples of x, which can be done in all cases by the method of Art. 112, or in simple cases without having recourse to that method. We then have

$$\sin^p x, \quad \cos^q x \quad \text{or} \quad \sin^p x \cos^q x$$

expressed in the form

$$\Sigma A_n \sin nx \quad \text{or} \quad \Sigma A_n \cos nx,$$

and each term may be integrated at once, giving

$$-\Sigma A_n \frac{\cos nx}{n} \quad \text{or} \quad \Sigma A_n \frac{\sin nx}{n}$$

as the integral.

118. Ex. 1. (A small even index.) $\displaystyle\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4}.$

Ex. 2. (A small odd index.) $\displaystyle\int \cos^3 x\,dx = \int \frac{3\cos x + \cos 3x}{4}\,dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x$

or otherwise $\displaystyle = \int (1-s^2)\,ds = \sin x - \frac{\sin^3 x}{3}$

Ex. 3. (A small even index.)
$$\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \int \frac{1+2\cos 2x + \frac{1+\cos 4x}{2}}{4}\,dx$$
$$= \int \left(\tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x\right) dx$$
$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x.$$

119. But for higher powers we adopt the method of Art. 112.

Ex. 4. (A large even index.) $\int \sin^8 x\,dx.$

Let $\cos x + \iota \sin x = y$, etc.
$$2^8\iota^8 \sin^8 x = \left(y - \frac{1}{y}\right)^8 = \left(y^8 + \frac{1}{y^8}\right) - 8\left(y^6 + \frac{1}{y^6}\right) + 28\left(y^4 + \frac{1}{y^4}\right) - 56\left(y^2 + \frac{1}{y^2}\right) + 70$$
$$= 2\cos 8x - 16\cos 6x + 56\cos 4x - 112\cos 2x + 70;$$
$$\therefore \int \sin^8 x\,dx = \frac{1}{2^7}\left[\frac{\sin 8x}{8} - \frac{8\sin 6x}{6} + \frac{28\sin 4x}{4} - \frac{56\sin 2x}{2} + 35x\right].$$

Ex. 5. (A large odd index.)
$$\int \sin^9 x\,dx = -\int (1-c^2)^4\,dc = -\int (1 - 4c^2 + 6c^4 - 4c^6 + c^8)\,dc$$
$$= -\cos x + \frac{4\cos^3 x}{3} - \frac{6\cos^5 x}{5} + \frac{4\cos^7 x}{7} - \frac{\cos^9 x}{9}.$$

Ex. 6. Find $\int \sin^8 x \cos^2 x\,dx.$ (Both indices even.)

Then, as in Art. 112,
$$2^8\iota^8 \sin^8 x \,.\, 2^2\cos^2 x = \left(y - \frac{1}{y}\right)^8 \left(y + \frac{1}{y}\right)^2$$

[and the working of the multiplication is

Coefficients in $(1-t)^8$	$1-8+28-56+70-56+28-\ 8+1$
Coefficients in $(1-t)^8(1+t)$	$1-7+20-28+14+14-28+20-7+1$
Coefficients in $(1-t)^8(1+t)^2$	$1-6+13-\ 8-14+28-14-\ 8+13-6+1$]

$\therefore\ 2^8\iota^8\sin^8 x \,.\, 2^2\cos^2 x$
$$= \left(y^{10} + \frac{1}{y^{10}}\right) - 6\left(y^8 + \frac{1}{y^8}\right) + 13\left(y^6 + \frac{1}{y^6}\right) - 8\left(y^4 + \frac{1}{y^4}\right) - 14\left(y^2 + \frac{1}{y^2}\right) + 28$$
$$= 2\cos 10x - 12\cos 8x + 26\cos 6x - 16\cos 4x - 28\cos 2x + 28;$$

$\therefore \int \sin^8 x \cos^2 x\,dx$
$$= \frac{1}{2^9}\left[\frac{\sin 10x}{10} - \frac{6\sin 8x}{8} + \frac{13\sin 6x}{6} - \frac{8\sin 4x}{4} - \frac{14\sin 2x}{2} + 14x\right]$$
$$= \frac{1}{2^9}\left[\frac{\sin 10x}{10} - \frac{3\sin 8x}{4} + \frac{13\sin 6x}{6} - 2\sin 4x - 7\sin 2x + 14x\right].$$

Ex. 7. Find $\int \sin^8 x \cos^3 x\, dx$.
(One index odd.)

$$\int \sin^8 x \cos^3 x\, dx = \int \sin^8 x (1 - \sin^2 x)\, d(\sin x).$$

$$= \frac{\sin^9 x}{9} - \frac{\sin^{11} x}{11}.$$

Ex. 8. $\int e^{2x} \sin^6 x \cos^2 x\, dx$
(An exponential factor.)

$$= -\frac{1}{2^7}\int e^{2x}[\cos 8x - 4\cos 6x + 4\cos 4x + 4\cos 2x - 5]\, dx$$

(Art. 112)

$$= -\frac{e^{2x}}{2^7}\left[\frac{\cos(8x - \tan^{-1}4)}{\sqrt{68}} - 2\frac{\cos(6x - \tan^{-1}3)}{\sqrt{10}}\right.$$

$$\left. + 2\frac{\cos(4x - \tan^{-1}2)}{\sqrt{5}} + 2\frac{\cos\left(2x - \frac{\pi}{4}\right)}{\sqrt{2}} - \frac{5}{2}\right].$$

Ex. 9. Consider $I \equiv \int e^x \sin nx \cos^3 x \sin^2 x\, dx.$
(An exponential factor and a trigonometrical factor $\sin nx$, in which n is not necessarily integral.)

As before, $\quad 2^3 \cos^3 x\, 2^2 \iota^2 \sin^2 x = \left(y + \frac{1}{y}\right)^3 \left(y - \frac{1}{y}\right)^2.$

Coefficients of $(1+t)^3$ $\quad 1+3+3+1,$

Coefficients of $(1+t)^3(1-t)$ $\quad 1+2+0-2-1,$

Coefficients of $(1+t)^3(1-t)^2$ $\quad 1+1-2-2+1+1;$

$$\therefore \cos^3 x \sin^2 x = -\frac{1}{2^4}(\cos 5x + \cos 3x - 2\cos x);$$

$$\therefore \sin nx \cos^3 x \sin^2 x = -\frac{1}{2^5}[2\sin nx \cos 5x + 2\sin nx \cos 3x - 4\sin nx \cos x]$$

$$= -\frac{1}{2^5}[\sin(n+5)x + \sin(n-5)x + \sin(n+3)x + \sin(n-3)x$$

$$- 2\sin(n+1)x - 2\sin(n-1)x];$$

whence $\quad \int e^x \sin nx \cos^3 x \sin^2 x\, dx$

$$= -\frac{1}{2^5}e^x\left[\frac{\sin\{(n+5)x - \tan^{-1}(n+5)\}}{\sqrt{(n+5)^2+1}} + \frac{\sin\{(n-5)x - \tan^{-1}(n-5)\}}{\sqrt{(n-5)^2+1}}\right.$$

$$+ \frac{\sin\{(n+3)x - \tan^{-1}(n+3)\}}{\sqrt{(n+3)^2+1}} + \frac{\sin\{(n-3)x - \tan^{-1}(n-3)\}}{\sqrt{(n-3)^2+1}}$$

$$\left. - 2\frac{\sin\{(n+1)x - \tan^{-1}(n+1)\}}{\sqrt{(n+1)^2+1}} - 2\frac{\sin\{(n-1)x - \tan^{-1}(n-1)\}}{\sqrt{(n-1)^2+1}}\right].$$

120. Integral Powers of a Secant or Cosecant.

Even positive powers of a secant or cosecant are even negative powers of a cosine or a sine, and come under the head discussed in Art. 116.

Thus, $\int \sec^2 x\,dx = \tan x$,

$$\int \sec^4 x\,dx = \int (1+\tan^2 x)\,d\tan x$$

$$= \tan x + \frac{\tan^3 x}{3},$$

$$\int \sec^6 x\,dx = \int (1+2\tan^2 x + \tan^4 x)\,d\tan x$$

$$= \tan x + \frac{2\tan^3 x}{3} + \frac{\tan^5 x}{5},$$

and generally

$$\int \sec^{2n+2} x\,dx = \int (1+t^2)^n\,dt, \text{ where } t = \tan x,$$

$$= \tan x + {}^nC_1\frac{\tan^3 x}{3} + {}^nC_2\frac{\tan^5 x}{5} + \ldots + {}^nC_n\frac{\tan^{2n+1} x}{2n+1}.$$

Similarly,

$$\int \operatorname{cosec}^2 x\,dx = -\cot x,$$

$$\int \operatorname{cosec}^4 x\,dx = -\int (1+\cot^2 x)\,d\cot x$$

$$= -\cot x - \frac{\cot^3 x}{3},$$

etc.,

and generally

$$\int \operatorname{cosec}^{2n+2} x\,dx = -\cot x - {}^nC_1\frac{\cot^3 x}{3} - {}^nC_2\frac{\cot^5 x}{5} - \ldots - {}^nC_n\frac{\cot^{2n+1} x}{2n+1}.$$

121. Exactly in the same way

$$\int \sec^p x \operatorname{cosec}^q x\,dx$$

can be integrated when $p+q$ is a positive even integer, either in terms of $\tan x$ or of $\cot x$.

This has been done already in Art. 116, for it may be written

$$\int \cos^{-p} x \sin^{-q} x\,dx,$$

where $-p-q$ is a negative even integer.

122. Odd Powers.

But for odd positive powers of a secant or a cosecant, we have to adopt another method, because the Binomial Series used would be non-terminating.

We now proceed as follows:

By differentiation,

$$(n+1)\sec^{n+2}x \quad -n\sec^n x \quad = \quad \frac{d}{dx}(\tan x\sec^n x)$$

and $$(n+1)\operatorname{cosec}^{n+2}x - n\operatorname{cosec}^n x = -\frac{d}{dx}(\cot x\operatorname{cosec}^n x);$$

whence

$$\left.\begin{aligned}(n+1)\int\sec^{n+2}x\,dx &= \tan x\sec^n x + n\int\sec^n x\,dx\\ \text{and}\quad (n+1)\int\operatorname{cosec}^{n+2}dx &= -\cot x\operatorname{cosec}^n x + n\int\operatorname{cosec}^n x\,dx.\end{aligned}\right\}\quad\text{(A)}$$

Hence, changing n to $n-2$,

$$\int\sec^n x\,dx = \frac{\tan x\sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\int\sec^{n-2}x\,dx,$$

$$\int\operatorname{cosec}^n x\,dx = -\frac{\cot x\operatorname{cosec}^{n-2}x}{n-1} + \frac{n-2}{n-1}\int\operatorname{cosec}^{n-2}x\,dx.$$

Now $$\int\sec x\,dx = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \operatorname{gd}^{-1}x,$$

$$\int\operatorname{cosec} x\,dx = \log\tan\frac{x}{2}.$$

Hence

$$\int\sec^3 x\,dx = \frac{\tan x\sec x}{2} + \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \quad \text{(see Art. 79)},$$

$$\int\sec^5 x\,dx = \frac{\tan x\sec^3 x}{4} + \frac{3}{4}\,\frac{\tan x\sec x}{2} + \frac{3}{4}\,\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right),$$

etc.,

and generally

$$\int\sec^n x\,dx = \frac{\tan x\sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\,\frac{\tan x\sec^{n-4}x}{n-3} + \frac{(n-2)(n-4)}{(n-1)(n-3)}\,\frac{\tan x\sec^{n-6}x}{n-5} + \dots + \frac{(n-2)(n-4)\dots 3.1}{(n-1)(n-3)\dots 4.2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

(n odd).

The same formula would equally apply if n be even, except that it would terminate differently, viz. the last term would be

$$\frac{(n-2)(n-4)\ldots 4.2}{(n-1)(n-3)\ldots 5.3}\tan x \qquad (n \text{ even}).$$

In the same way

$$\int \operatorname{cosec}^3 x\,dx = -\frac{\cot x \operatorname{cosec} x}{2} + \frac{1}{2}\log\tan\frac{x}{2},$$

$$\int \operatorname{cosec}^5 x\,dx = -\frac{\cot x \operatorname{cosec}^3 x}{4} - \frac{3}{4}\,\frac{\cot x \operatorname{cosec} x}{2} + \frac{3}{4}\,\frac{1}{2}\log\tan\frac{x}{2},$$

and generally,

$$\int \operatorname{cosec}^n x\,dx = -\frac{\cot x \operatorname{cosec}^{n-2} x}{n-1} - \frac{n-2}{n-1}\,\frac{\cot x \operatorname{cosec}^{n-4} x}{n-3}$$

$$-\frac{(n-2)(n-4)}{(n-1)(n-3)}\,\frac{\cot x \operatorname{cosec}^{n-6} x}{n-5} - \ldots$$

$$+\frac{(n-2)(n-4)\ldots 3.1}{(n-1)(n-3)\ldots 4.2}\log\tan\frac{x}{2} \qquad (n \text{ odd})$$

or

$$-\frac{(n-2)(n-4)\ldots 4.2}{(n-1)(n-3)\ldots 5.3}\cot x. \qquad (n \text{ even.})$$

But as explained above, if n be even we should not in general employ this method, but that of Art. 120.

123. Since positive or negative powers of secants and cosecants are negative or positive powers respectively of cosines and sines, it will appear that so long as p is an integer, whether positive or negative,

$$\int \sin^p x\,dx, \quad \int \cos^p x\,dx, \quad \int \sec^p x\,dx, \quad \int \operatorname{cosec}^p x\,dx$$

can be integrated. Also it appears that $\int \sin^p x \cos^q x\,dx$ can always be integrated *directly* if p and q are positive integers; also that, even if one of the two p or q be negative or fractional, the integration can still be *directly* effected if the other be a positive odd integer. And further, this integration can be *directly* effected if $p+q$ be a negative even integer, even though both p and q may be fractional.

For other cases of $\int \sin^p x \cos^q x\,dx$, where p, q are negative integers, a reduction formula is in general required (see Art. 228).

124. If the student has any difficulty in reproducing the formulae of connection marked (A), they may be obtained at once by integration by parts thus:

$$\int \sec^{n+2}x\,dx = \int \sec^n x \frac{d\tan x}{dx}\,dx$$

$$= \sec^n x \tan x - \int n \sec^n x \tan^2 x\,dx$$

$$= \sec^n x \tan x - n\int (\sec^{n+2}x - \sec^n x)\,dx,$$

i.e. $$(n+1)\int \sec^{n+2}x\,dx = \sec^n x \tan x + n\int \sec^n x\,dx.$$

And similarly for $\int \operatorname{cosec}^{n+2}x\,dx$,

$$(n+1)\int \operatorname{cosec}^{n+2}x\,dx = -\operatorname{cosec}^n x \cot x + n\int \operatorname{cosec}^n x\,dx.$$

125. **Integral Powers of tangents or cotangents.**

Any integral powers of tangents or cotangents may be readily integrated.

For $$\int \tan^n x\,dx = \int \tan^{n-2}x(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2}x\,d\tan x - \int \tan^{n-2}x\,dx$$

$$= \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx.$$

And since $\int \tan x\,dx = \log\sec x$

an $$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x,$$

we may integrate successively $\tan^3 x$, $\tan^4 x$, $\tan^5 x$, etc.

Thus we have

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \log\sec x,$$

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \tan x + x,$$

$$\int \tan^5 x\,dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log\sec x,$$

$$\int \tan^6 x\,dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x,$$

etc.,

and generally

$$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \frac{\tan^{n-3}x}{n-3} + \frac{\tan^{n-5}x}{n-5} - \dots + (-1)^{\frac{n+1}{2}}\frac{\tan^2 x}{2}$$

$$+(-1)^{\frac{n-1}{2}}\log\sec x \quad (n \text{ odd})$$

or

$$= \frac{\tan^{n-1}x}{n-1} - \frac{\tan^{n-3}x}{n-3} + \dots + (-1)^{\frac{n+2}{2}}\tan x + (-1)^{\frac{n}{2}} x$$

$(n$ even$)$.

126. Similarly for cotangents,

$$\int \cot^n x\,dx = \int \cot^{n-2} x(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx,$$

whilst $\int \cot x\,dx = \log\sin x,$

$$\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)\,dx = -\cot x - x.$$

Thus we have successively

$$\int \cot^3 x\,dx = -\frac{\cot^2 x}{2} - \log\sin x,$$

$$\int \cot^4 x\,dx = -\frac{\cot^3 x}{3} + \frac{\cot x}{1} + x,$$

$$\int \cot^5 x\,dx = -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log\sin x,$$

and generally

$$\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} + \frac{\cot^{n-3}x}{n-3} - \frac{\cot^{n-5}x}{n-5} + \dots$$

$$-(-1)^{\frac{n+1}{2}}\frac{\cot^2 x}{2} - (-1)^{\frac{n+1}{2}}\log\sin x \quad (n \text{ odd})$$

or

$$= -\frac{\cot^{n-1}x}{n-1} + \frac{\cot^{n-3}x}{n-3} - \frac{\cot^{n-5}x}{n-5} + \dots + (-1)^{\frac{n}{2}}\frac{\cot x}{1} + (-1)^{\frac{n}{2}} x$$

$(n$ even$)$.

Hence any odd or even positive or negative power of a tangent or cotangent can be integrated readily.

EXAMPLES.

1. Integrate $\sin^2 x$, $\sin^3 x$, $\sin^4 x$, $\sin^5 x$, $\sin^8 x$, $\sin^9 x$, $\sin^{2n} x$, $\sin^{2n+1} x$, doing those with odd indices in two ways.

2. Integrate $\sin^2 x\cos^2 x$, $\sin^3 x\cos^8 x$, $\sin^4 x\cos^4 x$, $\sin^8 x\cos^6 x$, $\sin^6 x\cos^8 x$, $\sin^6 x\cos^4 x$.

3. Integrate $\dfrac{\sin^2 x}{\cos^4 x}$, $\cos^2 x \operatorname{cosec}^4 x$, $\sec^2 x \operatorname{cosec}^2 x$, $\dfrac{1}{\sin^4 x \cos^4 x}$.

4. Evaluate $\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$, $\int_0^{\frac{\pi}{4}} \cos^5 x\, dx$, $\int_0^{\frac{\pi}{4}} \cos^6 x\, dx$.

5. Integrate $\sin ax \cos^2 bx$, $\sin 3x \cos^3 x$, $\sin nx \cos^2 x$.

6. Show that

$$\int \sin x \sin 2x \sin 3x\, dx = -\frac{1}{8}\cos 2x - \frac{1}{16}\cos 4x + \frac{1}{24}\cos 6x.$$

7. Show that

(i) $$\int \sin mx \cos nx\, dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}.$$

(ii) $$\int \sin mx \sin nx\, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}.$$

(iii) $$\int \cos mx \cos nx\, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$$

Deduce from (ii) and (iii) the values of

$$\int \sin^2 mx\, dx \quad \text{and} \quad \int \cos^2 mx\, dx,$$

and verify the results by independent integration.

8. Prove that $\int_0^{\pi} \sin mx \sin nx\, dx$ and $\int_0^{\pi} \cos mx \cos nx\, dx$ are both zero so long as m and n are integral and unequal. But if m and n are equal integers their values are each equal to $\frac{\pi}{2}$.

GENERAL EXAMPLES.

1. Prove that $\int u \dfrac{d^2 v}{dx^2} dx = u\dfrac{dv}{dx} - v\dfrac{du}{dx} + \int v \dfrac{d^2 u}{dx^2} dx.$

2. Perform the following integrations:

(i) $\int \cos^{-1} x\, dx.$

(ii) $\int \cos^{-1}\frac{1}{x}\, dx.$

(iii) $\int x^3 \tan^{-1} x\, dx.$

(iv) $\int x \sec^2 x\, dx.$

(v) $\int x \sec x \tan x\, dx.$

(vi) $\int (ax+b)\log(cx+d)\, dx.$

(vii) $\int \tan^{-1}\sqrt{1-x^2}\, dx.$ [ST. JOHN'S, 1886.]

(viii) $\int \dfrac{x^4}{1+x^2}\tan^{-1} x\, dx.$ [OX. II. P., 1889.]

(ix) $\int \sin^{-1}\sqrt{\frac{x}{a+x}}.$ (x) $\int x\sin^{-1}\sqrt{\frac{2a-x}{4a}}\,dx.$

(xi) $\int \cos^{-1}\sqrt{\frac{a}{a+x}}\,dx.$ (xii) $\int x^n \log x\,dx.$

3. Integrate

(i) $\int e^{a\sin^{-1}x}\,dx.$ (ii) $\int \frac{x\sin^{-1}x}{(1-x^2)^{\frac{1}{2}}}\,dx.$ (iii) $\int \frac{x^3\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}\,dx.$

4. Integrate

(i) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx.$ (ii) $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^{\frac{3}{2}}}\,dx.$ (iii) $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^2}\,dx.$

(iv) $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^{\frac{5}{2}}}\,dx.$ (v) $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)^{\frac{n}{2}+1}}\,dx$ ($n \equiv$ a positive integer)

5. Integrate (i) $\int xe^{bx}\cos ax\,dx.$

(ii) $\int x^2e^{ax}\sin bx\,dx.$ [a 1888.]

(iii) $\int xe^x\sin^2 x\,dx.$

6. Integrate

(i) $\int e^{ax}(\sin bx+\cos bx)\,dx.$ (v) $\int x^2 3^x \sin 4x\,dx.$

(ii) $\int e^{ax}(\sinh bx+\cosh bx)\,dx.$ (vi) $\int \cos\left(b\log\frac{x}{a}\right)dx.$

(iii) $\int e^{ax}\sinh bx\cosh ax\,dx.$ (vii) $\int \cosh\left(b\log\frac{x}{a}\right)dx.$

(iv) $\int e^{ax}\cosh ax\sin bx\,dx.$ (viii) $\int^{\pi} \theta\sin\theta\cosh(\cos\theta)\,d\theta.$

[a 1891.]

7. Integrate

(i) $\int \frac{xe^x}{(x+1)^2}\,dx.$ (ii) $\int e^x\frac{1+\sin x}{1+\cos x}\,dx.$ (iii) $\int e^x\frac{1-\sin x}{1-\cos x}\,dx.$

(iv) $\int \frac{\cosh x+\sinh x\sin x}{1+\cos x}\,dx.$ (v) $\int \frac{dx}{1+e^x}.$ [Mech. Sc. Trip.]

(vi) $\int \sqrt{1+e^{nx}}\,dx.$ (vii) $\int e^x\frac{1+x^2}{(1+x)^2}\,dx.$

[Ox. I. P., 1890.]

8. Integrate

(i) $\int (\log x)^2\, dx.$ [Ox. I. P., 1888.]

(ii) $\int \left(x + \frac{1}{x} + \frac{1}{x^2}\right) \log x\, dx.$ [Ox. I. P., 1889.]

(iii) $\int x^{-2} \tan^{-1} x\, dx.$ [Ox. II. P., 1887.]

(iv) $\int \log (x + \sqrt{a^2 + x^2})\, dx.$ [MATH. TRIP., 1882.]

(v) $\int x \log (x + \sqrt{a^2 + x^2})\, dx.$ [ST. JOHN'S, 1884.]

(vi) $\int (a + x) \sqrt{a^2 + x^2}\, dx.$ [ST. JOHN'S, 1888.]

(vii) $\int (a^2 + x^2) \sqrt{a + x}\, dx.$ [ST. JOHN'S, 1888.]

(viii) $\int e^{ax} x^2 \sin (bx + c)\, dx.$ [COLL., 1892.]

(ix) $\int x^3 (1 - x^{\frac{2}{3}})^{\frac{3}{2}}\, dx.$ [Ox. I. P., 1890.]

9. Integrate

(i) $\int x\, e^{ax} \sin bx \sin cx\, dx.$

(ii) $\int x\, e^{ax} \sin bx \sin^2 cx\, dx.$

10. Show that if u be a rational integral function of x,

$$\int e^{x/a} u\, dx = a e^{x/a} \left\{ u - a\frac{du}{dx} + a^2 \frac{d^2u}{dx^2} - a^3 \frac{d^3u}{dx^3} + \dots \right\},$$

where the series within the brackets is necessarily finite.

[TRIN. COLL., 1881.]

11. If $u \equiv \int e^{ax} \cos bx\, dx$, $v \equiv \int e^{ax} \sin bx\, dx$, prove that

$$\tan^{-1} \frac{v}{u} + \tan^{-1} \frac{b}{a} = bx,$$

and that $(a^2 + b^2)(u^2 + v^2) = e^{2ax}$.

12. Evaluate $\int x^2 \log (1 - x^2)\, dx$, and deduce that

$$\frac{1}{1 \cdot 5} + \frac{1}{2 \cdot 7} + \frac{1}{3 \cdot 9} + \dots = \frac{8}{9} - \frac{2}{3} \log_e 2.$$

[α, 1889.]

13. Integrate $\int \sec^{\frac{3}{5}}\theta \operatorname{cosec}^{\frac{7}{5}}\theta\, d\theta$, $\int \sec^{\frac{3}{5}}\theta \sin\theta\, d\theta$.

14. Find the value of

$$\int \left\{ u\frac{d^n v}{dx^n} - (-1)^n v \frac{d^n u}{dx^n} \right\} dx.$$

[γ, 1890.]

15. Evaluate

$$\int \left[\frac{d^3u}{dx^3}\left(\frac{dv}{dx} - \frac{dw}{dx}\right) + \frac{d^3v}{dx^3}\left(\frac{dw}{dx} - \frac{du}{dx}\right) + \frac{d^3w}{dx^3}\left(\frac{du}{dx} - \frac{dv}{dx}\right) \right] dx.$$

[γ, 1890.]

16. Establish the following formulae for integration by parts, u and v being functions of x, and accents denoting differentiations and suffixes integrations with respect to x, and $u^{(n)}$ denoting u with n accents:

(i) $$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots + (-1)^{n-1}u^{(n-1)}v_n + (-1)^n \int u^{(n)}\, dv_{n+1}.$$

(ii) $$\iint (uv)(dx)^2 = uv_2 - 2u'v_3 + 3u''v_4 - 4u'''v_5 + \dots + (-1)^{n-1}nu^{(n-1)}v_{n+1} + (-1)^n n \int u^{(n)}v_{n+1}\, dx + (-1)^n \int dx \int u^{(n)}v_n\, dx.$$

[a, 1888.]

17. If u be a function of x, and differentiations and integrations are respectively denoted by accents and suffixes, and (n) means n accents, show that

$$\log u = 1 - \left(\frac{1}{u}\right)' u_1 + \left(\frac{1}{u}\right)'' u_2 - \left(\frac{1}{u}\right)''' u_3 + \dots + (-1)^n \int \left(\frac{1}{u}\right)^{(n)} du_n$$

[ST. JOHN'S, 1889.]

18. If u, v, w be functions of x, and accents and suffixes denote differentiations and integrations respectively, show that

$$\begin{aligned} 2uvw = {} & (vw)'u_1 - (vw)''u_2 + (vw)'''u_3 - \dots + (-1)^{m-1}\int (vw)^{(m)} du_m \\ & + (wu)'v_1 - (wu)''v_2 + (wu)'''v_3 - \dots + (-1)^{n-1}\int (wu)^{(n)} dv_n \\ & + (uv)'w_1 - (uv)''w_2 + (uv)'''w_3 - \dots + (-1)^{p-1}\int (uv)^{(p)} dw_p. \end{aligned}$$

[ST. JOHN'S, 1889.]

19. Prove that

$$\int_0^1 v^{vx}\,dv = 1 - \frac{x}{2^2} + \frac{x^2}{3^3} - \frac{x^3}{4^4} + \frac{x^4}{5^5} - \dots \text{ etc.}$$

[MATH. TRIP., 1878.]

20. Find the value of $\int_0^1 x^x\,dx$ correct to five decimal places.

[J. M. SCH. OX., 1904.]

21. Prove that

$$e^{a^2x^2}\int_0^x e^{-a^2x^2}\,dx = x + \tfrac{2}{3}a^2x^3 + \frac{2^2a^4}{3\,.\,5}x^5 + \dots$$

$$+ \frac{(2a^2)^n}{3\,.\,5\,.\,\dots(2n-1)}e^{a^2x^2}\int_0^x x^{2n}e^{-a^2x^2}\,dx.$$

[OX. I. PUB., 1899.]

22. Find the sum of the series, supposed convergent,

$$\frac{x^5}{1\,.\,3\,.\,5} - \frac{x^7}{3\,.\,5\,.\,7} + \frac{x^9}{5\,.\,7\,.\,9} - \text{etc. to } \infty.$$

[COLL., 1881.]

23. If y and z be functions of x, and $u = yz' - zy'$, prove the following:

(i) $\int zu^{-2}(y'z'' - z'y'')\,dx = -y^{-1}(1 + y'zu^{-1})$,

(ii) the integration of $zy^{-1}u^{-2}(yz'' - zy'')$ can be reduced to that of y^{-2}. [ST. JOHN'S, 1886.]

24. Show how the method of integration by parts may be applied to find

$$\int f(x)\frac{d^{n+1}V}{dx^{n+1}}\,dx,$$

where $f(x)$ is a rational algebraical expression of the n^{th} degree.

Prove that $$\int_{-1}^1 f(x)\frac{d^{n+1}(x^2-1)^{n+1}}{dx^{n+1}}\,dx = 0.$$ [COLL., 1876.]

25. Prove that $\int(\cos x)^n\,dx$ may be expressed by the series,

$$\sin x - N_1\frac{\sin^3 x}{3} + N_2\frac{\sin^5 x}{5} - N_3\frac{\sin^7 x}{7} + \dots,\ \text{etc.},$$

N_1, N_2, N_3, ... being the coefficients of the expansion $(1+a)^{\frac{n-2}{2}}$, and n having any real value positive or negative. [SMITH'S PRIZE, 1876.]

26. Prove that

$$\int x^n e^x \sin x\,dx = e^x\sum_{r=0}^{r=n}(-1)^r\frac{n!}{(n-r)!}x^{n-r}\,2^{-\frac{r+1}{2}}\sin\left\{x - \frac{(r+1)\pi}{4}\right\}.$$

27. Express the infinite series

$$\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3}+\dots$$

as a definite integral, and find its value. [Ox. II. P., 1902.]

28. Show that

$$2^m\int\cos mx\cos^m x\,dx=A+x+m\frac{\sin 2x}{2}+\frac{m(m-1)}{1.2}\frac{\sin 4x}{4}+\dots$$
$$+\frac{\sin 2mx}{2m},$$

where m is an integer and A is independent of x. [Coll. a, 1885.]

29. Evaluate the integral

$$\int_0^T a_1\sin\frac{2\pi t}{T}\cdot a_2\sin\frac{2\pi}{T}(t+\lambda)\,dt,$$

and draw curves showing how its value depends on that of λ.

[Mech. Sc. Trip., 1899.]

30. Prove that if $y=f(x)$ and $x=\phi(y)$ are equivalent relations, then, between any corresponding limits,

$$\int\sqrt{f'(x)}\,dx=\int\sqrt{\phi'(y)}\,dy.$$

Hence, or otherwise, prove that if $\tan\beta=\sqrt{1-c}\tan\alpha$,

$$\int_0^\alpha\frac{dx}{\sqrt{1-c\sin^2 x}}=\int_0^\beta\frac{dx}{\sqrt{1-c\cos^2 x}}.$$ [Ox. II. P., 1886.]

31. Prove that the remainder R in the series

$$\theta=\tan\theta-\tfrac{1}{3}\tan^3\theta+\dots+\frac{(-1)^n}{2n+1}\tan^{2n+1}\theta+R$$

may be written as a definite integral,

$$(-1)^{n+1}\int_0^\theta\tan^{2n+2}\theta\,d\theta.$$ [Coll., 1881.]

32. Show that the integrals $\int_0^x f(z)\,dz$, $\int_0^x z^n f^{(n)}(z)\,dz$ are connected thus:

$$\int_0^x f(z)\,dz=xf(x)-\frac{x^2}{2!}f'(x)+\frac{x^3}{3!}f''(x)-\dots$$
$$+(-1)^{n-1}\frac{x^n}{n!}f^{(n-1)}(x)+(-1)^n\int_0^x z^n f^{(n)}(z)\,dz,$$

and that if one can be integrated the other can also be integrated.

[Bernoulli.]

33. Integrate

$$\int \{(2n+1)\cos(2n+\tfrac{1}{2})\theta + (2n-2)\cos(2n-\tfrac{3}{2})\theta\}(\cos\theta)^{\frac{1}{2}}\,d\theta,$$

and prove that when n is a positive integer,

$$\int_0^{\frac{\pi}{2}} \cos(2n+\tfrac{1}{2})\theta\,(\cos\theta)^{\frac{1}{2}}\,d\theta = 0.$$

[OXFORD II. PUB., 1913.]

34. Find the sum of the areas included between the axis of x and the arc of the curve $y = x\sin(x/a)$ from the ordinate $x=0$ to the ordinate $x = n\pi a$, n being any positive integer, odd or even.

[OXF. I. P., 1911.]

35. Evaluate $\displaystyle\int_0^{2a} \frac{x^n}{\sqrt{2ax-x^2}}\,dx$ when n is any positive integer.

[OXF. I. P., 1916.]

36. Show that $\displaystyle\int_0^1 x\log(1+\tfrac{1}{2}x)\,dx = \tfrac{3}{4}(1-2\log\tfrac{3}{2})$, and prove that this is less than $\displaystyle\int_0^1 \tfrac{1}{2}x^2\,dx$.

[MATH. TRIP., PART I., 1913.]

37. If $T_n = \displaystyle\int_0^x \tan^n x\,dx$, show that $(n-1)(T_n + T_{n-2}) = \tan^{n-1}x$.

Given that $\pi = 3{\cdot}141592\ldots$, $\log_e 2 = 0{\cdot}693147\ldots$, show that

$$\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = 0{\cdot}09657\ldots, \qquad \int_0^{\frac{\pi}{4}} \tan^4 x\,dx = 0{\cdot}11873\ldots.$$

[MATH. TRIP. I., 1915.]

38. Prove that $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}\,dx = \frac{\pi}{4} - \frac{1}{2}\log_e 2.$

[MATH TRIP. I., 1917.]

39. Find the area A between the curve

$$y = a(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x)$$

and the axis of x between the limits 0 and π; and the volume V obtained by rotating this area about the axis of x.

Prove that $4V = \pi^2 aA$.

[MATH. TRIP. I., 1913.]

40. Show that

$$\int_0^1 x^{2p-1}\log(1+x)\,dx = \frac{1}{2p}\left\{\frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{(2p-1)\,2p}\right\}.$$

[MATH. TRIP., PT. I., 1916.]

CHAPTER V.

RATIONAL ALGEBRAIC FRACTIONAL FORMS.

127. Integration of

$$\frac{1}{a^2-x^2} \quad (x<a) \quad \text{and} \quad \frac{1}{x^2-a^2} \quad (x>a).$$

Either of these forms should be thrown into Partial Fractions, which can be done by inspection.

$$\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\int\left(\frac{1}{a+x}+\frac{1}{a-x}\right)dx$$

$$=\frac{1}{2a}[\log(a+x)-\log(a-x)]=\frac{1}{2a}\log\frac{a+x}{a-x}$$

or $$=\frac{1}{a}\tanh^{-1}\frac{x}{a} \quad (x<a).$$

$$\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx$$

$$=\frac{1}{2a}[\log(x-a)-\log(x+a)]=\frac{1}{2a}\log\frac{x-a}{x+a}$$

or $$=-\frac{1}{a}\coth^{-1}\frac{x}{a} \quad \text{or} \quad \frac{-1}{a}\tanh^{-1}\frac{a}{x} \quad (x>a).$$

The Partial Fractions are so simple that the results are not usually committed to memory.

128. These inverse hyperbolic forms should be compared with

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}=\frac{1}{a}\cos^{-1}\frac{a}{\sqrt{a^2+x^2}}=\frac{1}{a}\sec^{-1}\frac{\sqrt{a^2+x^2}}{a}.$$

The three results are:

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}\quad\text{or}\quad-\frac{1}{a}\cot^{-1}\frac{x}{a}.$$

$$\int\frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\frac{x}{a}\quad(x<a),$$

$$\int\frac{dx}{x^2-a^2}=-\frac{1}{a}\coth^{-1}\frac{x}{a}\quad(x>a),$$

or

$$-\frac{1}{a}\tanh^{-1}\frac{a}{x}.$$

129. **Extension of above rule.**

In the same way, α and β being real,

$$\int\frac{dx}{\beta^2+(x+\alpha)^2}=\frac{1}{\beta}\tan^{-1}\frac{x+\alpha}{\beta},$$

$$\int\frac{dx}{\beta^2-(x+\alpha)^2}=\frac{1}{\beta}\tanh^{-1}\frac{x+\alpha}{\beta},\quad i.e.\ \frac{1}{2\beta}\log\frac{\beta+(x+\alpha)}{\beta-(x+\alpha)}\quad(x+\alpha<\beta),$$

$$\int\frac{dx}{(x+\alpha)^2-\beta^2}=-\frac{1}{\beta}\coth^{-1}\frac{x+\alpha}{\beta}$$

or

$$-\frac{1}{\beta}\tanh^{-1}\frac{\beta}{x+\alpha},\quad i.e.\ \frac{1}{2\beta}\log\frac{x+\alpha-\beta}{x+\alpha+\beta}\quad(x+\alpha>\beta).$$

130. **Integration of**

$$I\equiv\int\frac{dx}{ax^2+bx+c}.$$

Since ax^2+bx+c can always be written as

$$a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right],\quad i.e.\ \text{of form } a\left[(x+\alpha)^2+\beta^2\right],$$

or as

$$a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right],\quad i.e.\ \text{of form } a\left[(x+\alpha)^2-\beta^2\right];$$

taking the first or the second according as $b^2<4ac$ or $b^2>4ac$ the rules of the former article apply.

Thus

131. Case I. $b^2 < 4ac.$

$$I \equiv \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}$$

$$= \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}}$$

or $$= -\frac{2}{\sqrt{4ac-b^2}} \cot^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}}$$

or $$= \frac{2}{\sqrt{4ac-b^2}} \sec^{-1} \frac{\sqrt{4a}\sqrt{ax^2+bx+c}}{\sqrt{4ac-b^2}}, \text{ etc.}$$

132. Case II. $b^2 > 4ac.$

$$I \equiv \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}}$$

$$= \frac{1}{\sqrt{b^2-4ac}} \log \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}$$

or $$= -\frac{2}{\sqrt{b^2-4ac}} \coth^{-1} \frac{2ax+b}{\sqrt{b^2-4ac}}$$

or $$= -\frac{2}{\sqrt{b^2-4ac}} \operatorname{cosech}^{-1} \frac{\sqrt{4a}\sqrt{ax^2+bx+c}}{\sqrt{b^2-4ac}},$$

which is a real form if $2ax+b > \sqrt{b^2-4ac}$,

or $$I \equiv -\frac{1}{a}\int \frac{dx}{\frac{b^2-4ac}{4a^2} - \left(x+\frac{b}{2a}\right)^2}$$

$$= -\frac{1}{\sqrt{b^2-4ac}} \log \frac{\sqrt{b^2-4ac}+(2ax+b)}{\sqrt{b^2-4ac}-(2ax+b)},$$

$$\text{i.e.} \quad -\frac{2}{\sqrt{b^2-4ac}} \tanh^{-1} \frac{2ax+b}{\sqrt{b^2-4ac}} = \text{etc.},$$

which is a real form if $2ax+b < \sqrt{b^2-4ac}.$

133. Of these several forms the real one is to be chosen in each numerical case. The general forms are equivalent, except that they differ by a constant which may be unreal.

134. **Another Method.**

As the factors in the second case are real, say

$$a(x-x_1)(x-x_2),$$

the usual proceeding is to write the work as follows without the formal completing of the square in the denominator:

$$\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dx}{(x-x_1)(x-x_2)}$$

$$=\frac{1}{a(x_1-x_2)}\int\frac{dx}{x-x_1}+\frac{1}{a(x_2-x_1)}\int\frac{dx}{x-x_2}$$

$$=\frac{1}{a(x_1-x_2)}\log(x-x_1)+\frac{1}{a(x_2-x_1)}\log(x-x_2)$$

$$=\frac{1}{a(x_1-x_2)}\log\frac{x-x_1}{x-x_2}.$$

135. **Other forms of the above results.**

Other forms of these results may be exhibited. For instance, taking $R\equiv ax^2+bx+c$, and $4ac-b^2=4a^2\kappa^2=-4a^2\kappa'^2$; then

$$2\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}=\sin^{-1}\left(\kappa\frac{2ax+b}{ax^2+bx+c}\right)\qquad=\sin^{-1}\left(\kappa\frac{d}{dx}\log R\right)$$

and

$$2\tanh^{-1}\frac{2ax+b}{\sqrt{b^2-4ac}}=\sinh^{-1}\left(-\kappa'\frac{2ax+b}{ax^2+bx+c}\right)=-\sinh^{-1}\left(\kappa'\frac{d}{dx}\log R\right)$$

whence $I=\dfrac{1}{2a\kappa}\sin^{-1}\left(\kappa\dfrac{d}{dx}\log R\right)$ or $\dfrac{1}{2a\kappa'}\sinh^{-1}\left(\kappa'\dfrac{d}{dx}\log R\right)$

the real form to be chosen.

136. **Integrals of expressions of the form**

$$\frac{px+q}{ax^2+bx+c},\quad i.e.\ \frac{px+q}{R},$$

can be obtained at once by throwing $px+q$ into the form

$$px+q\equiv\lambda R'+\mu,\quad i.e.\equiv\lambda(2ax+b)+\mu,$$

where λ, μ are constants to be found;

for then

$$I=\int\frac{px+q}{ax^2+bx+c}\,dx=\int\frac{\lambda R'+\mu}{R}\,dx=\lambda\int\frac{R'}{R}\,dx+\mu\int\frac{dx}{R}$$
$$=\lambda\log R+\mu\int\frac{dx}{R},$$

and the second member of the right side has been discussed.

137. This transformation is one very frequently required. It may be performed either by inspection, or by comparing coefficients.

(i) By inspection,

$$px+q\equiv\frac{p}{2a}(2ax+b)+\left(q-\frac{pb}{2a}\right).$$

(ii) By comparing coefficients,

$$\left.\begin{aligned}2a\lambda&=p,\\ b\lambda+\mu&=q,\end{aligned}\right\}\text{ giving }\lambda=\frac{p}{2a}\text{ and }\mu=q-\frac{pb}{2a}.$$

Thus

$$\int\frac{px+q}{ax^2+bx+c}\,dx=\frac{p}{2a}\int\frac{2ax+b}{ax^2+bx+c}\,dx+\left(q-\frac{pb}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$
$$=\frac{p}{2a}\log(ax^2+bx+c)+\left(q-\frac{pb}{2a}\right)\int\frac{dx}{ax^2+bx+c}.$$

It is *essential* that the numerator of the first partial fraction shall be the *differential coefficient of the denominator*, and that the x's *of the numerator of the given fraction are thereby exhausted.*

138. Ex. 1. $\displaystyle\int\frac{9-7x}{x^2+12x+38}\,dx=\int\frac{51-\frac{7}{2}(2x+12)}{(x+6)^2+2}\,dx$

$$=51\int\frac{dx}{2+(x+6)^2}-\frac{7}{2}\int\frac{2x+12}{x^2+12x+38}\,dx$$
$$=\frac{51}{\sqrt{2}}\tan^{-1}\frac{x+6}{\sqrt{2}}-\frac{7}{2}\log(x^2+12x+38).$$

Ex. 2. $\displaystyle\int\frac{9-7x}{35+2x-x^2}\,dx=\int\frac{9-7x}{(7-x)(5+x)}\,dx$

$$=\int\left(-\frac{10}{3}\frac{1}{7-x}+\frac{11}{3}\frac{1}{5+x}\right)dx$$
$$=\frac{10}{3}\log(7-x)+\frac{11}{3}\log(5+x).$$

This difference is to be noted in such examples as the two preceding; in the first the form of the result is real for all real values of x; in the second the form given is only real if x lies between -5 and $+7$. For values of $x > 7$ we should write it

$$\frac{10}{3}\log(x-7)+\frac{11}{3}\log(x+5),$$

and for values of $x < -5$,

$$\frac{10}{3}\log(7-x)+\frac{11}{3}\log(-5-x).$$

These three forms differ by unreal constants.

EXAMPLES.

1. $\displaystyle\int\frac{x\,dx}{x^2+2x+3}$.

2. $\displaystyle\int\frac{x\,dx}{x^2+2x+1}$.

3. $\displaystyle\int\frac{x+1}{x^2+4x+5}\,dx$.

4. $\displaystyle\int\frac{(x+1)\,dx}{3+2x-x^2}$.

5. $\displaystyle\int\frac{(x-1)^2\,dx}{x^2+2x+2}$.

6. $\displaystyle\int\frac{2x^2+3x+4}{x^2+6x+10}\,dx$.

7. $\displaystyle\int\frac{dx}{(ax+b)^2+(cx+d)^2}$.

8. $\displaystyle\int\frac{dx}{(ax+b)^2-(cx+d)^2}$.

9. $\displaystyle\int\frac{x\,dx}{(ax^2+b)^2+(cx^2+d)^2}$.

10. $\displaystyle\int\frac{x\,dx}{(ax^2+b)^2+(cx^2+d)^2+(ex^2+f)^2}$.

11. $\displaystyle\int\frac{dx}{x\left(ax+\frac{b}{x}\right)\left(cx+\frac{d}{x}\right)}$.

12. $\displaystyle\int\frac{e^{2x}\,dx}{e^{2x}+2e^x+3}$.

NOTE ON PARTIAL FRACTIONS.

139. In the author's *Differential Calculus* (p. 72) a Note was inserted on the methods to be pursued in the case of finding the n^{th} Differential Coefficient of an algebraical fraction when it was necessary to resolve the fraction into its simple or partial fractions. It is now necessary to repeat this Note, with some additions and alterations, as success in the integration of complicated rational algebraic fractions will depend upon the ability of the student to obtain the equivalent partial fractions with facility. Moreover, many subsequent articles will depend upon the general theory.

140. Let $\dfrac{f(x)}{\phi(x)}$ be the fraction in its lowest terms which is to be resolved into its simple component or partial fractions, $f(x)$ and $\phi(x)$ being supposed rational integral algebraic

functions of x, the coefficients being real and, unless the contrary be stated, rational.

Then if the degree of $f(x)$ be not already less than the degree of $\phi(x)$, we can, by ordinary division, express $\dfrac{f(x)}{\phi(x)}$ in the form

$$a_0x^n+a_1x^{n-1}+\dots+a_n+\frac{\chi(x)}{\phi(x)},$$

where $a_0x^n+a_1x^{n-1}+\dots+a_n$ is the quotient, and $\chi(x)$ is the remainder, of lower degree than $\phi(x)$.

Hence the integration of

$$\int\frac{f(x)}{\phi(x)}dx \quad\text{is}\quad \frac{a_0x^{n+1}}{n+1}+a_1\frac{x^n}{n}+\dots+a_nx+\int\frac{\chi(x)}{\phi(x)}dx,$$

and we only have to attend to $\displaystyle\int\frac{\chi(x)}{\phi(x)}dx$.

Hence we may confine our attention to the case when $f(x)$ is of lower degree than $\phi(x)$.

Also we may, without loss of generality, consider the coefficient of the highest power of x in $\phi(x)$ to be unity.

141. It is proved in Theory of Equations that if $\phi(x)=0$ be a rational algebraical equation of degree n,

(1) there are n roots, real or imaginary,

(2) that imaginary roots occur in pairs, $\alpha\pm\iota\beta$, $\gamma\pm\iota\delta$, etc.

Any of these roots may be repeated.

Then the general form of ϕ is of the nature

$$\phi\equiv(x-a)(x-b)^p\{(x-\alpha)^2+\beta^2\}\{(x-\gamma)^2+\delta^2\}^q,$$

where we have taken the case of

(1) a real linear factor *occurring once only;*

(2) a real linear factor *occurring p times;*

(3) a *pair of unreal factors, each occurring once;*

(4) a *pair of unreal factors, each occurring q times.*

Any other factors which there may be in ϕ must be of one or other of these categories.

We consider these four cases separately.

And as we are going to suppose that $\dfrac{f(x)}{\phi(x)}$ is a fraction in its lowest terms, none of the factors described above will be factors of $f(x)$ also.

142. I. To obtain the partial fraction corresponding to the factor $x-a$ occurring once only.

Let $\phi(x)\equiv(x-a)\psi(x)$ for short. Then $\psi(x)$ does not contain $x-a$ as a factor, and $\psi(a)$ does not vanish.

Let $\dfrac{f(x)}{(x-a)\psi(x)}=\dfrac{A}{x-a}+\dfrac{\chi(x)}{\psi(x)}$, an assumption justifiable if we succeed in finding A, supposed independent of x.

Then $\dfrac{f(x)}{\psi(x)}=A+\dfrac{\chi(x)}{\psi(x)}(x-a)$ is an identity and true for all values of x.

Hence putting $x=a$, $\dfrac{f(a)}{\psi(a)}=A$.

Therefore $$\frac{f(x)}{(x-a)\psi(x)}=\frac{f(a)}{(x-a)\psi(a)}+\frac{\chi(x)}{\psi(x)}.$$

Hence our rule to find A is,

"*Write a for x in every portion of the fraction* $\dfrac{f(x)}{(x-a)\psi(x)}$ *except in the factor* $(x-a)$ *itself.*"

And this process may be applied to every partial fraction corresponding to a factor of $\phi(x)$, *which only occurs once.*

Moreover, since

$$\phi(x)=(x-a)\psi(x),\quad \phi'(x)=(x-a)\psi'(x)+\psi(x),$$

and $\psi'(a)$ is finite, $\therefore\ \phi'(a)=\psi(a)$.

Hence we may also write A in the form $\dfrac{f(a)}{\phi'(a)}$.

143. Ex. 1. $\dfrac{x}{(x-1)(x-2)(x-3)}$

$$=\frac{1}{(x-1)(1-2)(1-3)}+\frac{2}{(2-1)(x-2)(2-3)}+\frac{3}{(3-1)(3-2)(x-3)}$$

$$=\frac{1}{2(x-1)}-\frac{2}{x-2}+\frac{3}{2(x-3)}.$$

Thus, here, three partial fractions must occur. No others can occur. For if there were a fourth fraction $\dfrac{D}{x-\delta}$, say, the denominator of their sum must be $(x-1)(x-2)(x-3)(x-\delta)$, which is not so.

Hence we have obtained the whole expression.

Ex. 2. $\dfrac{x^3}{(x-a)(x-b)}$. Here the numerator not being of lower degree than the denominator, we must divide by the denominator. The result will then be expressible as

$$\frac{x^3}{(x-a)(x-b)}=x+(a+b)+\frac{A}{x-a}+\frac{B}{x-b},$$

where A and B are to be found.

Since $\dfrac{x^3}{x-b}\equiv(x-a)[x+a+b]+A+\dfrac{B}{x-b}(x-a)$, putting $x=a$ we get $A=\dfrac{a^3}{a-b}$, and similarly $B=\dfrac{b^3}{b-a}$.

We may here stop to remark that A and B can be written down by the rule "Put $x=a$ everywhere except in $x-a$ itself" just as well *in the original expression* $\dfrac{x^3}{(x-a)(x-b)}$ as in $\dfrac{x^3}{(x-a)(x-b)}-(x+a+b)$.

This remark is general, and will usually save much trouble.

Thus $$\frac{x^3}{(x-a)(x-b)}\equiv x+(a+b)+\frac{a^3}{a-b}\,\frac{1}{x-a}+\frac{b^3}{b-a}\cdot\frac{1}{x-b}.$$

Ex. 3. Let the roots of $x^n=1$ be $\alpha, \beta, \gamma, \ldots$ and $F(x)$ a rational integral algebraic expression of degree lower than n; then, by the second rule of Art. 142,

$$\frac{F(x)}{x^n-1}=\frac{F(\alpha)}{n\alpha^{n-1}}\,\frac{1}{x-\alpha}+\frac{F(\beta)}{n\beta^{n-1}}\,\frac{1}{x-\beta}+\ldots$$
$$=\frac{1}{n}\left(\frac{\alpha F(\alpha)}{x-\alpha}+\frac{\beta F(\beta)}{x-\beta}+\ldots\right)=\frac{1}{n}\sum\frac{\alpha F(\alpha)}{x-\alpha},$$

where the summation is for all the roots.

This may be also further expressed as

$$\frac{1}{2n}\sum\frac{(x+\alpha)-(x-\alpha)}{x-\alpha}F(\alpha),$$

or $$\frac{1}{2n}\sum F(\alpha)\frac{x+\alpha}{x-\alpha}-\frac{1}{2n}\sum F(\alpha).$$

If $F(x)$ be written as $Ax^m+Bx^{m-1}+\ldots+K\,(m<n)$, then, since the sum of the r^{th} powers of the n^{th} roots of unity is zero when $0<r<n$, we have

$$\sum F(\alpha)=nK=nF(0);$$

$$\therefore\ \frac{F(x)}{x^n-1}=\frac{1}{n}\sum\frac{\alpha F(\alpha)}{x-\alpha}=\frac{1}{2n}\sum\frac{x+\alpha}{x-\alpha}F(\alpha)-\frac{1}{2}F(0).$$

By taking $F(x)=x$ and putting $x=e^{2\iota\theta}$, deduce that

$$\frac{\sin(n-2)x}{\sin nx}=-\frac{1}{n}\sum_{r=1}^{r=(n-1)}\sin\frac{2r\pi}{n}\cot\left(x-\frac{r\pi}{n}\right).$$

[MATH. TRIP., PART II., 1919.]

144. II. Next suppose **the factor $(x-a)$ in the denominator to be repeated r times** and no more, so that we may write

$$\phi(x)=(x-a)^r\psi(x) \text{ where } \psi(a) \text{ does not vanish.}$$

Put $x-a=y$.

Then $\dfrac{f(x)}{\phi(x)} = \dfrac{1}{y^r} \cdot \dfrac{f(a+y)}{\psi(a+y)}$, or expanding each function by any means in *ascending powers of* y,

$$= \frac{1}{y^r} \frac{A_0 + A_1 y + A_2 y^2 + \dots}{B_0 + B_1 y + B_2 y^2 + \dots}.$$

Divide out thus:

$$B_0 + B_1 y + B_2 y^2 + \dots) \, A_0 + A_1 y + A_2 y^2 + \dots (\, C_0 + C_1 y + C_2 y^2 + \dots$$

etc.,

and let the division be *continued until* y^r *is a factor of the remainder.*

Let the remainder be $y^r \chi(y)$.

Hence

$$\frac{f(x)}{\phi(x)} = \frac{C_0}{y^r} + \frac{C_1}{y^{r-1}} + \frac{C_2}{y^{r-2}} + \dots + \frac{C_{r-1}}{y} + \frac{\chi(y)}{\psi(a+y)}$$

$$= \frac{C_0}{(x-a)^r} + \frac{C_1}{(x-a)^{r-1}} + \frac{C_2}{(x-a)^{r-2}} + \dots + \frac{C_{r-1}}{x-a} + \frac{\chi(x-a)}{\psi(x)}.$$

Hence the partial fractions corresponding to $(x-a)^r$ are determined by a "long division" sum.

145. Ex. (i). Take $\dfrac{x^2}{(x-1)^3(x+1)}$. Put $x-1=y$.

Then the fraction $= \dfrac{1}{y^3} \cdot \dfrac{(1+y)^2}{2+y}$.

$$\begin{array}{l} 2+y\,)\,1+2y+y^2\,(\,\tfrac{1}{2}+\tfrac{3}{4}y+\tfrac{1}{8}y^2-\tfrac{1}{8}\dfrac{y^3}{2+y} \\ \underline{1+\tfrac{1}{2}y} \\ \tfrac{3}{2}y+y^2 \\ \underline{\tfrac{3}{2}y+\tfrac{3}{4}y^2} \\ \phantom{2+y\,)\,1+\tfrac{3}{2}y+}\tfrac{1}{4}y^2 \\ \phantom{2+y\,)\,1+\tfrac{3}{2}y+}\underline{\tfrac{1}{4}y^2+\tfrac{1}{8}y^3} \\ \phantom{2+y\,)\,1+\tfrac{3}{2}y+\tfrac{1}{4}y^2}-\tfrac{1}{8}y^3 \end{array}$$

Therefore the fraction $= \dfrac{1}{2y^3} + \dfrac{3}{4y^2} + \dfrac{1}{8y} - \dfrac{1}{8(2+y)}$

$$= \frac{1}{2(x-1)^3} + \frac{3}{4(x-1)^2} + \frac{1}{8(x-1)} - \frac{1}{8(x+1)}.$$

146. **Remarks.**

(1) In practice it is desirable to perform the division by the "detached coefficients" method, and the above work appears as

$$\begin{array}{l} 2+1\,)\,1+2+1\,(\,\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{1}{8} \\ \underline{1+\tfrac{1}{2}} \\ \tfrac{3}{2}+1 \\ \underline{\tfrac{3}{2}+\tfrac{3}{4}} \\ \phantom{2+1\,)\,1+\tfrac{3}{2}+}\tfrac{1}{4} \\ \phantom{2+1\,)\,1+\tfrac{3}{2}+}\underline{\tfrac{1}{4}+\tfrac{1}{8}} \\ \phantom{2+1\,)\,1+\tfrac{3}{2}+\tfrac{1}{4}}-\tfrac{1}{8} \end{array}$$

(2) In cases where there is but one other linear or quadratic factor in the denominator $\phi(x)$ and that not a repeated one, this process will *finish the whole operation.*

Ex. (ii). $\dfrac{x^2+2x}{(x-1)^5(x^2+1)}$. Put $x=1+y$.

$$\text{The fraction}=\frac{1}{y^5}\,\frac{3+4y+y^2}{2+2y+y^2}.$$

$$\begin{array}{l}
2+2+1\,)\,3+4+1\,(\,\tfrac{3}{2}+\tfrac{1}{2}-\tfrac{3}{4}+\tfrac{1}{2}-\tfrac{1}{8}\\
\qquad\qquad\underline{3+3+\tfrac{3}{2}}\\
\qquad\qquad\quad 1-\tfrac{1}{2}\\
\qquad\qquad\quad \underline{1+1+\tfrac{1}{2}}\\
\qquad\qquad\qquad -\tfrac{3}{2}-\tfrac{1}{2}\\
\qquad\qquad\qquad \underline{-\tfrac{3}{2}-\tfrac{3}{2}-\tfrac{3}{4}}\\
\qquad\qquad\qquad\qquad 1+\tfrac{3}{4}\\
\qquad\qquad\qquad\qquad \underline{1+1+\tfrac{1}{2}}\\
\qquad\qquad\qquad\qquad\quad -\tfrac{1}{4}-\tfrac{1}{2}\\
\qquad\qquad\qquad\qquad\quad \underline{-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{8}}\\
\qquad\qquad\qquad\qquad\qquad -\tfrac{1}{4}+\tfrac{1}{8}
\end{array}$$

Hence $\quad$ the fraction $=\dfrac{3}{2y^5}+\dfrac{1}{2y^4}-\dfrac{3}{4y^3}+\dfrac{1}{2y^2}-\dfrac{1}{8y}-\dfrac{\frac{1}{4}-\frac{1}{8}y}{2+2y+y^2}$

$$=\frac{3}{2(x-1)^5}+\frac{1}{2(x-1)^4}-\frac{3}{4(x-1)^3}+\frac{1}{2(x-1)^2}-\frac{1}{8(x-1)}+\frac{1}{8}\,\frac{x-3}{1+x^2},$$

and is then ready for integration.

Ex. (iii). $\dfrac{x}{(x-1)^3(x-2)^2}$. In such a case we find the three partial fractions corresponding to $x-1$, and then, either *from the remainder* or *beginning over again*, the two corresponding to $(x-2)^2$.

147. Instead of expanding out $f(a+y)$ and $\psi(a+y)$ separately, as shown above (which is however usually best in practical cases), we may expand $\dfrac{f(a+y)}{\psi(a+y)}$ as though it were $F(a+y)$ by Taylor's theorem, or otherwise, which shows a compact theoretical form for the several coefficients, C_0, C_1, C_2, ... , of Art. 144.

Thus

$$\frac{f(a+y)}{\psi(a+y)}=\frac{f(a)}{\psi(a)}+y\frac{d}{da}\left(\frac{fa}{\psi a}\right)+\dots+\frac{y^r}{\underline{|r}}\frac{d^r}{da^r}\left(\frac{fa}{\psi a}\right)+\dots,$$

So that

$$C_0=\frac{f(a)}{\psi(a)},\quad C_1=\frac{d}{da}\left(\frac{fa}{\psi a}\right),\quad C_2=\frac{1}{\underline{|2}}\frac{d^2}{da^2}\left(\frac{fa}{\psi a}\right)\dots,$$

$$C_{r-1}=\frac{1}{\underline{|r-1}}\frac{d^{r-1}}{da^{r-1}}\left(\frac{fa}{\psi a}\right).$$

148. Nothing has been assumed so far as to the reality of the several roots, a, b, etc., of $\phi(x)=0$. Hence the rules obtained equally apply for unreal or for real roots.

If then
$$\phi(x)\equiv(x-a)^p(x-b)^q(x-c)^r\dots,$$
whether a, b, c be real or unreal, so that $p+q+r+\dots=n$, the degree of $\phi(x)$, we obtain, by methods explained above, a result of form

$$\begin{aligned}\frac{f(x)}{\phi(x)}=\frac{A_0}{(x-a)^p}+\frac{A_1}{(x-a)^{p-1}}+\frac{A_2}{(x-a)^{p-2}}+\dots+\frac{A_{p-1}}{x-a}\\ +\frac{B_0}{(x-b)^q}+\frac{B_1}{(x-b)^{q-1}}+\frac{B_2}{(x-b)^{q-2}}+\dots+\frac{B_{q-1}}{x-b}\\ +\frac{C_0}{(x-c)^r}+\frac{C_1}{(x-c)^{r-1}}+\frac{C_2}{(x-c)^{r-2}}+\dots+\frac{C_{r-1}}{x-c}\\ +\dots;\end{aligned}$$

and imagining these fractions to be reduced to a common denominator and added up to get back to the form $\frac{f(x)}{\phi(x)}$, the coefficient of x^{n-1} is obviously $A_{p-1}+B_{q-1}+C_{r-1}+\dots$.

The integral will be

$$\begin{aligned}\int\frac{f(x)}{\phi(x)}dx\\ =-\frac{A_0}{(p-1)(x-a)^{p-1}}-\frac{A_1}{(p-2)(x-a)^{p-2}}-\dots-\frac{A_{p-2}}{x-a}+A_{p-1}\log(x-a)\\ -\frac{B_0}{(q-1)(x-b)^{q-1}}-\frac{B_1}{(q-2)(x-b)^{q-2}}-\dots-\frac{B_{q-2}}{x-b}+B_{q-1}\log(x-b)\\ -\frac{C_0}{(r-1)(x-c)^{r-1}}-\frac{C_1}{(r-2)(x-c)^{r-2}}-\dots-\frac{C_{r-2}}{x-c}+C_{r-1}\log(x-c),\\ \text{etc.,}\end{aligned}$$

i.e. in general partly algebraic and partly logarithmic.

149. The conditions necessary that the integral should be purely algebraic are clearly
$$A_{p-1}=B_{q-1}=C_{r-1}=\dots=0,$$
and in number the same as the number of *different* roots of $\phi(x)=0$. But the coefficient of x^{n-1} in $f(x)/\phi(x)$ has been seen to be
$$A_{p-1}+B_{q-1}+C_{r-1}+\dots,$$
and this must vanish when the above conditions are satisfied.

Hence the index of the highest power of x in the numerator must be at least 2 less than that of the highest power of x in the denominator.

If then the number of different roots of $\phi(x)=0$, viz. $a, b, c, \ldots$, be k, say; and if the degree of $f(x)$ be lower by 2 than the degree of $\phi(x)$, we must *necessarily* have

$$A_{p-1}+B_{q-1}+C_{r-1}+\ldots=0,$$

and one of the k conditions, $A_{p-1}=B_{q-1}=\ldots=0$, must be included in the others, and there are then only $k-1$ independent conditions to be satisfied for $\int\frac{f(x)}{\phi(x)}dx=0$ to be entirely algebraic

150. III. Consider next the **case of an irreducible quadratic factor,**

$$(x-a)^2+\beta^2,$$

not repeated, occurring in the denominator, $\phi(x)$, and let

$$\phi(x)\equiv[(x-a)^2+\beta^2]\psi(x).$$

Then the partial fractions of $\frac{f(x)}{\phi(x)}$, *i.e.* of

$$\frac{f(x)}{(x-a-\iota\beta)(x-a+\iota\beta)\psi(x)},$$

corresponding to these unreal factors, are

$$\frac{f(a+\iota\beta)}{(2\iota\beta)\psi(a+\iota\beta)}\,\frac{1}{x-a-\iota\beta}+\frac{f(a-\iota\beta)}{(-2\iota\beta)\psi(a-\iota\beta)}\,\frac{1}{x-a+\iota\beta},$$

or, separating out the real and unreal parts of $\frac{f(a+\iota\beta)}{(2\iota\beta)\psi(a+\iota\beta)}$ as $P+\iota Q$, these partial fractions are

$$\frac{P+\iota Q}{x-a-\iota\beta}+\frac{P-\iota Q}{x-a+\iota\beta},\quad\text{or}\quad\frac{2P(x-a)-2Q\beta}{(x-a)^2+\beta^2},$$

which is of form $$\frac{Lx+M}{(x-a)^2+\beta^2},$$

$$\left.\begin{aligned}\text{where}\quad P&=\tfrac{1}{4}\left[\frac{f(a+\iota\beta)}{\iota\beta\psi(a+\iota\beta)}-\frac{f(a-\iota\beta)}{\iota\beta\psi(a-\iota\beta)}\right]\\ \text{and}\quad Q&=-\tfrac{1}{4}\left[\frac{f(a+\iota\beta)}{\beta\psi(a+\iota\beta)}+\frac{f(a-\iota\beta)}{\beta\psi(a-\iota\beta)}\right],\end{aligned}\right\}\text{which are both real,}$$

and $\quad L=2P,\quad M=-2Pa-2Q\beta.$

151. IV. Case of the factor $(x-a)^2+\beta^2$ repeated r times.

Let $$\phi(x) \equiv [(x-a)^2+\beta^2]^r \psi(x).$$

Then it will be possible to write

$$\frac{f(x)}{\phi(x)} \equiv \frac{f(x)}{[(x-a)^2+\beta^2]^r \psi(x)} = \frac{P_r x + Q_r}{[(x-a)^2+\beta^2]^r} + \frac{\chi_r(x)}{[(x-a)^2+\beta^2]^{r-1}\psi(x)}.$$

For this is equivalent to determining P_r and Q_r, so that

$$f(x)-(P_r x+Q_r)\psi(x) \equiv \chi_r(x)[(x-a)^2+\beta^2],$$

i.e. so that $$f(x)-(P_r x+Q_r)\psi(x)$$

contains $x-a-\iota\beta$ and $x-a+\iota\beta$ as factors, and this will be effected by taking P_r and Q_r such that

$$\frac{f(a+\iota\beta)}{\psi(a+\iota\beta)} = P_r(a+\iota\beta)+Q_r \quad \text{and} \quad \frac{f(a-\iota\beta)}{\psi(a-\iota\beta)} = P_r(a-\iota\beta)+Q_r;$$

and if $\dfrac{f(a+\iota\beta)}{\psi(a+\iota\beta)}$, when separated into real and unreal parts, becomes $A+\iota B$, then $P_r a+Q_r=A$ and $P_r\beta=B$,

i.e. $$P_r=\frac{B}{\beta} \quad \text{and} \quad Q_r=A-\frac{Ba}{\beta}=\frac{A\beta-Ba}{\beta}.$$

Thus P_r, Q_r, and therefore χ_r are determinate.

This being so, it is obvious that

$$\frac{\chi_r(x)}{[(x-a)^2+\beta^2]^{r-1}\psi(x)}$$

can itself be expressed as

$$\frac{P_{r-1}x+Q_{r-1}}{[(x-a)^2+\beta^2]^{r-1}} + \frac{\chi_{r-1}(x)}{[(x-a)^2+\beta^2]^{r-2}\psi(x)},$$

and by continued repetition of the argument we get finally that

$$\frac{f(x)}{\phi(x)} \equiv \frac{P_r x+Q_r}{[(x-a)^2+\beta^2]^r} + \frac{P_{r-1}x+Q_{r-1}}{[(x-a)^2+\beta^2]^{r-1}} + \frac{P_{r-2}x+Q_{r-2}}{[(x-a)^2+\beta^2]^{r-2}} + \dots + \frac{P_1 x+Q_1}{(x-a)^2+\beta^2} + \frac{\chi_1(x)}{\psi(x)},$$

and the values of the r pairs of quantities,

$$P_r \text{ and } Q_r, \quad P_{r-1} \text{ and } Q_{r-1}, \ \dots, \quad P_1 \text{ and } Q_1,$$

are successively obtainable as described.

The general form of the result is thus established. But this mode of finding the numerical value of the P's and Q's is laborious, except when r is small.

152. It now appears that the general result of putting $\frac{f(x)}{\phi(x)}$ into partial fractions, where $\phi(x)$ is, say,

$$(x-a)(x-b)^{\lambda}(x^2+px+q)(x^2+rx+s)^{\mu},$$

the last two factors being irreducible to real linear factors, and $f(x)$ is any rational integral function of x of any degree, will be of the form

$$\frac{f(x)}{\phi(x)} = \text{an integral algebraic quotient}$$

$$+\frac{A}{x-a}$$

$$+\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\frac{B_3}{(x-b)^3}+\dots+\frac{B_\lambda}{(x-b)^\lambda}$$

$$+\frac{Px+Q}{x^2+px+q}$$

$$+\frac{R_1x+S_1}{x^2+rx+s}+\frac{R_2x+S_2}{(x^2+rx+s)^2}+\frac{R_3x+S_3}{(x^2+rx+s)^3}+\dots+\frac{R_\mu x+S_\mu}{(x^2+rx+s)^\mu}. \quad \text{I.}$$

This is the general typical form of the result. If other factors occur in $\phi(x)$, other partial fractions will occur in the result. But all others will be of the types exhibited.

153. The integration can therefore be effected.

For (1) The integrals of the algebraic terms are of type

$$\int A_s x^s\,dx = A_s\frac{x^{s+1}}{s+1}.$$

(2) The integral of $\int\frac{A}{x-a}dx$ is $A\log(x-a)$.

(3) The integral of $\int\frac{B_\lambda}{(x-b)^\lambda}dx$ is $-\frac{B_\lambda}{\lambda-1}\frac{1}{(x-b)^{\lambda-1}}$.

(4) The integration of $\int\frac{Px+Q}{x^2+px+q}dx$ has been effected in Art. 136.

(5) The integration of $\int\frac{R_\mu x+S_\mu}{(x^2+rx+s)^\mu}dx$ can be effected by means of a reduction formula, as will be explained in a subsequent article.

Hence we may then regard the integration $\int\frac{f(x)}{\phi(x)}dx$ as complete whenever $\frac{f(x)}{\phi(x)}$ is a rational algebraic function of x.

154. In practice, when irresoluble quadratic factors are present in the denominator we may first of all determine the

partial fractions corresponding to the real linear factors, single and repeated. Then, if there be only one quadratic factor, and that not repeated, it will appear without further trouble in the remainder of $\dfrac{f(x)}{\phi(x)}$. But if there be several such factors or a repeated factor, we may subtract the simple partial fractions when obtained and then after simplification discuss the remainder.

155. **Use of "Undetermined or Indeterminate Coefficients."** We may often with advantage apply the method of "indeterminate coefficients."

When the fraction has been reduced by division till the numerator is of lower degree than the denominator, *i.e.* of degree $n-1$ at most, and we get, as in I.,

$$\frac{f(x)}{\phi(x)} \equiv \frac{A}{x-a} + \sum_{s=1}^{s=\lambda} \frac{B_s}{(x-b)^s} + \frac{Px+Q}{x^2+px+q} + \sum_{k=1}^{k=\mu} \frac{R_k x + S_k}{(x^2+rx+s)^k}, \quad \text{II.}$$

we have, upon multiplying up by $\phi(x)$ an identity in which the right-hand side is of degree $n-1$ and consists of n terms when arranged in powers of x, and the left side is of degree $n-1$ at most, viz. $f(x)$.

Now $\phi(x)$ is of degree $1+\lambda+2+2\mu$, which must $=n$, and the number of quantities

$$\begin{array}{lccccccc} & A, & (B_1, B_2, \ldots), & (P, Q), & (R_1, S_1, R_2, S_2, \ldots) & \\ \text{is} & 1 & + \quad \lambda & + \quad 2 & + \quad 2\mu, & i.e. = n. \end{array}$$

Hence, upon equating coefficients of the n terms on the right-hand side to the corresponding coefficients in $f(x)$, we have just enough equations to obtain the n quantities, provided that these equations are all independent. But as we have established *otherwise* a means of finding these quantities we may *infer the consistence of the equations obtained by equating coefficients.*

156. Many of the coefficients, or all, may be found by the substitution in the identity of numerical values for x. Obviously any number of equations of this kind could be obtained, but only n would be independent. The most suitable values to take for this purpose will be such as will make one of the factors $x-a$, $x-b$, x^2+px+q or x^2+rx+s vanish, for such values would cause many of the terms of the identity to disappear.

In substituting roots of x^2+px+q, viz. $\alpha \pm \iota\beta$ say, only one root need be substituted. Then the real and unreal parts on each side of the identity may be equated.

All the B's and A, *i.e.* $\lambda+1$ of the quantities, can be found by the easy rules given above (Arts. 140 to 147). Hence $\lambda+1$ of the equations obtained by equating coefficients will not be independent of the others when the values of A, B_1, B_2, ... B_λ, which have been found, are substituted. But there will still remain $2+2\mu$ *independent* relations from the equating of coefficients. The substitution of a root of x^2+px+q and of a root of $x^2+rx+s=0$ with the equating of real and unreal parts will furnish four other relations and reduce the number of independent "equated coefficient equations" to $2\mu-2$, which are linear and to be solved in the easiest way available. The student will perceive that in practice it will be best to combine several methods to determine the coefficients and to use redundant equations to check numerical results.*

157. If none but even powers of x occur in both numerator and denominator, we may put $x^2=y$, and thereby reduce the labour considerably. In such fractions, the quadratic factors becoming linear by this substitution, their occurrence may be termed pseudo-quadratic or quasi-linear.

Ex. 1. $\dfrac{x^2+1}{(x^2+4)(x^2+9)^2}$.

This is of form $\dfrac{y+1}{(y+4)(y+9)^2}$.

Putting, then, x^2 (or y) $=z-9$,

$$\frac{x^2+1}{(x^2+4)(x^2+9)^2}=\frac{-8+z}{z^2(-5+z)}$$

$$\begin{array}{r|l} -5+z & -8+z\left(\dfrac{8}{5}+\dfrac{3z}{25}\right. \\ & -8+\dfrac{8z}{5} \\ \hline & \quad -\dfrac{3z}{5} \\ & \quad -\dfrac{3z}{5}+\dfrac{3z^2}{25} \\ \hline & \qquad -\dfrac{3z^2}{25} \end{array}$$

$$\therefore \frac{x^2+1}{(x^2+4)(x^2+9)^2}=\frac{8}{5z^2}+\frac{3}{25z}-\frac{3}{25}\,\frac{1}{-5+z}$$
$$=\frac{8}{5}\,\frac{1}{(x^2+9)^2}+\frac{3}{25}\,\frac{1}{x^2+9}-\frac{3}{25}\,\frac{1}{x^2+4}.$$

* See also Art. 1891, Vol. II.

Ex. 2. $\dfrac{1}{(x-1)(x^2+1)(x^2+4)^2}$.

The partial fractions are of form

$$\frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+4)}+\frac{Fx+G}{(x^2+4)^2}.$$

Multiplying up we have the identity

$$1 \equiv A(x^2+1)(x^2+4)^2+(Bx+C)(x-1)(x^2+4)^2$$
$$+(Dx+E)(x-1)(x^2+1)(x^2+4)$$
$$+(Fx+G)(x-1)(x^2+1).$$

Putting $x=1$, $\quad 1=50A$.

Putting $x=\iota$, $\quad 1=(B\iota+C)(\iota-1)9$;

$$\therefore \left.\begin{array}{r}-B+C=0,\\ -B-C=\frac{1}{9},\end{array}\right\}\text{ whence } B=C=-\tfrac{1}{18}.$$

Putting $x=2\iota$, $\quad 1=(2F\iota+G)(2\iota-1)(-3)$;

$$\therefore \left.\begin{array}{r}4F+G=\frac{1}{3},\\ F-G=0,\end{array}\right\}\text{ whence } F=G=\tfrac{1}{15}.$$

Equating coefficients of x^6,

$$A+B+D=0;$$
$$\therefore D=-\tfrac{1}{50}+\tfrac{1}{18}=\tfrac{8}{225}.$$

Equating absolute terms,

$$16A-16C-4E-G=1, \text{ whence } E=\tfrac{8}{225};$$

$$\therefore \frac{1}{(x-1)(x^2+1)(x^2+4)^2}=\frac{1}{50}\frac{1}{x-1}-\frac{1}{18}\frac{x+1}{x^2+1}+\frac{8}{225}\frac{x+1}{x^2+4}+\frac{1}{15}\frac{x+1}{(x^2+4)^2}.$$

158. Case when the numerator is an odd function of x and the denominator is even.

$$I=\int\frac{f(x)}{\phi(x)}dx \text{ takes the form } \int\frac{xF(x^2)}{\Phi(x^2)}dx,$$

and putting $x^2=y$, $\quad I=\dfrac{1}{2}\displaystyle\int\frac{F(y)}{\Phi(y)}dy,$

and the factors in the denominator which were quadratic factors in x are linear in y.

Ex. Thus
$$\int\frac{x^3+3x}{(x^2-1)(x^2+1)^2}dx=\frac{1}{2}\int\frac{y+3}{(y-1)(y+1)^2}dy$$
$$=\frac{1}{2}\int\left[\frac{1}{y-1}-\frac{1}{y+1}-\frac{1}{(y+1)^2}\right]dy$$
$$=\frac{1}{2}\log\frac{y-1}{y+1}+\frac{1}{2}\frac{1}{y+1}$$
$$=\frac{1}{2}\log\frac{x^2-1}{x^2+1}+\frac{1}{2}\frac{1}{x^2+1}.$$

159. **Case when the denominator is odd and the numerator even.** The same process may be adopted.

Thus
$$\int \frac{x^2+1}{x(x^2+4)}\,dx = \frac{1}{2}\int \frac{y+1}{y(y+4)}\,dy$$
$$= \frac{1}{8}\int \left(\frac{1}{y}+\frac{3}{y+4}\right)dy = \frac{1}{8}\log y\,(y+4)^3$$
$$= \frac{1}{8}\log x^2(x^2+4)^3.$$

160. Integration of $\displaystyle\int \frac{z^{2q}\,dz}{(z^2+a_1^2)(z^2+a_2^2)(z^2+a_3^2)\dots(z^2+a_n^2)}$, where $q<n$.

The partial fractions are of the form
$$\sum_1^n \frac{A_r}{z^2+a_r^2},$$
and the integral is
$$\sum_1^n \frac{A_r}{a_r}\tan^{-1}\frac{z}{a_r}.$$

The value of A_r is
$$\frac{(-a_r^2)^q}{(a_1^2-a_r^2)(a_2^2-a_r^2)\dots(a_{r-1}^2-a_r^2)(a_{r+1}^2-a_r^2)\dots(a_n^2-a_r^2)}.$$

The denominator factorized may be written as
$$(a_1-a_r)(a_2-a_r)\dots(a_{r-1}-a_r)\quad(a_{r+1}-a_r)(a_{r+2}-a_r)\dots(a_n-a_r)$$
$$\times(a_1+a_r)(a_2+a_r)\dots(a_{r-1}+a_r)\quad(a_{r+1}+a_r)(a_{r+2}+a_r)\dots(a_n+a_r).$$

Taking the case when $a_1, a_2, a_3, \dots a_n$ form an A.P., with common difference b, this denominator D, say, is
$$D=(-1)^{r-1}(r-1)b(r-2)b\dots 2b\,.\,b,\quad b\,.\,2b\,.\,3b\dots(n-r)b\times\prod_{k=1}^{k=n}(a_k+a_r)/2a_r,$$
where in forming the product of the factors in the lower line the missing term (a_r+a_r) has been supplied;
$$D=(-1)^{r-1}b^{r-1}(r-1)!\,b^{n-r}(n-r)!\prod_{k=1}^{k=n}(a_k+a_r)/2a_r$$
and
$$A_r=(-1)^{q-r+1}2a_r^{2q+1}/b^{n-1}(r-1)!(n-r)!\prod_{k=1}^{k=n}(a_k+a_r).$$

If $b=a_1$, we have
$$a_k+a_r=(r+k)a_1,$$
and
$$\prod_1^n(a_k+a_r)=a_1^n(r+1)(r+2)\dots(r+n)=a_1^n\frac{(r+n)!}{r!},$$
giving for this case the partial fractions
$$\frac{2}{a_1^{2n-2q-2}}\sum_{r=1}^{r=n}(-1)^{q-r+1}\frac{r^{2q+2}}{(n+r)!\,(n-r)!}\,\frac{1}{z^2+a_r^2},$$
and the integral
$$\frac{2}{a_1^{2n-2q-1}}\sum_{r=1}^{r=n}(-1)^{q-r+1}\frac{r^{2q+1}}{(n+r)!\,(n-r)!}\tan^{-1}\frac{z}{a_r},$$

161. Obviously we should also have in the same case

$$\int \frac{z^{2q+1}\,dz}{(z^2+a_1^2)(z^2+a_2^2)\dots(z^2+a_n^2)}$$
$$=\frac{2}{a_1^{2n-2q-2}}\int\sum_{r=1}^{r=n}(-1)^{q-r+1}\frac{r^{2q+2}}{(n+r)!\,(n-r)!}\,\frac{z}{z^2+a_r^2}$$
$$=\frac{1}{a_1^{2n-2q-2}}\sum_{r=1}^{r=n}(-1)^{q-r+1}\frac{r^{2q+2}}{(n+r)!\,(n-r)!}\log(z^2+a_r^2).$$

162. Taking the case $a_1=2$, $b=2$, and therefore $a_r=2r$,

$$\frac{z^{2q}}{(z^2+2^2)(z^2+4^2)(z^2+6^2)\dots(z^2+2^2n^2)}$$
$$=\frac{1}{(2n)!}\sum_n^1(-1)^{q-r+1}\frac{r^{2q+2}}{2^{2n-2q-3}}\,{}^{2n}C_{n-r}\frac{1}{z^2+2^2r^2}$$
$$=\frac{(-1)^{q+n-1}}{(2n)!}\frac{1}{2^{2n-1}}\left[{}^{2n}C_0\frac{(2n)^{2q+2}}{z^2+2^2n^2}-{}^{2n}C_1\frac{(2n-2)^{2q+2}}{z^2+(2n-2)^2}+{}^{2n}C_2\frac{(2n-4)^{2q+2}}{z^2+(2n-4)^2}\right.$$
$$\left.+\dots+(-1)^{n-1}\,{}^{2n}C_{n-1}\frac{2^{2q+2}}{z^2+2^2}\right];$$

and its integral

$$=\frac{(-1)^{q+n-1}}{(2n)!}\frac{1}{2^{2n-1}}$$
$$\left[{}^{2n}C_0(2n)^{2q+1}\tan^{-1}\frac{z}{2n}-{}^{2n}C_1(2n-2)^{2q+1}\tan^{-1}\frac{z}{2n-2}\right.$$
$$\left.+\dots+(-1)^{n-1}\,{}^{2n}C_{n-1}\,2^{2q+1}\tan^{-1}\frac{z}{2}\right]. \quad\dots\dots(\text{A})$$

163. And similarly, if the index of z in the numerator had been $2q+1$ instead of $2q$, the same work shows

$$\frac{z^{2q+1}}{(z^2+2^2)(z^2+4^2)\dots(z^2+2^2n^2)}$$
$$=\frac{(-1)^{q+n-1}}{(2n)!}\frac{1}{2^{2n-1}}$$
$$\left[{}^{2n}C_0(2n)^{2q+2}\frac{z}{z^2+2^2n^2}-{}^{2n}C_1(2n-2)^{2q+2}\frac{z}{z^2+(2n-2)^2}\right.$$
$$\left.+\dots+(-1)^{n-1}\,{}^{2n}C_{n-1}\,2^{2q+2}\frac{z}{z^2+2^2}\right];$$

and its integral

$$=\frac{(-1)^{q+n-1}}{(2n)!}\frac{1}{2^{2n}}$$
$$[{}^{2n}C_0(2n)^{2q+2}\log(z^2+2^2n^2)-{}^{2n}C_1(2n-2)^{2q+2}\log\{z^2+(2n-2)^2\}$$
$$+\dots+(-1)^{n-1}\,{}^{2n}C_{n-1}2^{2q+2}\log(z^2+2^2)]. \quad\dots(\text{B})$$

164. Taking the case $a_1=1$, $b=2$, and therefore $a_r=2r-1$,

$$\frac{z^{2q}}{(z^2+1^2)(z^2+3^2)(z^2+5^2)\dots[z^2+(2n-1)^2]}$$

$$=\frac{1}{(2n-1)!}\sum_n^1(-1)^{q-r+1}\frac{(2r-1)^{2q+1}}{2^{2n-2}}\,{}^{2n-1}C_{n-r}\frac{1}{z^2+(2r-1)^2}$$

$$=\frac{(-1)^{q-n+1}}{(2n-1)!}\frac{1}{2^{2n-2}}\left[{}^{2n-1}C_0\frac{(2n-1)^{2q+1}}{z^2+(2n-1)^2}-{}^{2n-1}C_1\frac{(2n-3)^{2q+1}}{z^2+(2n-3)^2}\right.$$

$$\left.+{}^{2n-1}C_2\frac{(2n-5)^{2q+1}}{z^2+(2n-5)^2}-\dots+(-1)^{n-1}\,{}^{2n-1}C_{n-1}\frac{1^{2q+1}}{z^2+1^2}\right]:$$

and its integral

$$=\frac{(-1)^{q-n+1}}{(2n-1)!}\frac{1}{2^{2n-2}}$$

$$\left[{}^{2n-1}C_0(2n-1)^{2q}\tan^{-1}\frac{z}{2n-1}-{}^{2n-1}C_1(2n-3)^{2q}\tan^{-1}\frac{z}{2n-3}\right.$$

$$\left.+\dots+(-1)^{n-1}\,{}^{2n-1}C_{n-1}1^{2q}\tan^{-1}\frac{z}{1}\right]. \quad \text{(C)}$$

165. And for $\dfrac{z^{2q+1}}{(z^2+1^2)\dots[z^2+(2n-1)^2]}$ the integral will be

$$=\frac{(-1)^{q-n+1}}{(2n-1)!}\frac{1}{2^{2n-1}}[{}^{2n-1}C_0(2n-1)^{2q+1}\log\{z^2+(2n-1)^2\}$$

$$-{}^{2n-1}C_1(2n-3)^{2q+1}\log\{z^2+(2n-3)^2\}$$

$$+\dots+(-1)^{n-1}\,{}^{2n-1}C_{n-1}1^{2q+1}\log\{z^2+1^2\}]. \quad \text{(D)}$$

166. Consider the integral $\displaystyle\int\frac{x^m\,dx}{x^{2n}-2a^nx^n\cos na+a^{2n}}.\quad (m<2n)$

Here $\quad f(x)=x^m,\quad \phi(x)=x^{2n}-2a^nx^n\cos na+a^{2n}\quad$ (Art. 142)

$$=\prod_{r=0}^{r=n-1}\left[x^2-2ax\cos\left(a+\frac{2r\pi}{n}\right)+a^2\right],$$

$$\phi'(x)=2nx^{n-1}(x^n-a^n\cos na).$$

Let $$a+\frac{2r\pi}{n}=\chi.$$

The factor $\quad x^2-2ax\cos\chi+a^2=(x-ae^{\iota\chi})(x-ae^{-\iota\chi})$,
and gives rise to the partial fractions

$$\frac{f(ae^{\iota\chi})}{\phi'(ae^{\iota\chi})}\frac{1}{x-ae^{\iota\chi}}+\frac{f(ae^{-\iota\chi})}{\phi'(ae^{-\iota\chi})}\frac{1}{x-ae^{-\iota\chi}}.$$

Now $$\frac{f(ae^{\iota\chi})}{\phi'(ae^{\iota\chi})}=\frac{a^me^{\iota m\chi}}{2na^{2n-1}e^{\iota(n-1)\chi}(e^{\iota n\chi}-\cos na)}$$

$$=\frac{a^me^{\iota m\chi}}{2na^{2n-1}e^{\iota(n-1)\chi}\iota\sin na}=\frac{e^{-\iota(n-m-1)\chi}}{2\iota na^{2n-m-1}\sin na}.$$

Hence the two partial fractions

$$=\frac{1}{2\iota n a^{2n-m-1}\sin n\alpha}\left[\frac{e^{-\iota(n-m-1)\chi}}{x-ae^{\iota\chi}}-\frac{e^{\iota(n-m-1)\chi}}{x-ae^{-\iota\chi}}\right]$$

$$=\frac{1}{2\iota n a^{2n-m-1}\sin n\alpha}\left[\frac{e^{-\iota(n-m-1)\chi}(x-ae^{-\iota\chi})-e^{\iota(n-m-1)\chi}(x-ae^{\iota\chi})}{x^2-2ax\cos\chi+a^2}\right]$$

$$=\frac{1}{2\iota n\sin n\alpha\, a^{2n-m-1}}\left[\frac{2a\iota\sin(n-m)\chi-2x\iota\sin(n-m-1)\chi}{x^2-2ax\cos\chi+a^2}\right];$$

$$\therefore\quad \frac{x^m}{x^{2n}-2a^nx^n\cos n\alpha+a^{2n}}$$

$$\equiv\frac{1}{2n\sin n\alpha\, a^{2n-m-1}}\sum_{r=1}^{r=n}\left[\frac{2a\sin\chi\cos(n-m-1)\chi-2(x-a\cos\chi)\sin(n-m-1)\chi}{(x-a\cos\chi)^2+a^2\sin^2\chi}\right]$$

Hence $\displaystyle\int\frac{x^m\,dx}{x^{2n}-2a^nx^n\cos n\alpha+a^{2n}}$, $(m<2n)$,

$$=\frac{1}{n\sin n\alpha}\frac{1}{a^{2n-m-1}}\sum_0^{n-1}\cos(n-m-1)\left(\alpha+\frac{2r\pi}{n}\right)\tan^{-1}\frac{x-a\cos\left(\alpha+\frac{2r\pi}{n}\right)}{a\sin\left(\alpha+\frac{2r\pi}{n}\right)}$$

$$-\frac{1}{2n\sin n\alpha}\frac{1}{a^{2n-m-1}}\sum_0^{n-1}\sin(n-m-1)\left(\alpha+\frac{2r\pi}{n}\right)\log\left[x^2-2ax\cos\left(\alpha+\frac{2r\pi}{n}\right)+a^2\right].$$

In the same way $x^{\mu-1}/(x^n\pm a^n)$ may be integrated. The results are given in Exs. 39 and 40, pages 166 and 167.

167. Ex. Calculate $\displaystyle\int_0^\infty\frac{dx}{x^4+2a^2x^2\cos 2\beta+a^4}$.

Here (Art. 166) $\qquad \beta=\frac{\pi}{2}-\alpha,\quad m=0,\quad n=2.$

The indefinite integral is

$$\frac{1}{2\sin 2\beta}\frac{1}{a^3}\left[\cos\left(\frac{\pi}{2}-\beta\right)\tan^{-1}\frac{x-a\cos\left(\frac{\pi}{2}-\beta\right)}{a\sin\left(\frac{\pi}{2}-\beta\right)}\right.$$

$$+\cos\left(\frac{3\pi}{2}-\beta\right)\tan^{-1}\frac{x-a\cos\left(\frac{3\pi}{2}-\beta\right)}{a\sin\left(\frac{3\pi}{2}-\beta\right)}$$

$$-\tfrac{1}{2}\sin\left(\frac{\pi}{2}-\beta\right)\log\left\{x^2-2ax\cos\left(\frac{\pi}{2}-\beta\right)+a^2\right\}$$

$$\left.-\tfrac{1}{2}\sin\left(\frac{3\pi}{2}-\beta\right)\log\left\{x^2-2ax\cos\left(\frac{3\pi}{2}-\beta\right)+a^2\right\}\right]$$

$$=\frac{1}{2a^3\sin 2\beta}\left[\sin\beta\tan^{-1}\frac{x-a\sin\beta}{a\cos\beta}+\sin\beta\tan^{-1}\frac{x+a\sin\beta}{a\cos\beta}\right.$$

$$\left.-\tfrac{1}{2}\cos\beta\log(x^2-2ax\sin\beta+a^2)+\tfrac{1}{2}\cos\beta\log(x^2+2ax\sin\beta+a^2)\right],$$

and taken between limits 0 and ∞

$$=\frac{1}{2a^3\sin 2\beta}\left[\sin\beta\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right]=\frac{\pi}{4a^3\cos\beta}.$$

The indefinite integral may also be written as

$$\frac{1}{2a^3\sin 2\beta}\left[\sin\beta\tan^{-1}\frac{2ax\cos\beta}{a^2-x^2}+\cos\beta\tanh^{-1}\frac{2ax\sin\beta}{a^2+x^2}\right].$$

168. An integral of the form

$$\int\frac{a+bx^{\frac{p}{q}}}{c+dx^{\frac{r}{s}}}dx$$

can always be integrated as follows:

Let l be the L.C.M. of q and s, and let $\frac{p}{q}=\frac{\lambda}{l}$ and $\frac{r}{s}=\frac{\mu}{l}$.

Let $x=z^l$, $dx=lz^{l-1}dz$.

Then $$\int\frac{a+bx^{\frac{p}{q}}}{c+dx^{\frac{r}{s}}}dx=l\int\frac{a+bz^\lambda}{c+dz^\mu}z^{l-1}dz,$$

and the expression to be integrated is now rational, and when expressed in partial fractions each term can be integrated.

Ex. $$\int\frac{1+x^{\frac{1}{3}}}{1+x^{\frac{1}{2}}}dx\quad(\text{Let } x=z^6.)\quad=\int\frac{1+z^2}{1+z^3}6z^5\,dz$$

$$=6\int\frac{z^5+z^7}{1+z^3}dz=6\int\left(z^4+z^2-z-\frac{z^2-z}{z^3+1}\right)dz$$

$$=6\int\left[z^4+z^2-z-\frac{\frac{2}{3}}{z+1}-\frac{1}{6}\frac{2z-1-3}{z^2-z+1}\right]dz$$

$$=\frac{6z^5}{5}+2z^3-3z^2-4\log(z+1)-\log(z^2-z+1)+2\sqrt{3}\tan^{-1}\frac{2z-1}{\sqrt{3}}$$

$$=\tfrac{6}{5}x^{\frac{5}{6}}+2x^{\frac{1}{2}}-3x^{\frac{1}{3}}-4\log(1+x^{\frac{1}{6}})-\log(1-x^{\frac{1}{6}}+x^{\frac{1}{3}})$$
$$+2\sqrt{3}\tan^{-1}\frac{2x^{\frac{1}{6}}-1}{\sqrt{3}}.$$

169. In exactly the same way the integration of

$$\int\frac{a+b(\alpha+\beta x)^{\frac{p}{q}}}{c+d(\alpha+\beta x)^{\frac{r}{s}}}dx,$$

can be effected by putting $\alpha+\beta x=z^l$ when l is the L.C.M. of q and s, and more generally that of

$$\int\frac{f[(\alpha+\beta x)^{\frac{p}{q}}]}{\phi[(\alpha+\beta x)^{\frac{r}{s}}]}dx,$$

where $f(t)$ and $\phi(t)$ are any rational algebraic functions of t; for, putting $\alpha+\beta x=z^l$, as before, the integral becomes

$$\int \frac{f(z^\lambda)}{\phi(z^\mu)}\cdot\frac{lz^{l-1}}{\beta}dz;$$

and the integrand being now rational and algebraic, we can in any such case proceed to put it into partial fractions and then integrate.

EXAMPLES.

Integrate with regard to x the expressions in the following seven groups:

1. Linear unrepeated factors

(i) $\dfrac{1}{x(x^2-1)}$. (ii) $\dfrac{1}{(x-1)(x-3)(x-5)}$.

(iii) $\dfrac{1-3x^2}{3x-x^3}$. (iv) $\dfrac{x^2+x+1}{(x^2-1)(x^2-4)}$.

(v) $\dfrac{(x-1)(x-2)}{(x^2-9)(x^2-16)}$. (vi) $\dfrac{(x-a)(x-b)(x-c)}{(x-a_1)(x-b_1)(x-c_1)}$.

(vii) $\dfrac{(x-a)(x-b)(x-c)}{(x^2-a_1^2)(x^2-b_1^2)(x^2-c_1^2)}$. (viii) $\dfrac{x+1}{x^2+10x-75}$.

(ix) $\dfrac{x+1}{x^2+10x-119}$. (x) $\dfrac{x+1}{x^3-31x^2+311x-1001}$.

2. Linear repeated factors:

(i) $\dfrac{1}{(x-1)^3(x+1)}$. (ii) $\dfrac{1}{(x-1)^4(x+1)^4}$.

(iii) $\dfrac{x+1}{x^4(x-1)^4}$. (iv) $(ax^2+bx^3)^{-1}$.

(v) $(x^2-7x+12)^{-2}$. (vi) $\dfrac{x^3}{(x-a)^2(x-b)}$.

(vii) $\dfrac{x^2-3x+3}{x^3-7x^2+16x-12}$. [I. C. S., 1900.]

3. Quasi-linear occurrence of factors. Powers of x *all even*:

(i) $\displaystyle\int\frac{dx}{(x^2+a^2)(x^2+b^2)}$. (ii) $\displaystyle\int\frac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)}dx$.

(iii) $\displaystyle\int\frac{x^2(x^2+a^2)}{(x^2+c^2)}dx$. (iv) $\displaystyle\int\frac{x^2dx}{(x^2+1)(2x^2+1)}$.

(v) $\displaystyle\int\frac{ax^2+b}{(cx^2+d)(ex^2+f)}dx$. (vi) $\displaystyle\int\frac{ax^2+b}{x^2(cx^2+d)(ex^2+f)(gx^2+h)}dx$.

In the last two c, d, e, f, g, h may be considered positive.

4. Quasi-linear factors. Numerator an odd function, Denominator even, or Numerator even, Denominator odd:

(i) $\displaystyle\int\frac{dx}{x(x^2+1)}$. (ii) $\displaystyle\int\frac{x^2+2}{x(x^4-1)}\,dx$.

(iii) $\displaystyle\int\frac{dx}{x^7-6x^5+11x^3-6x}$. (iv) $\displaystyle\int\frac{x\,dx}{(ax^2+bx+c)^2+(ax^2-bx+c)^2}$.

5. Quadratic factors not repeated:

(i) $\displaystyle\int\frac{dx}{x^4+x^2+1}$. (ii) $\displaystyle\int\frac{(x+1)^2}{x^4+x^2+1}\,dx$.

(iii) $\displaystyle\int\frac{x^2+1}{x^4+1}\,dx$. (iv) $\displaystyle\int\frac{x^2+1}{x^4-x^2+1}\,dx$.

(v) $\displaystyle\int(x^2+a^2)(x^4+a^2x^2+a^4)^{-1}\,dx$. (vi) $\displaystyle\int(x^2-a^2)(x^4+a^2x^2+a^4)^{-1}dx$.

(vii) $\displaystyle\int\frac{x^2+3x+1}{x^4+x^2+1}\,dx$. (viii) $\displaystyle\int\frac{dx}{x^4+1}$.

6. Linear factors repeated. Quadratic factors not repeated.

(i) $\dfrac{x^2dx}{(x-1)^2(x^2-2x+4)}$. (ii) $\dfrac{dx}{(1+x)^2(1+2x+4x^2)}$.

(iii) $\dfrac{x^4\,dx}{(x-1)^2(x^2+4)}$. (iv) $\dfrac{dx}{(x+1)^2(x^2+1)}$.

(v) $\dfrac{dx}{(x-1)^2(x^2+1)}$. (vi) $\dfrac{dx}{x(x-1)^2(x^2+1)}$.

(vii) $\dfrac{dx}{x^3(a^2+x^2)}$. (viii) $\dfrac{dx}{x^3(a^2+x^2)(b^2+x^2)}$.

(ix) $\dfrac{dx}{(x-1)^3(x^2+x+1)}$. (x) $\dfrac{dx}{(2x-3)^2(4x^2+5)}$.

7. Repeated quadratic factors:

(i) $\dfrac{dx}{x(x^2+1)^2}$. (ii) $\dfrac{dx}{(x-1)^2(x^2+1)^2}$.

(iii) $\dfrac{(x+1)\,dx}{(x^2+1)^2}$. (iv) $\dfrac{(x+a)(x+b)}{(x^2+c^2)^3}\,dx$.

8. Evaluate $\displaystyle\int_0^{\frac{\pi}{4}}\sqrt{\tan\theta}\,d\theta$ and $\displaystyle\int_0^{\frac{\pi}{4}}\sqrt{\cot\theta}\,d\theta$.

9. Evaluate (i) $\displaystyle\int_0^{\frac{\pi}{4}}\frac{dx}{\cos^4x-\cos^2x\sin^2x+\sin^4x}$,

(ii) $\displaystyle\int_0^{\frac{\pi}{4}}\frac{dx}{\cos^4x+\cos^2x\sin^2x+\sin^4x}$.

10. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$.

11. Show that $\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}$.

12. Show that $\int_{-\infty}^{+\infty} \frac{dx}{(x^2 \pm ax + a^2)(x^2 \pm bx + b^2)} = \frac{2\pi}{\sqrt{3}} \frac{a+b}{ab(a^2+ab+b^2)}$.

[γ, 1891.]

13. Show that the sum of the infinite series

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots \quad (a>0,\ b>0)$$

can be expressed as a definite integral, viz.

$$\int_0^1 \frac{t^{a-1}}{1+t^b}\,dt.$$

And hence prove that

$$\tfrac{1}{1} - \tfrac{1}{4} + \tfrac{1}{7} - \tfrac{1}{10} + \tfrac{1}{13} - \tfrac{1}{16} + \dots = \tfrac{1}{3}(\pi 3^{-\frac{1}{2}} + \log_e 2).$$

[Oxford, 1887.]

14. Integrate:

(i) $\int \frac{25\,dx}{2x^4 + 3x^3 + 3x - 2}$. [Colleges, 1882.]

(ii) $\int \frac{x^5+2}{x^5-x}\,dx$. [St. John's, 1881.]

(iii) $\int \frac{(1+x^2)\,dx}{1 - 2x^2\cos\alpha + x^4}$. [Colleges, 1882.]

(iv) $\int \frac{dx}{1+x^5}$. [Colleges α, 1891.]

(v) $\int_0^1 \frac{1+x^2}{(1-x^2)^2 + a^2x^2}\,dx$.

15. Prove that $\int_0^{\infty} \frac{dx}{1+x^6} = \frac{\pi}{3}$. [St. John's, 1881.]

16. Prove that $\int \frac{dx}{(x-a)^p(x-b)^q}$

$$= \sum_{r=0}^{r=p-2} \frac{Q_r}{(a-b)^{q+r}} \frac{(x-a)^{-p+r+1}}{-p+r+1} + \sum_{r=0}^{r=q-2} \frac{P_r}{(b-a)^{p+r}} \frac{(x-b)^{-q+r+1}}{-q+r+1}$$

$$+ \frac{1}{(a-b)^{p+q-1}} Q_{p-1} \log(x-a) + \frac{1}{(b-a)^{p+q-1}} P_{q-1} \log(x-b),$$

where P_r and Q_r are the coefficients of z^r in $(1+z)^{-p}$ and $(1+z)^{-q}$ respectively.

17. Integrate (i) $\displaystyle\int\frac{dx}{(5x^3-3x)^2(x^2-1)}$. [MATH. TRIP., 1878.]

(ii) $\displaystyle\int\frac{dx}{(5x^3-3x^2)^2(x^2-1)}$.

(iii) $\displaystyle\int\frac{\sqrt{x}\,dx}{(1+x)(2+x)(3+x)}$. [OXFORD I., 1888.]

18. Prove (i) $\displaystyle\int_{-\infty}^{\infty}\frac{2x^2-x+1}{(x^2+x+1)^3}dx=\tfrac{10}{9}\pi\sqrt{3}$. [COLLEGES β, 1891.]

(ii) $\displaystyle\int_0^{\infty}\frac{\sqrt{x}}{1+x^2}dx=\frac{\pi}{\sqrt{2}}$. [TRINITY, 1882.]

(iii) $\displaystyle\int_x^{\frac{1}{x}}\frac{dx}{1+x^4}dx=\frac{1}{2\sqrt{2}}\left\{\pi-2\tan^{-1}\frac{\sqrt{2}}{x^{-1}-x}\right\}$.

[TRINITY, 1895.]

19. Integrate $\displaystyle\int\frac{x\,dx}{x^3+1}$.

Prove that $\displaystyle\frac{1}{2\,.\,5}+\frac{1}{8\,.\,11}+\frac{1}{14\,.\,17}+\dots\text{ to }\infty=\frac{1}{9}\left[\frac{\pi}{\sqrt{3}}-\log 2\right]$.

[COLLEGES, 1896.]

20. Integrate $\displaystyle\int\frac{(\sqrt{\cot x}-\sqrt{\tan x})\,dx}{1+3\sin 2x}$. [COLLEGES β, 1890.]

21. Integrate (i) $\displaystyle\int\tan^{-1}\sqrt{\frac{a^2x-1}{b^2x-1}}\,dx$.

(ii) $\displaystyle\int\sqrt{a^2+\sqrt{b^2+c/x}}\,dx$. [MATH. TRIP., 1898.]

22. Integrate $\displaystyle\int\frac{(\sqrt{a}-\sqrt{x})^2\,dx}{(a^2+ax+x^2)\sqrt{x}}$. [COLLEGES, 1896.]

23. Integrate $\displaystyle\int\frac{5x^3+3x-1}{(x^3+3x+1)^3}dx$. [J. M. SCH., OX., 1904.]

24. Evaluate $\displaystyle\int_0^{\frac{\pi}{4}}\frac{\sin^4 x}{\cos^5 x}dx$. [ST. JOHN'S, 1892.]

25. Integrate $\displaystyle\int\frac{dx}{x(x-a)^n}$, n being a positive integer.

[ST. JOHN'S, 1882.]

26. Integrate $\displaystyle\int\left(\frac{x-b}{x-a}\right)^{\frac{n}{2}}dx$. [COLLEGES α, 1885.]

27. Integrate $\displaystyle\int\frac{dx}{1+3e^x+2e^{2x}}$. [MATH. TRIP., 1895.]

28. Sum the series

$$\frac{x^3}{1.3}+\frac{x^5}{3.5}+\frac{x^7}{5.7}+\dots\ ad\ inf.,$$

assuming it to be convergent.

Deduce that

$$\frac{1}{1.3}\cdot\frac{1}{2^3}+\frac{1}{3.5}\cdot\frac{1}{2^5}+\frac{1}{5.7}\cdot\frac{1}{2^7}+\dots\ ad\ inf.=\frac{1}{4}-\frac{3}{16}\log 3.$$

[I. C. S., 1899.]

29. Prove that

$$1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\dots\ ad\ inf.=\frac{\pi}{8}(1+\sqrt{2}).$$

[COLLEGES β, 1888.]

30. Evaluate $\displaystyle\int x^2\log(1-x^2)\,dx,$

and deduce that

$$\frac{1}{1.5}+\frac{1}{2.7}+\frac{1}{3.9}+\dots=\frac{8}{9}-\frac{2}{3}\log 2.$$

[COLLEGES α, 1889.]

31. Integrate $\displaystyle\int(a^4+x^4)^{-1}\,dx.$

Prove that

$$\frac{1}{1.5}+\frac{1}{9.13}+\frac{1}{17.21}+\frac{1}{25.29}+\dots=\frac{\sqrt{2}}{32}\left\{\pi+\log(3+2\sqrt{2})\right\}.$$

[MATH. TRIP., 1896.]

32. Show that

$$\int\frac{dx}{x(x+1)(x+2)(x+3)\dots(x+n)}=\frac{1}{\underline{|n}}\sum_{r=0}^{r=n}(-1)^r\,{}^nC_r\log(x+r).$$

33. Show that

$$\int\frac{(1+x)^n}{(1-2x)^3}dx=\frac{3^n}{2^{n+2}}\frac{1}{(1-2x)^2}-n\frac{3^{n-1}}{2^{n+1}}\frac{1}{1-2x}-\frac{n(n-1)3^{n-2}}{2^{n+2}}\log(1-2x)$$

+ a rational integral algebraic expression of a finite number of terms.

34. Show that if $c<1$, $\displaystyle\int\frac{dx}{(1-x)(1-cx)(1-c^2x)\dots\text{to }\infty}$

$$=x+\frac{1}{2}\frac{x^2}{1-c}+\frac{1}{3}\frac{x^3}{(1-c)(1-c^2)}+\frac{1}{4}\frac{x^4}{(1-c)(1-c^2)(1-c^3)}+\dots\text{ to }\infty.$$

35. Show that $\displaystyle\int\frac{x^{n+2}\,dx}{(x-a_1)(x-a_2)(x-a_3)\dots(x-a_n)}$

$$=\frac{x^3}{3}+H_1\frac{x^2}{2}+H_2x+\sum\frac{a_1{}^{n+2}}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)}\log(x-a_1),$$

where H_r is the sum of the homogeneous products r at a time of $a_1, a_2, \dots, a_n$.

36. Show that the part of the indefinite integral

$$\int \frac{1}{x^3}\frac{f(x)}{\phi(x)}\,dx$$

which becomes infinite when $x=0$, f and ϕ being rational integral functions of x which do not vanish when $x=0$, is

$$-\frac{1}{2x^2}\frac{f(0)}{\phi(0)}-\frac{1}{x}\frac{f'(0)\phi(0)-\phi'(0)f(0)}{[\phi(0)]^2}$$
$$+\frac{1}{2}\log x\frac{f''(0)\{\phi(0)\}^2-2f'(0)\phi'(0)\phi(0)-f(0)[\phi(0)\phi''(0)-2\{\phi'(0)\}^2]}{[\phi(0)]^3}.$$

[Ox. I. P., 1901.]

37. Show that when a rational fraction is decomposed into its simple or "partial" fractions, the decomposition is unique.

38. If $F(x)$ be a function of the $(n-1)^{\text{th}}$ degree which assumes the values $u_1, u_2, u_3, \ldots, u_n$ when $x=x_0, x_1, x_2, \ldots, x_n$ respectively, show that

$$\begin{aligned}F(x)=u_1&\frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}\\ +u_2&\frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}\\ +&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ +u_n&\frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}.\end{aligned}$$

39. Prove that if $p<n+1$,

$$\begin{aligned}na^{n-p}\int\frac{x^{p-1}\,dx}{x^n-a^n}&=\log(x-a)+\sum_{r=1}^{r=\frac{n-1}{2}}\cos\frac{2rp\pi}{n}\log\left(x^2-2ax\cos\frac{2r\pi}{n}+a^2\right)\\ &\quad-2\sum_{r=1}^{r=\frac{n-1}{2}}\sin\frac{2rp\pi}{n}\tan^{-1}\frac{x-a\cos\frac{2r\pi}{n}}{a\sin\frac{2r\pi}{n}}\end{aligned}$$

if n be odd,

and

$$\begin{aligned}&=\log(x-a)+(-1)^p\log(x+a)\\ &\quad+\sum_{r=1}^{r=\frac{n-2}{2}}\cos\frac{2rp\pi}{n}\log\left(x^2-2ax\cos\frac{2r\pi}{n}+a^2\right)\\ &\quad-2\sum_{r=1}^{r=\frac{n-2}{2}}\sin\frac{2rp\pi}{n}\tan^{-1}\frac{x-a\cos\frac{2r\pi}{n}}{a\sin\frac{2r\pi}{n}}\end{aligned}$$

if n be even.

40. Prove that if $p<n+1$,

$$na^{n-p}\int\frac{x^{p-1}}{x^n+a^n}dx=(-1)^{p-1}\log(x+a)$$

$$-\sum_{r=1}^{r=\frac{n-1}{2}}\cos(2r-1)\frac{p\pi}{n}\log\left\{x^2-2ax\cos(2r-1)\frac{\pi}{n}+a^2\right\}$$

$$+2\sum_{r=1}^{r=\frac{n-1}{2}}\sin(2r-1)\frac{p\pi}{n}\tan^{-1}\frac{x-a\cos(2r-1)\frac{\pi}{n}}{a\sin(2r-1)\frac{\pi}{n}}$$

if n be odd,

and $$=-\sum_{r=1}^{r=n}\cos(2r-1)\frac{p\pi}{n}\log\left\{x^2-2ax\cos(2r-1)\frac{\pi}{n}+a^2\right\}$$

$$+2\sum_{r=1}^{r=\frac{n}{2}}\sin(2r-1)\frac{p\pi}{n}\tan^{-1}\frac{x-a\cos(2r-1)\frac{\pi}{n}}{a\sin(2r-1)\frac{\pi}{n}}$$ if n be even.

41. Prove that

$$\int_0^x\frac{dx}{1-x^{2n}}=\frac{1}{2n}\sum_{r=0}^{r=n-1}\left(\cos\frac{r\pi}{n}\tanh^{-1}\frac{2x\cos\frac{r\pi}{n}}{1+x^2}+\sin\frac{r\pi}{n}\tan^{-1}\frac{2x\sin\frac{r\pi}{n}}{1-x^2}\right).$$

[MATH. TRIP., 1884.]

42. Show that $$\int_0^\infty\frac{t}{t^5+1}dt=\frac{4\pi}{5\sqrt{10+2\sqrt{5}}}.$$

43. (i) Show that the remainder left after dividing the rational integral function $f(x)$ by $(x-c)^2+b^2$ is

$$\left[f(c)-\frac{b^2}{2!}f^{(2)}(c)+\frac{b^4}{4!}f^{(4)}(c)-\dots+(-1)^r\frac{b^{2r}}{(2r)!}f^{(2r)}(c)+\dots\right]$$

$$+(x-c)\left[f'(c)-\frac{b^3}{3!}f^{(3)}(c)+\frac{b^5}{5!}f^{(5)}(c)-\dots\right.$$

$$\left.+(-1)^r\frac{b^{2r+1}}{(2r+1)!}f^{(2r+1)}(c)+\dots\right],$$

where $f^{(s)}(c)$ denotes $\dfrac{d^sf(c)}{dc^s}$.

(ii) If $f(x)$ and $\phi(x)$ are rational integral functions of x, and $\phi(x)$ does not contain $(x-c)^2+b^2$ as a factor, show that it is possible to determine finite values for the constants P and Q in such a manner that
$$f(x)-[P(x-c)+Q]\phi(x)$$
is divisible, without remainder, by $(x-c)^2+b^2$.

(iii) Apply the last result to show (or prove in any manner) that

$$\frac{f(x)}{[(x-c)^2+b^2]^r \phi(x)}$$

can be expressed in the form

$$\frac{\chi(x)}{\phi(x)}+\sum_{n=1}^{n=r}\frac{P_n(x-c)+Q_n}{[(x-c)^2+b^2]^n},$$

r being a positive integer, $\chi(x)$ a rational integral function of x, and P_n and Q_n constants. [I.C.S., 1892.]

44. If
$$\frac{F(x)}{f(x)} \equiv \phi(x)+\frac{\psi(x)}{f(x)},$$
where F, f, ϕ, ψ are rational polynomials of degrees $m+n$, n, m, $n-1$ respectively, show that if $a_1, a_2, a_3, \ldots, a_n$ be the roots of $f(x)=0$, considered all different, $\psi(x)$ will be determinable from

$$\begin{vmatrix} \psi(x), & x^{n-1}, & x^{n-2}, \ldots, & x, & 1 \\ F(a_1), & a_1^{n-1}, & a_1^{n-2}, \ldots, & a_1, & 1 \\ F(a_2), & a_2^{n-1}, & a_2^{n-2}, \ldots, & a_2, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F(a_n), & a_n^{n-1}, & a_n^{n-2}, \ldots, & a_n, & 1 \end{vmatrix} = 0.$$

Also determine $\psi(x)$ when $f(x)=0$ has equal roots.

[Oxford I. P., 1913.]

45. Integrate $\displaystyle\int\left\{\frac{ax^2+bx+c}{(x-\alpha)(x-\beta)(x-\gamma)}\right\}^2 dx.$ [Oxford I. P., 1917.]

46. Prove that if n be a positive integer,

(i) $\displaystyle\int\frac{\cos(n-2p)\theta}{\sin n\theta}\,d\theta=\frac{1}{n}\sum_{r=1}^{r=n}\cos\frac{2pr\pi}{n}\log\sin\left(\theta-\frac{r\pi}{n}\right),$

(ii) $\displaystyle\int\frac{\sin(n-2p)\theta}{\sin n\theta}\,d\theta=\frac{1}{n}\sum_{r=1}^{r=(n-1)}\sin\frac{2pr\pi}{n}\log\operatorname{cosec}\left(\theta-\frac{r\pi}{n}\right).$

47. Integrate (i) $\displaystyle\int\frac{dx}{1+e^x}$, (ii) $\displaystyle\int\frac{dx}{\sqrt{\sin^3x\sin(x+\alpha)}}$,

and prove that (iii) $\displaystyle\int_0^\infty\frac{x\,dx}{(1+x)(1+x^2)}=\frac{\pi}{4}$,

(iv) $\displaystyle\int_1^\infty\frac{(x^2+3)\,dx}{x^6(x^2+1)}=\frac{1}{30}(58-15\pi).$

[Math. Trip. I., 1917.]

48. Obtain the rational part of $\displaystyle\int\frac{dx}{(x^5+1)^3}$. [Math. Trip. II., 1915.]

49. Prove that

$$\frac{2n}{(1+x)^{2n}+(1-x)^{2n}}=\sum_{r=0}^{r=n}\frac{\sin a_r \cos^{2n-2} a_r}{\sin(2n-1)a_r}\cdot\frac{1}{x^2+\tan^2 a_r},$$

where $a_r=(2r+1)\pi/4n$. [Oxf. II. P., 1899.]

Write down the values of the integrals

$$\int\frac{dx}{(1+x)^{2n}+(1-x)^{2n}},\quad \int\frac{x\,dx}{(1+x)^{2n}+(1-x)^{2n}}.$$

50. Show that

$$\int_0^\infty\frac{x\,dx}{(a+x)^n-(a-x)^n}=\frac{\pi}{2na^{n-2}}\Sigma(-1)^{\lambda-1}\sin\frac{\lambda\pi}{n}\cos^{n-3}\frac{\lambda\pi}{n},$$

the summation extending from $\lambda=1$ to $\lambda=\frac{n-1}{2}$ or to $\lambda=\frac{n-2}{2}$, according as n is odd or even.

[Cf. Wolstenholme's *Problems*, No. 1912.]

Write down the value of the integral

$$\int\frac{x^2dx}{(a+x)^n-(a-x)^n}.$$

51. Show that if n be even and $x+y=1$,

$$\int\frac{dx}{x^ny^n}=\frac{1}{n-1}\left[\frac{1}{y^{n-1}}-\frac{1}{x^{n-1}}\right]+\frac{n}{1}\cdot\frac{1}{n-2}\left[\frac{1}{y^{n-2}}-\frac{1}{x^{n-2}}\right]$$
$$+\frac{n(n+1)}{1\,.\,2}\cdot\frac{1}{n-3}\left[\frac{1}{y^{n-3}}-\frac{1}{x^{n-3}}\right]+\dots+\frac{n(n+1)\dots(2n-2)}{1\,.\,2\dots(n-1)}\log\frac{x}{y}.$$

[Murphy, *Camb. Tr.*, vi.]

52. Show that if $p<q$,

$$Lt_{n=\infty}\frac{\prod_{r=1}^{r=n}\left(1-\frac{p^2x^2}{r^2\pi^2}\right)}{\prod_{r=1}^{r=n}\left(1-\frac{q^2x^2}{r^2\pi^2}\right)}=\sum_{r=1}^{r=\infty}\frac{2r\pi}{pq}\,\frac{\sin\left(\frac{p}{q}r\pi\right)}{\cos r\pi}\,\frac{1}{x^2-\frac{r^2\pi^2}{q^2}}.$$

[Todhunter, *I.C.*, p. 38.]

Deduce that if $p<q$,

$$\int\frac{\sin px}{\sin qx}dx=-2\sum_{r=1}^{r=\infty}\frac{1}{q}\,\frac{\sin\left(\frac{p}{q}r\pi\right)}{\cos r\pi}\tanh^{-1}\frac{qx}{r\pi}\quad(-\pi<qx<\pi).$$

CHAPTER VI.

INTEGRALS OF FORMS $\int \frac{dx}{(a+b\cos x+c\sin x)^n}$, etc

170. Integration of forms

$$\int \frac{dx}{a+b\cos x},\quad \int \frac{dx}{a+b\sin x},\quad \int \frac{dx}{a+b\cos x+c\sin x}, \text{ etc}$$

To integrate $\int \frac{dx}{a+b\cos x}$, we may write $a+b\cos x$ as

$$a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right),$$

$$\textit{i.e.}\ (a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2},$$

or
$$(a-b)\cos^2\frac{x}{2}\left[\frac{a+b}{a-b}+\tan^2\frac{x}{2}\right].$$

Thus
$$\int \frac{dx}{a+b\cos x}=\frac{2}{a-b}\int \frac{\frac{1}{2}\sec^2\frac{x}{2}\,dx}{\frac{a+b}{a-b}+\tan^2\frac{x}{2}}. \qquad \text{(1)}$$

171. Case I. If $a^2>b^2$, this becomes

$$\frac{2}{a-b}\frac{1}{\sqrt{\frac{a+b}{a-b}}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{\frac{a+b}{a-b}}}$$

or
$$\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2},$$

$$\textit{i.e.}\ \frac{2}{a\sin\alpha}\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{x}{2}\right), \text{ where } b=a\cos\alpha.$$

This may be written in other forms:

e.g. since $$2\tan^{-1} z=\cos^{-1}\frac{1-z^2}{1+z^2},$$

we may write the result as

$$\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{1-\dfrac{a-b}{a+b}\tan^2\dfrac{x}{2}}{1+\dfrac{a-b}{a+b}\tan^2\dfrac{x}{2}}$$

or $$\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b+a\cos x}{a+b\cos x}=\frac{1}{a\sin\alpha}\cos^{-1}\frac{\cos\alpha+\cos x}{1+\cos\alpha\cos x}.$$

Further forms are:

$$\frac{2}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a-b}\sin\dfrac{x}{2}}{\sqrt{a+b\cos x}}\quad\text{or}\quad\frac{2}{\sqrt{a^2-b^2}}\cos^{-1}\frac{\sqrt{a+b}\cos\dfrac{x}{2}}{\sqrt{a+b\cos x}},$$

or $$\frac{1}{\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}\sin x}{a+b\cos x},\text{ etc.}$$

172. Case II. If $a^2<b^2$, writing the integral in the form

$$\frac{2}{b-a}\int\frac{d\tan\dfrac{x}{2}}{\dfrac{b+a}{b-a}-\tan^2\dfrac{x}{2}}$$

in place of the form (1), we have by Art. 127,

$$\int\frac{dx}{a+b\cos x}=\frac{2}{b-a}\,\frac{1}{2\sqrt{\dfrac{b+a}{b-a}}}\log\frac{\sqrt{\dfrac{b+a}{b-a}}+\tan\dfrac{x}{2}}{\sqrt{\dfrac{b+a}{b-a}}-\tan\dfrac{x}{2}}$$

$$=\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b+a}+\sqrt{b-a}\tan\dfrac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\dfrac{x}{2}}$$

$$=\frac{1}{a\tan\alpha}\log\frac{\cos\dfrac{x-\alpha}{2}}{\cos\dfrac{x+\alpha}{2}},$$

where $b=a\sec\alpha$.

By Art. 64, this may also be written as

$$\frac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\sqrt{\frac{b-a}{b+a}}\tan\frac{x}{2} \quad \text{or} \quad \frac{2}{a\tan\alpha}\tanh^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{x}{2}\right)$$

or, since $$2\tanh^{-1}z=\cosh^{-1}\frac{1+z^2}{1-z^2},$$

we may still further exhibit the result as

$$\frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{1+\dfrac{b-a}{b+a}\tan^2\dfrac{x}{2}}{1-\dfrac{b-a}{b+a}\tan^2\dfrac{x}{2}}$$

or

$$\frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\cos x}{a+b\cos x}, \quad \textit{i.e.} \quad \frac{1}{a\tan\alpha}\cosh^{-1}\frac{1+\cos\alpha\cos x}{\cos\alpha+\cos x},$$

and in other but equivalent forms as in Case I.

173. We therefore have

$$\int\frac{dx}{a+b\cos x}=\begin{cases}\left.\begin{array}{l}\dfrac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2},\\ \textit{i.e.}\ \dfrac{1}{\sqrt{a^2-b^2}}\cos^{-1}\dfrac{b+a\cos x}{a+b\cos x},\end{array}\right\}a^2>b^2,\\ \qquad\qquad\text{or}\\ \left.\begin{array}{l}\dfrac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\sqrt{\dfrac{b-a}{b+a}}\tan\dfrac{x}{2},\\ \textit{i.e.}\ \dfrac{1}{\sqrt{b^2-a^2}}\log\dfrac{\sqrt{b+a}+\sqrt{b-a}\tan\dfrac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\dfrac{x}{2}},\\ \text{or}\ \dfrac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\dfrac{b+a\cos x}{a+b\cos x},\end{array}\right\}a^2<b^2,\end{cases}$$

with many other forms.

174. In the cases $b=\pm a$, the integral is at once obtainable, for

$$\int\frac{dx}{a+a\cos x}=\frac{1}{2a}\int\sec^2\frac{x}{2}\,dx=\frac{1}{a}\tan\frac{x}{2}$$

and $$\int\frac{dx}{a-a\cos x}=\frac{1}{2a}\int\operatorname{cosec}^2\frac{x}{2}\,dx=-\frac{1}{a}\cot\frac{x}{2}.$$

175. The integration of $\int \frac{dx}{a+b\sin x}$ is reduced to the foregoing forms by the substitution $x=\frac{\pi}{2}+y$, when we have

$$\int \frac{dx}{a+b\sin x}=\int \frac{dy}{a+b\cos y}$$

$$\left.\begin{aligned} &=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\\ \text{or}\quad &=\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b+a\sin x}{a+b\sin x}\\ &=\frac{1}{a\cos\alpha}\cos^{-1}\frac{\sin\alpha+\sin x}{1+\sin\alpha\sin x},\end{aligned}\right\} a^2>b^2,$$

where $b=a\sin\alpha$,

$$\left.\begin{aligned} \text{or}\quad &\frac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\sqrt{\frac{b-a}{b+a}}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\\ \text{or}\quad &\frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}\\ \text{or}\quad &\frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\sin x}{a+b\sin x}\\ &=\frac{1}{a\cot\alpha}\cosh^{-1}\frac{1+\sin\alpha\sin x}{\sin\alpha+\sin\alpha},\end{aligned}\right\} a^2<b^2,$$

where $b=a\operatorname{cosec}\alpha$, with many other forms.

176. We might also treat $\int \frac{dx}{a+b\sin x}$ independently.

Proceeding in the same way as for $\int \frac{dx}{a+b\cos x}$, we write

$$a+b\sin x=a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+2b\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=a\cos^2\frac{x}{2}\left[\left(\tan\frac{x}{2}+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a^2}\right].$$

Thus,

$$\int \frac{dx}{a+b\sin x}=\frac{2}{a}\int \frac{d\left(\tan\frac{x}{2}\right)}{\left(\tan\frac{x}{2}+\frac{b}{a}\right)^2+\frac{a^2-b^2}{a^2}},$$

and two cases arise as before, viz. $a \gtrless b$, when we apply Art. 127;

$$\therefore \int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}, \quad a^2 > b^2,$$

or

$$-\frac{1}{\sqrt{b^2-a^2}} \coth^{-1} \frac{a\tan\frac{x}{2}+b}{\sqrt{b^2-a^2}}, \quad a^2 < b^2,$$

showing the result in different forms from those already given, but of course differing from them only by quantities independent of x. The student should consider this statement and reconcile the results, as it is a matter of some little ingenuity.

177. **Extension.** Again, since $b\cos x + c\sin x$ may be written as $R\cos(x-\gamma)$, where $R=\sqrt{b^2+c^2}$ and $\tan\gamma = \frac{c}{b}$, we may deduce $\int \frac{dx}{a+b\cos x+c\sin x}$ from $\int \frac{dx}{a+b\cos x}$, or we may proceed independently, at our pleasure. Adopting the former course, we have

$$\int \frac{dx}{a+b\cos x+c\sin x}$$

$$= \int \frac{d(x-\gamma)}{a+R\cos(x-\gamma)}$$

$$\left.\begin{aligned} &= \frac{2}{\sqrt{a^2-R^2}} \tan^{-1} \sqrt{\frac{a-R}{a+R}} \tan\frac{x-\gamma}{2} \\ \text{or} \quad &= \frac{1}{\sqrt{a^2-R^2}} \cos^{-1} \frac{R+a\cos\overline{x-\gamma}}{a+R\cos\overline{x-\gamma}} \end{aligned}\right\} \text{if } a^2 > R^2$$

$$\left.\begin{aligned} \text{or} \quad & \frac{2}{\sqrt{R^2-a^2}} \tanh^{-1} \sqrt{\frac{R-a}{R+a}} \tan\frac{x-\gamma}{2}, \\ \textit{i.e.} \quad & \frac{1}{\sqrt{R^2-a^2}} \log \frac{\sqrt{R+a}+\sqrt{R-a}\tan\frac{x-\gamma}{2}}{\sqrt{R+a}-\sqrt{R-a}\tan\frac{x-\gamma}{2}} \\ \text{or} \quad & \frac{1}{\sqrt{R^2-a^2}} \cosh^{-1} \frac{R+a\cos\overline{x-\gamma}}{a+R\cos\overline{x-\gamma}}, \end{aligned}\right\} \text{if } a^2 < R^2$$

with other forms.

And these of course include the forms of Arts. 171 to 176 as particular cases, viz. when $c=0$ or $b=0$.

178. The reduction to the form

$$\int \frac{dx}{a+b\cos x}$$

has the advantage of making the integral depend upon the integration of

$$\int \frac{dx}{x^2 \pm k^2},$$

whilst the independent treatment throws the integration upon the form

$$\int \frac{dx}{ax^2+2bx+c},$$

and involves the completion of the square in the denominator.

179. **Illustrative Examples.**

Ex. 1.

$$\int \frac{dx}{3+5\cos x} = \int \frac{dx}{3\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+5\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{8\cos^2\frac{x}{2}-2\sin^2\frac{x}{2}}$$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{4-\tan^2\frac{x}{2}}\,dx$$

$$= \frac{1}{4}\int\left(\frac{1}{2-\tan\frac{x}{2}}+\frac{1}{2+\tan\frac{x}{2}}\right)d\tan\frac{x}{2}$$

$$= \frac{1}{4}\log\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}} = \frac{1}{2}\tanh^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right) = \frac{1}{4}\cosh^{-1}\frac{5+3\cos x}{3+5\cos x}.$$

Ex. 2.

$$\int \frac{dx}{3-5\cos x} = \int \frac{dy}{3+5\cos y}, \text{ where } x=\pi+y$$

$$= \tfrac{1}{4}\cosh^{-1}\frac{5+3\cos y}{3+5\cos y} = \tfrac{1}{4}\cosh^{-1}\frac{5-3\cos x}{3-5\cos x}.$$

Ex. 3.

$$\int\frac{dx}{5+3\cos x}=\int\frac{dx}{5\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+3\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=\int\frac{dx}{8\cos^2\frac{x}{2}+2\sin^2\frac{x}{2}}=\frac{1}{2}\int\frac{\sec^2\frac{x}{2}dx}{4+\tan^2\frac{x}{2}}$$

$$=\tfrac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)=\tfrac{1}{4}\cos^{-1}\frac{3+5\cos x}{5+3\cos x}.$$

Ex 4. $$\int\frac{dx}{5+3\sin x}=\int\frac{dy}{5+3\cos y},\quad\text{where } x=\frac{\pi}{2}+y,$$

$$=\tfrac{1}{4}\cos^{-1}\frac{3+5\cos y}{5+3\cos y}=\tfrac{1}{4}\cos^{-1}\frac{3+5\sin x}{5+3\sin x}.$$

Ex. 5. $$\int\frac{dx}{13+3\cos x+4\sin x}=\int\frac{dx}{13+5\cos(x-a)},\quad\text{where }\tan a=\tfrac{4}{3},$$

$$=\frac{1}{12}\cos^{-1}\frac{5+13\cos(x-a)}{13+5\cos(x-a)}=\frac{1}{6}\tan^{-1}\left(\frac{2}{3}\tan\frac{x-a}{2}\right).$$

180. The integrals

$$\int\frac{dx}{a+b\cosh x},\quad\int\frac{dx}{a+b\sinh x},\quad\int\frac{dx}{a+b\cosh x+c\sinh x}$$

may be treated similarly.

Thus,

$$\int\frac{dx}{a+b\cosh x}=\int\frac{dx}{a\left(\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}\right)+b\left(\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}\right)}$$

$$=\frac{2}{b-a}\int\frac{d\tanh\frac{x}{2}}{\frac{b+a}{b-a}+\tanh^2\frac{x}{2}}$$

or $$\frac{2}{a-b}\int\frac{d\tanh\frac{x}{2}}{\frac{a+b}{a-b}-\tanh^2\frac{x}{2}}.$$

Hence, if $a^2<b^2$, we have the forms

$$\frac{2}{\sqrt{b^2-a^2}}\tan^{-1}\sqrt{\frac{b-a}{b+a}}\tanh\frac{x}{2}\quad\text{or}\quad\frac{1}{\sqrt{b^2-a^2}}\cos^{-1}\frac{b+a\cosh x}{a+b\cosh x};$$

and if $a^2>b^2$,

$$\frac{2}{\sqrt{a^2-b^2}}\tanh^{-1}\sqrt{\frac{a-b}{a+b}}\tanh\frac{x}{2}\quad\text{or}\quad\frac{1}{\sqrt{a^2-b^2}}\cosh^{-1}\frac{b+a\cosh x}{a+b\cosh x}.$$

Again,

$$\int \frac{dx}{a+b\sinh x} = \int \frac{dx}{a\left(\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}\right)+2b\sinh\frac{x}{2}\cosh\frac{x}{2}}$$

$$= \frac{2}{a}\int \frac{d\tanh\frac{x}{2}}{\frac{a^2+b^2}{a^2}-\left(\tanh\frac{x}{2}-\frac{b}{a}\right)^2}$$

$$= \frac{2}{\sqrt{a^2+b^2}}\tanh^{-1}\left(\frac{a\tanh\frac{x}{2}-b}{\sqrt{a^2+b^2}}\right);$$

and other forms will be exhibited later.

Similarly, in the **general case,**

$$\int \frac{dx}{a+b\cosh x+c\sinh x}$$

$$= \int \frac{dx}{a\left(\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}\right)+b\left(\cosh^2\frac{x}{2}+\sinh^2\frac{x}{2}\right)+2c\sinh\frac{x}{2}\cosh\frac{x}{2}}$$

$$= \int \frac{\operatorname{sech}^2\frac{x}{2}\,dx}{a+b+2c\tanh\frac{x}{2}-(a-b)\tanh^2\frac{x}{2}}$$

$$= \frac{2}{a-b}\int \frac{d\tanh\frac{x}{2}}{\left\{\frac{a+b}{a-b}+\frac{c^2}{(a-b)^2}\right\}-\left(\tanh\frac{x}{2}-\frac{c}{a-b}\right)^2}$$

or $$\frac{2}{b-a}\int \frac{d\tanh\frac{x}{2}}{\left\{\frac{a+b}{b-a}-\frac{c^2}{(b-a)^2}\right\}+\left(\tanh\frac{x}{2}+\frac{c}{b-a}\right)^2}$$

$$= \frac{2}{\sqrt{a^2-b^2+c^2}}\tanh^{-1}\frac{(a-b)\tanh\frac{x}{2}-c}{\sqrt{a^2-b^2+c^2}}, \quad a^2+c^2>b^2,$$

or $$\frac{2}{\sqrt{b^2-a^2-c^2}}\tan^{-1}\frac{(b-a)\tanh\frac{x}{2}+c}{\sqrt{b^2-a^2-c^2}}, \quad a^2+c^2<b^2.$$

But we notice also that just as $a+b\cos\theta+c\sin\theta$ may be written

$$a+R\cos\overline{\theta-a}$$

by putting $b=R\cos a$ and $c=R\sin a$, where $R=\sqrt{b^2+c^2}$ and $\tan a=\frac{c}{b}$, we may write

$$a+b\cosh x+c\sinh x \quad \text{as} \quad a+R\cosh\overline{x+\gamma}$$

by putting $b=R\cosh\gamma$ and $c=R\sinh\gamma$ if $b^2>c^2$, where $R=\sqrt{b^2-c^2}$ and $\tanh\gamma=\frac{c}{b}$, or as

$$a+R\sinh\overline{x+\gamma}$$

by putting $b=R\sinh\gamma$, $c=R\cosh\gamma$, where $R=\sqrt{c^2-b^2}$ and $\tanh\gamma=\frac{b}{c}$ when $b^2<c^2$, and therefore the case may be regarded as one of the previous ones or *vice versâ*.

181. **Another Method.** A further method of treatment will be obvious if we remember that these hyperbolic functions are merely functions of a real exponential.

Taking the general integral in this way, we have

$$\int\frac{dx}{a+b\cosh x+c\sinh x}=\int\frac{2dx}{2a+b(e^x+e^{-x})+c(e^x-e^{-x})}$$

$$=\int\frac{2e^x\,dx}{(b+c)e^{2x}+2ae^x+b-c}$$

$$=\frac{2}{b+c}\int\frac{de^x}{\left(e^x+\frac{a}{b+c}\right)^2+\frac{b^2-c^2-a^2}{(b+c)^2}}$$

or

$$\frac{2}{b+c}\int\frac{de^x}{\left(e^x+\frac{a}{b+c}\right)^2-\frac{a^2+c^2-b^2}{(b+c)^2}},$$

giving the forms

$$\frac{2}{\sqrt{b^2-c^2-a^2}}\tan^{-1}\frac{(b+c)e^x+a}{\sqrt{b^2-c^2-a^2}} \quad \text{if } b^2>a^2+c^2$$

or

$$-\frac{2}{\sqrt{a^2+c^2-b^2}}\coth^{-1}\frac{(b+c)e^x+a}{\sqrt{a^2+c^2-b^2}} \quad \text{if } b^2<a^2+c^2.$$

Comparing with the results of Art. 180, it will be remarked that the integrals of such expressions differ much in appearance

according to the method adopted in integration. Integrals of the same expression, however, can only differ by a quantity (real or unreal) which does not contain x, and it will be a useful exercise to deduce one form from another; and, as has been said previously, this will sometimes require some ingenuity.

182. **The Integration expressed in terms of the Integrand.** Far more symmetry, however, will be obtained in the results if we attempt to express the integration in terms of the integrand, as we now proceed to show.

These integrals may be deduced from the form

$$\int \frac{dx}{\sqrt{Ax^2+2Bx+C}},$$

which is

$$\frac{1}{\sqrt{A}} \cosh^{-1} \frac{Ax+B}{\sqrt{B^2-AC}}, \quad A>0,\ B^2>AC \text{ (Arts. 80 and 81)},$$

or $$-\frac{1}{\sqrt{-A}} \cos^{-1} \frac{Ax+B}{\sqrt{B^2-AC}}, \quad A<0,\ B^2>AC,$$

or $$\frac{1}{\sqrt{A}} \sinh^{-1} \frac{Ax+B}{\sqrt{AC-B^2}}, \quad A>0,\ B^2<AC,$$

the case $A<0$, $B^2<AC$ being omitted because the radical in the integrand becomes unreal in that case.

The rule is to *substitute y for the integrand in all cases and integrate in terms of y.* This method leads to remarkable symmetry of form, and expresses the result in terms of the integrand itself, and yields new forms for the integration.

Thus, considering the general case, and writing

$$\int \frac{d\theta}{a+b\cos\theta+c\sin\theta} = \int y\, d\theta,$$

where $$\frac{1}{a+b\cos\theta+c\sin\theta} = y,$$

we have $$b\cos\theta+c\sin\theta = \frac{1}{y} - a;$$

and therefore $$b\sin\theta - c\cos\theta = \frac{1}{y^2}\frac{dy}{d\theta}.$$

Squaring and adding,

$$b^2+c^2-a^2=\frac{1}{y^2}-\frac{2a}{y}+\frac{1}{y^4}\left(\frac{dy}{d\theta}\right)^2.$$

Hence

$$\int y\,d\theta=\pm\int\frac{dy}{\sqrt{(b^2+c^2-a^2)\,y^2+2ay-1}}$$

$$=\pm\frac{1}{\sqrt{b^2+c^2-a^2}}\cosh^{-1}\frac{(b^2+c^2-a^2)\,y+a}{\sqrt{b^2+c^2}}\quad\text{if } b^2+c^2>a^2$$

or $$=\pm\frac{1}{\sqrt{a^2-b^2-c^2}}\cos^{-1}\frac{(b^2+c^2-a^2)\,y+a}{\sqrt{b^2+c^2}}\quad\text{if } b^2+c^2<a^2,$$

where $y^{-1}=a+b\cos\theta+c\sin\theta$.

The sign is to be determined by examining whether y increases or decreases with θ.

If y and θ increase together, $\frac{dy}{d\theta}$ is $+$; *e.g.* in $\int\frac{d\theta}{a+b\cos\theta}$, provided it be a case where b is $+^{\text{ve}}$ and in which $0<\theta<\frac{\pi}{2}$ throughout the integration, we use a $+$, for in the first quadrant as θ increases $\cos\theta$ diminishes; $\therefore\ \frac{1}{a+b\cos\theta}$ increases, that is, y increases.

In $\int\frac{d\theta}{a+b\sin\theta}$, supposing θ to lie in the first quadrant throughout the integration, we should use the $-$ sign.

183. In the same way, to integrate

$$\int\frac{dx}{a+b\cosh x+c\sinh x}\quad\text{or}\quad\int y\,dx,\text{ say,}$$

where $$\frac{1}{a+b\cosh x+c\sinh x}=y,$$

we have $$b\cosh x+c\sinh x=\frac{1}{y}-a,$$

$$b\sinh x+c\cosh x=-\frac{1}{y^2}\frac{dy}{dx}.$$

Squaring and subtracting,

$$b^2-c^2-a^2=\frac{1}{y^2}-\frac{2a}{y}-\frac{1}{y^4}\left(\frac{dy}{dx}\right)^2,$$

and taking the case b and c both positive, y decreases as x increases;

$$\therefore \int y\,dx = -\int \frac{dy}{\sqrt{(a^2+c^2-b^2)\,y^2-2ay+1}}$$

$$= -\frac{1}{\sqrt{a^2+c^2-b^2}}\cosh^{-1}\frac{(a^2+c^2-b^2)\,y-a}{\sqrt{b^2-c^2}}$$

$$= \frac{1}{\sqrt{a^2+c^2-b^2}}\cosh^{-1}\frac{a-(a^2+c^2-b^2)\,y}{\sqrt{b^2-c^2}}+\text{const.}$$

if $b^2>c^2$ and $a^2+c^2>b^2$

or $$= -\frac{1}{\sqrt{-a^2-c^2+b^2}}\cos^{-1}\frac{(a^2+c^2-b^2)\,y-a}{\sqrt{b^2-c^2}}$$

$$= \frac{1}{\sqrt{-a^2-c^2+b^2}}\cos^{-1}\frac{a-(a^2+c^2-b^2)y}{\sqrt{b^2-c^2}}+\text{const.}$$

if $a^2+c^2<b^2$

or $$= -\frac{1}{\sqrt{a^2+c^2-b^2}}\sinh^{-1}\frac{(a^2+c^2-b^2)\,y-a}{\sqrt{c^2-b^2}}$$

$$= \frac{1}{\sqrt{a^2+c^2-b^2}}\sinh^{-1}\frac{a-(a^2+c^2-b^2)\,y}{\sqrt{c^2-b^2}} \quad \text{if } b^2<c^2,$$

where $y^{-1}\equiv a+b\cosh x+c\sinh x$.

184. Hence we get the following **particular results** by putting b or $c=0$ in the general results of Arts. 182, 183,

$$\int\frac{d\theta}{a+b\cos\theta} = \frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\cos\theta}{a+b\cos\theta} \quad (b^2>a^2)$$

$$\text{or} = \frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b+a\cos\theta}{a+b\cos\theta} \quad (b^2<a^2),$$

$$\int\frac{d\theta}{a+b\sin\theta} = -\frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\sin\theta}{a+b\sin\theta} \quad (b^2>a^2)$$

$$\text{or} = \frac{1}{\sqrt{a^2-b^2}}\sin^{-1}\frac{b+a\sin\theta}{a+b\sin\theta} \quad (b^2<a^2),$$

$$\int\frac{dx}{a+b\cosh x} = \frac{1}{\sqrt{b^2-a^2}}\cos^{-1}\frac{b+a\cosh x}{a+b\cosh x} \quad (b^2>a^2)$$

$$\text{or} = \frac{1}{\sqrt{a^2-b^2}}\cosh^{-1}\frac{b+a\cosh x}{a+b\cosh x} \quad (b^2<a^2),$$

$$\int\frac{dx}{a+b\sinh x} = \frac{1}{\sqrt{a^2+b^2}}\sinh^{-1}\frac{-b+a\sinh x}{a+b\sinh x}.$$

The symmetrical form of the several results was given (without proof) by Greenhill in his *Chapter on the Integral Calculus*, p. 34.

When $a=0$ we arrive at results obtained earlier in other forms, viz.

$$\int\frac{d\theta}{\cos\theta}=\cosh^{-1}(\sec\theta)\ \text{(compare Art. 74)}$$

$$\int\frac{d\theta}{\sin\theta}=-\cosh^{-1}(\operatorname{cosec}\theta),$$

$$\int\frac{dx}{\cosh x}=\cos^{-1}(\operatorname{sech} x),$$

$$\int\frac{dx}{\sinh x}=\sinh^{-1}(-\operatorname{cosech} x)=-\sinh^{-1}(\operatorname{cosech} x);$$

and from the general results

$$\int\frac{d\theta}{b\cos\theta+c\sin\theta}=\frac{1}{\sqrt{b^2+c^2}}\operatorname{sech}^{-1}\frac{b\cos\theta+c\sin\theta}{\sqrt{b^2+c^2}},$$

$$\int\frac{d\theta}{b\cosh x+c\sinh x}=\frac{1}{\sqrt{b^2-c^2}}\sec^{-1}\frac{b\cosh x+c\sinh x}{\sqrt{b^2-c^2}}\quad\text{if } b^2>c^2$$

$$=-\frac{1}{\sqrt{c^2-b^2}}\operatorname{cosech}^{-1}\frac{b\cosh x+c\sinh x}{\sqrt{c^2-b^2}}\quad\text{if } b^2<c^2$$

or again, $$=\frac{2}{\sqrt{b^2-c^2}}\tan^{-1}\sqrt{\frac{b+c}{b-c}}e^x\quad\text{if } b^2>c^2,$$

or $$-\frac{2}{\sqrt{c^2-b^2}}\coth^{-1}\sqrt{\frac{c+b}{c-b}}e^x\quad\text{if } b^2<c^2,$$

forms which the student should compare with those previously obtained.

185. **Reduction formulae for integrals of form $I_n\equiv\int\frac{dx}{X^n}$, where**

$$X=a+b\,{}^{\cos}_{\sin}\,x.$$

Let us consider the case

$$I_2\equiv\int\frac{dx}{(a+b\cos x)^2}.$$

We shall connect the integral with another, viz.

$$I_1\equiv\int\frac{dx}{a+b\cos x}.$$

Put $$P\equiv\frac{\sin x}{a+b\cos x}.$$

[*Note.*—That is, to form P, $\sin x$ is introduced into the numerator of the integrand of I_2, and the index of the denominator is lowered by unity.]

Thus $$\frac{dP}{dx}=\frac{\cos x(a+b\cos x)+b(1-\cos^2 x)}{(a+b\cos x)^2}$$

$$=\frac{b+a\cos x}{(a+b\cos x)^2}=\frac{b-\frac{a^2}{b}+\frac{a}{b}(a+b\cos x)}{(a+b\cos x)^2}$$

$$=\frac{a}{b}\frac{1}{a+b\cos x}-\frac{a^2-b^2}{b}\frac{1}{(a+b\cos x)^2}.$$

Therefore integrating,

$$\frac{\sin x}{a+b\cos x}=\frac{a}{b}I_1-\frac{a^2-b^2}{b}I_2.$$

Hence $$I_2=-\frac{b}{a^2-b^2}\frac{\sin x}{a+b\cos x}+\frac{a}{a^2-b^2}I_1,$$

and I_1 has been given in various forms in Art. 173, *e.g.*

$$\frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\frac{b+a\cos x}{a+b\cos x} \quad \text{or} \quad \frac{1}{\sqrt{b^2-a^2}}\cosh^{-1}\frac{b+a\cos x}{a+b\cos x},$$

according as a^2 is greater or less than b^2.

$$\therefore\ I_2=-\frac{b}{a^2-b^2}\frac{\sin x}{a+b\cos x}+\frac{a}{(a^2-b^2)^{\frac{3}{2}}}\cos^{-1}\frac{b+a\cos x}{a+b\cos x}(a^2>b^2),$$

or $$=-\frac{b}{a^2-b^2}\frac{\sin x}{a+b\cos x}-\frac{a}{(b^2-a^2)^{\frac{3}{2}}}\cosh^{-1}\frac{b+a\cos x}{a+b\cos x}\ (a^2<b^2).$$

186. Again, in the general case, if

$$I_n\equiv\int\frac{dx}{(a+b\cos x)^n},$$

put $$P\equiv\frac{\sin x}{(a+b\cos x)^{n-1}}.$$

Then

$$\frac{dP}{dx}=\frac{\cos x(a+b\cos x)+(n-1)b(1-\cos^2 x)}{(a+b\cos x)^n}$$

$$=\frac{A+B(a+b\cos x)+C(a+b\cos x)^2}{(a+b\cos x)^n}, \text{ say,}$$

where $$A+Ba+Ca^2=(n-1)b,$$

$$Bb+2Cab=a,$$

$$Cb^2=(2-n)b$$

giving $C=-\frac{(n-2)}{b}$, $B=\frac{a}{b}+2\frac{a}{b}(n-2)=(2n-3)\frac{a}{b}$,

$$A=\quad (n-1)b-(2n-3)\frac{a^2}{b}+(n-2)\frac{a^2}{b}$$
$$=-(n-1)\frac{a^2-b^2}{b}.$$

Hence, substituting these values and integrating,

$$\frac{\sin x}{(a+b\cos x)^{n-1}}=-(n-1)\frac{a^2-b^2}{b}I_n+(2n-3)\frac{a}{b}I_{n-1}-\frac{n-2}{b}I_{n-2}.$$

The reduction formula is then

$$I_n=-\frac{b}{(n-1)(a^2-b^2)}\frac{\sin x}{(a+b\cos x)^{n-1}}+\frac{2n-3}{n-1}\frac{a}{a^2-b^2}I_{n-1}$$
$$-\frac{n-2}{n-1}\frac{1}{a^2-b^2}I_{n-2}.$$

Thus, as I_1 and I_2 have already been found in finite terms, we can successively deduce the values of I_3, I_4, etc.

It will be noted that I_n is in this case shown to be dependent upon *two* integrals of lower order, viz. I_{n-1} and I_{n-2}, except when $n=2$.

Also, the result of Art. 185 could have been obtained by putting $n=2$ in the present result.

187. **Generalization of above method.**

As $\int\frac{dx}{(a+b\sin x)^n}$ reduces to $\int\frac{dy}{(a+b\cos y)^n}$ on substituting $\frac{\pi}{2}+y$ for x, and

$$\int\frac{dx}{(a+b\cos x+c\sin x)^n}$$

may be written as $\int\frac{d(x-\gamma)}{[a+R\cos(x-\gamma)]^n}$, where $R=\sqrt{b^2+c^2}$ and $\gamma=\tan^{-1}\frac{c}{b}$, it is usual to refer these integrals to the case considered in Art. 186. We may, however, establish a reduction formula independently for each case.

Taking $$I_n\equiv\int\frac{dx}{(a+b\cos x+c\sin x)^n}.$$

Let $$P\equiv\frac{-b\sin x+c\cos x}{(a+b\cos x+c\sin x)^{n-1}}$$

$$\left[i.e.\text{ if } D\equiv a+b\cos x+c\sin x,\ P=\frac{D'}{D^{n-1}}\right].$$

Then

$$\frac{dP}{dx}=\frac{-b\cos x-c\sin x}{(a+b\cos x+c\sin x)^{n-1}}-(n-1)\frac{(-b\sin x+c\cos x)^2}{(a+b\cos x+c\sin x)^n}$$

$$=\frac{-a(b\cos x+c\sin x)-(b\cos x+c\sin x)^2-(n-1)[b^2+c^2-(b\cos x+c\sin x)^2]}{(a+b\cos x+c\sin x)^n}$$

$$=\frac{A+B(a+b\cos x+c\sin x)+C(a+b\cos x+c\sin x)^2}{(a+b\cos x+c\sin x)^n}, \text{ say,}$$

where A, B, C are constants to be determined so that

$$\left.\begin{aligned} A+Ba+Ca^2&=-(n-1)(b^2+c^2),\\ B+2aC&=-a,\\ C&=n-2,\end{aligned}\right\}$$

whence $A=(n-1)(a^2-b^2-c^2)$, $B=-(2n-3)a$, $C=n-2$.

Therefore the proper reduction formula for I_n is

$$\frac{-b\sin x+c\cos x}{(a+b\cos x+c\sin x)^{n-1}}$$
$$=(n-1)(a^2-b^2-c^2)\,I_n-(2n-3)\,aI_{n-1}+(n-2)\,I_{n-2}.$$

We note that when $n=2$, the last term disappears, and

$$(a^2-b^2-c^2)\,I_2=aI_1+\frac{-b\sin x+c\cos x}{(a+b\cos x+c\sin x)},$$

i.e. $$(a^2-b^2-c^2)\int\frac{dx}{(a+b\cos x+c\sin x)^2}=\frac{-b\sin x+c\cos x}{a+b\cos x+c\sin x}+aI_1,$$

the real form of I_1 being selected from the various forms in Art. 177.

Also I_1 and I_2 now having been found, we can proceed to deduce I_3, I_4, etc., successively by aid of the reduction formula established.

188. **Corresponding formulae for the case of Hyperbolic Functions.**

In like manner reduction formulae for

$$\int\frac{dx}{(a+b\cosh x)^n},\quad \int\frac{dx}{(a+b\sinh x)^n},\quad \int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$

may be constructed.

As the last includes the first two as particular cases, we consider that one in particular, and proceed as before.

Put $$P \equiv \frac{b \sinh x + c \cosh x}{(a + b \cosh x + c \sinh x)^{n-1}}.$$

Then

$$\frac{dP}{dx} = \frac{(b \cosh x + c \sinh x)(a + b \cosh x + c \sinh x) - (n-1)(b \sinh x + c \cosh x)^2}{(a + b \cosh x + c \sinh x)^n}$$

$$= \frac{a(b \cosh x + c \sinh x) + (b \cosh x + c \sinh x)^2 - (n-1)[(b \cosh x + c \sinh x)^2 - (b^2 - c^2)]}{(a + b \cosh x + c \sinh x)^n}$$

$$= \frac{A + B(a + b \cosh x + c \sinh x) + C(a + b \cosh x + c \sinh x)^2}{(a + b \cosh x + c \sinh x)^n},$$

say,

where $$\left.\begin{aligned} A + Ba + Ca^2 &= (n-1)(b^2 - c^2), \\ B + 2Ca &= a, \\ C &= -(n-2), \end{aligned}\right\}$$

whence $A = (n-1)(-a^2 + b^2 - c^2)$, $B = (2n-3)a$, $C = -(n-2)$.

And the proper reduction formula is

$$\frac{b \sinh x + c \cosh x}{(a + b \cosh x + c \sinh x)^{n-1}} = (n-1)(-a^2 + b^2 - c^2) I_n + (2n-3) a I_{n-1} - (n-2) I_{n-2}.$$

As before, the last term disappears in the case $n = 2$.

Hence

$$(-a^2 + b^2 - c^2) I_2 = \frac{b \sinh x + c \cosh x}{a + b \cosh x + c \sinh x} - a I_1,$$

the real form of I_1 being selected from the various forms shown in Art. 180.

I_1 and I_2 being now known, we can proceed as before to deduce successively I_3, I_4, etc., by aid of the reduction formula.

189. Special Cases.

We notice also that, putting $a = 0$, $b = 0$ or $c = 0$, or two of them, in these reduction formulae, we have a mode of reduction for such expressions as

$$\int \operatorname{sech}^n x \, dx, \quad \int \operatorname{cosech}^n x \, dx, \quad \int \frac{dx}{(b \cosh x + c \sinh x)^n},$$

$$\int \frac{dx}{(b \cos x + c \sin x)^n}, \quad \int \frac{dx}{(a + b \sinh x)^n}, \quad \text{etc.}$$

190. Fractions of form

$$\frac{a+b\cos x+c\sin x}{a_1+b_1\cos x+c_1\sin x}.$$

The numerator of this fraction can be thrown into the form

$$A(a_1+b_1\cos x+c_1\sin x)+B(-b_1\sin x+c_1\cos x)+C$$

i.e. $\qquad A(\text{denr.})+B(\text{diff. co. of denr.})+C,$

by taking $\quad Aa_1+C=a,\quad Ab_1+Bc_1=b,\quad Ac_1-Bb_1=c,$
which determine A, B and C.

The fraction then takes the form

$$A+B\frac{-b_1\sin x+c_1\cos x}{a_1+b_1\cos x+c_1\sin x}+\frac{C}{a_1+b_1\cos x+c_1\sin x},$$

and the integral is

$$Ax+B\log(a_1+b_1\cos x+c_1\sin x)+C\int\frac{dx}{a_1+b_1\cos x+c_1\sin x},$$

and the last integral has been evaluated.

191. Extension of above Method.

In the same way $\dfrac{a+b\cos x+c\sin x}{(a_1+b_1\cos x+c_1\sin x)^n}$ may be arranged as

$$\frac{A}{(a_1+b_1\cos x+b_1\sin x)^{n-1}}+B\frac{-b_1\sin x+c_1\cos x}{(a_1+b_1\cos x+c_1\sin x)^n}$$
$$+\frac{C}{(a_1+b_1\cos x+c_1\sin x)^n}.$$

The integrals of the first and last fractions may be deduced by the reduction formula of Art. 187, and that of the second fraction is

$$-\frac{B}{n-1}\frac{1}{(a_1+b_1\cos x+c_1\sin x)^{n-1}}\quad (n>1).$$

192. Case of Hyperbolic Functions.

Exactly in the same way fractions of the forms

$$\frac{a+b\cosh x+c\sinh x}{a_1+b_1\cosh x+c_1\sinh x},\quad \frac{a+b\cosh x+c\sinh x}{(a_1+b_1\cosh x+c_1\sinh x)^n}$$

may be integrated.

193. Further Generalization.

If $$\prod_{r=1}^{r=n}(a_r+b_r\cos\theta+c_r\sin\theta)$$

stands for the product of n factors, some of which may be

repeated, and of which the one exhibited is a type, and if $\phi(x, y)$ be any rational integral algebraic function of x and y, the integral of

$$\int \frac{\phi(\cos\theta, \sin\theta)}{\prod_{r=1}^{r=n}(a_r + b_r\cos\theta + c_r\sin\theta)}\,d\theta$$

can now be found. For expressing $\cos\theta$ and $\sin\theta$ in terms of the tangent of the half angle, and writing $t = \tan\frac{\theta}{2}$,

$$\phi(\cos\theta, \sin\theta) = \phi\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) = \frac{\chi(t)}{(1+t^2)^p},$$

where p is the degree of $\phi(x, y)$ in x and y, not necessarily homogeneous, and $\chi(t)$ is a rational and integral algebraic function of t of degree $2p$ at most.

Also $\quad a_r + b_r\cos\theta + c_r\sin\theta = a_r + b_r\frac{1-t^2}{1+t^2} + c_r\frac{2t}{1+t^2}$,

whence

$$\prod_{r=1}^{r=n}(a_r + b_r\cos\theta + c_r\sin\theta) = \frac{\prod_{r=1}^{r=n}[a_r + b_r + 2c_r t + (a_r - b_r)t^2]}{(1+t^2)^n}.$$

also $$d\theta = \frac{2dt}{1+t^2}.$$

Hence

$$\frac{\phi(\cos\theta, \sin\theta)\,d\theta}{\prod_{r=1}(a_r + b_r\cos\theta + c_r\sin\theta)}$$

$$= \frac{2\chi(t)\,dt}{(1+t^2)^{-n+p+1}\prod_{r=1}^{r=n}(a_r + b_r + 2c_r t + \overline{a_r - b_r}t^2)},$$

and supposing $a_r \neq b_r$ for any of the values of r, the degree of $\chi(t)$ in t, *i.e.* $2p$, is lower than that of the denominator, which is $2(p+1-n)+2n$, *i.e.* $2p+2$.

This expression may then be put into partial fractions, some of type $\frac{A+Bt}{(1+t^2)^\lambda}$, others of type $\frac{C+Dt}{(F+Gt+Ht^2)^\mu}$.

The proper reduction formulae for such cases will be found in the next chapter. The integration can now be effected.

The reader may consider for himself the effect of $a_r = b_r$ for any value or values of r.

194. A different Method.

To obtain integrals of form

$$\int \frac{d\theta}{(a+b\cos\theta+c\sin\theta)^n} \quad \text{or} \quad \int \frac{dx}{(a+b\cosh x+c\sinh x)^n}$$

and their particular cases, **we may avoid the reduction formulae** referred to, and proceed as follows, using a reduction of different nature.

Consider the first of these.

Case $b^2+c^2>a^2$.

Taking

$$\int \frac{d\theta}{a+b\cos\theta+c\sin\theta}$$

$$= \pm\frac{1}{\sqrt{b^2+c^2-a^2}}\cosh^{-1}\frac{(b^2+c^2-a^2)y+a}{\sqrt{b^2+c^2}} = \frac{u}{\sqrt{b^2+c^2-a^2}}, \text{ say,}$$

where $y^{-1}=a+b\cos\theta+c\sin\theta$ and $b^2+c^2>a^2$ (Art. 182),

$$y\,d\theta=\frac{du}{\sqrt{b^2+c^2-a^2}} \quad\text{and}\quad (b^2+c^2-a^2)y=\sqrt{b^2+c^2}\cosh u-a;$$

$$\therefore\ y^n\,d\theta=\frac{(\sqrt{b^2+c^2}\cosh u-a)^{n-1}}{(b^2+c^2-a^2)^{\frac{2n-1}{2}}}du,$$

$$\int y^n\,d\theta=\frac{1}{(b^2+c^2-a^2)^{\frac{2n-1}{2}}}\int(\sqrt{b^2+c^2}\cosh u-a)^{n-1}\,du,$$

i.e. $$\int\frac{d\theta}{(a+b\cos\theta+c\sin\theta)^n}$$

$$=\frac{1}{(b^2+c^2-a^2)^{\frac{2n-1}{2}}}\int(\sqrt{b^2+c^2}\cosh u-a)^{n-1}\,du.$$

We may then expand $(\sqrt{b^2+c^2}\cosh u-a)^{n-1}$ and integrate each term, finally substituting back for u its value

$$\pm\cosh^{-1}\frac{(b^2+c^2-a^2)y+a}{\sqrt{b^2+c^2}},$$

i.e. $$\pm\cosh^{-1}\frac{1}{\sqrt{b^2+c^2}}\left[\frac{b^2+c^2-a^2}{a+b\cos\theta+c\sin\theta}+a\right],$$

the proper sign having been selected as indicated in Art. 182.

Case $b^2+c^2<a^2$.

$$\int\frac{d\theta}{a+b\cos\theta+c\sin\theta}$$

$$=\pm\frac{1}{\sqrt{a^2-b^2-c^2}}\cos^{-1}\frac{(b^2+c^2-a^2)y+a}{\sqrt{b^2+c^2}}=\frac{u}{\sqrt{a^2-b^2-c^2}},\text{ say}$$

i.e. $$y\,d\theta=\frac{du}{\sqrt{a^2-b^2-c^2}}\quad\text{and}\quad(a^2-b^2-c^2)\,y=a-\sqrt{b^2+c^2}\cos u;$$

$$\therefore\ y^n\,d\theta=\frac{(a-\sqrt{b^2+c^2}\cos u)^{n-1}}{(a^2-b^2-c^2)^{\frac{2n-1}{2}}}du,$$

i.e. $$\int\frac{d\theta}{(a+b\cos\theta+c\sin\theta)^n}$$

$$=\frac{1}{(a^2-b^2-c^2)^{\frac{2n-1}{2}}}\int(a-\sqrt{b^2+c^2}\cos u)^{n-1}\,du.$$

195. In exactly the same way, from the three forms (where $y^{-1}=a+b\cosh x+c\sinh x$)

$$\int\frac{dx}{a+b\cosh x+c\sinh x}$$

$$=\frac{1}{\sqrt{-a^2-c^2+b^2}}\cos^{-1}\frac{a-(a^2+c^2-b^2)y}{\sqrt{b^2-c^2}}=\frac{u}{\sqrt{-a^2-c^2+b^2}},$$

where $b^2>a^2+c^2$;

or $$=\frac{1}{\sqrt{a^2+c^2-b^2}}\cosh^{-1}\frac{a-(a^2+c^2-b^2)y}{\sqrt{b^2-c^2}}=\frac{u}{\sqrt{a^2+c^2-b^2}},$$

where $a^2+c^2>b^2>c^2$;

or $$=\frac{1}{\sqrt{a^2+c^2-b^2}}\sinh^{-1}\frac{a-(a^2+c^2-b^2)y}{\sqrt{c^2-b^2}}=\frac{u}{\sqrt{a^2+c^2-b^2}},$$

where $b^2<c^2$

we obtain respectively,

Case $b^2>a^2+c^2$,

$$\int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$

$$=\frac{1}{(-a^2-c^2+b^2)^{\frac{2n-1}{2}}}\int(\sqrt{b^2-c^2}\cos u-a)^{n-1}\,du,$$

where $$a-\frac{a^2+c^2-b^2}{a+b\cosh x+c\sinh x}=\sqrt{b^2-c^2}\cos u.$$

Case $a^2+c^2>b^2>c^2$,

$$\int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$
$$=\frac{1}{(a^2+c^2-b^2)^{\frac{2n-1}{2}}}\int(a-\sqrt{b^2-c^2}\cosh u)^{n-1}\,du,$$
$$\text{where } a-\frac{a^2+c^2-b^2}{a+b\cosh x+c\sinh x}=\sqrt{b^2-c^2}\cosh u.$$

Case $c^2>b^2$,

$$\int\frac{dx}{(a+b\cosh x+c\sinh x)^n}$$
$$=\frac{1}{(a^2+c^2-b^2)^{\frac{2n-1}{2}}}\int(a-\sqrt{c^2-b^2}\sinh u)^{n-1}\,du,$$
$$\text{where } a-\frac{a^2+c^2-b^2}{a+b\cosh x+c\sinh x}=\sqrt{c^2-b^2}\sinh u.$$

196. IMPORTANT PARTICULAR CASES.

The particular cases (according as b or $c=0$ in the general formulae, and which should be worked *ab initio* by the student) are

$$\int\frac{d\theta}{(a+b\cos\theta)^n}=\frac{1}{(b^2-a^2)^{\frac{2n-1}{2}}}\int(b\cosh u-a)^{n-1}\,du,$$
$$b^2>a^2,\ \left(\frac{b+a\cos\theta}{a+b\cos\theta}=\cosh u\right),$$
$$=\frac{1}{(a^2-b^2)^{\frac{2n-1}{2}}}\int(a-b\cos u)^{n-1}\,du,$$
$$b^2<a^2,\ \left(\frac{b+a\cos\theta}{a+b\cos\theta}=\cos u\right).$$

$$\int\frac{d\theta}{(a+b\sin\theta)^n}=\frac{1}{(b^2-a^2)^{\frac{2n-1}{2}}}\int(b\cosh u-a)^{n-1}\,du,$$
$$b^2>a^2,\ \left(\frac{b+a\sin\theta}{a+b\sin\theta}=\cosh u\right),$$
$$=\frac{1}{(a^2-b^2)^{\frac{2n-1}{2}}}\int(a-b\sin u)^{n-1}\,du,$$
$$b^2<a^2,\ \left(\frac{b+a\sin\theta}{a+b\sin\theta}=\sin u\right).$$

$$\int\frac{dx}{(a+b\cosh x)^n}=\frac{1}{(b^2-a^2)^{\frac{2n-1}{2}}}\int(b\cos u-a)^{n-1}\,du,$$

$$b^2>a^2,\quad\left(\frac{b+a\cosh x}{a+b\cosh x}=\cos u\right),$$

$$=\frac{1}{(a^2-b^2)^{\frac{2n-1}{2}}}\int(a-b\cosh u)^{n-1}\,du,$$

$$b^2<a^2,\quad\left(\frac{b+a\cosh x}{a+b\cosh x}=\cosh u\right).$$

$$\int\frac{du}{(a+b\sinh x)^n}=\frac{1}{(a^2+b^2)^{\frac{2n-1}{2}}}\int(a-b\sinh u)^{n-1}\,du,$$

$$\left(\frac{-b+a\sinh x}{a+b\sinh x}=\sinh u\right)$$

197. We have the further results, from putting $a=0$ and $b=1$ in the above, viz.

$$\int\sec^n\theta\,d\theta\ =\int\cosh^{n-1}u\,du,\quad\text{where }\theta=\sec^{-1}\cosh u;$$

$$\int\operatorname{cosec}^n\theta\,d\theta=\int\cosh^{n-1}u\,du,\quad\text{where }\theta=\operatorname{cosec}^{-1}\cosh u.$$

Hence either integral may be expressed in the form

$$\frac{1}{2^{n-1}}\int(e^u+e^{-u})^{n-1}\,du=\frac{1}{2^{n-2}}\int[\cosh(n-1)u+{}^{n-1}C_1\cosh(n-3)u$$
$$+{}^{n-1}C_2\cosh(n-5)u+\text{etc.}]du$$

$$=\frac{1}{2^{n-2}}\left[\frac{\sinh(n-1)u}{n-1}+{}^{n-1}C_1\frac{\sinh(n-3)u}{n-3}+{}^{n-1}C_2\frac{\sinh(n-5)u}{n-5}\right.$$
$$\left.+\dots+\tfrac{1}{2}\,{}^{n-1}C_{\frac{n-1}{2}}u\quad\text{or}\quad+{}^{n-1}C_{\frac{n}{2}}\sinh u\right].$$

(Compare the forms in Art. 122.)

198. Further, if in the results of Art. 196 we write $n-1=-m$, we have

$$\int\frac{du}{(b\cosh u-a)^m}=\frac{1}{(b^2-a^2)^{\frac{2m-1}{2}}}\int(a+b\cos\theta)^{m-1}\,d\theta,\qquad b^2>a^2,$$

$$\int\frac{du}{(a-b\cos u)^m}=\frac{1}{(a^2-b^2)^{\frac{2m-1}{2}}}\int(a+b\cos\theta)^{m-1}\,d\theta,\qquad b^2<a^2,$$

etc.

Several of these results are given in Greenhill's *Chapter on the Integral Calculus.* The geometrical significance of some of these transformations will appear later.

199. Cases required for the time in an Elliptic Orbit.

The cases of $\int \frac{d\theta}{(a+b\cos\theta)^n}$, where $a=1$, $b=e$, $n=2$, are required in the theory of Planetary Motion in finding the time in an assigned portion of an elliptic (or hyperbolic) orbit. We may either quote the results from Art. 185, or proceed independently as follows.

If $e<1$, by Art. 171,

$$\int \frac{d\theta}{1+e\cos\theta} = \frac{1}{\sqrt{1-e^2}}\cos^{-1}\frac{e+\cos\theta}{1+e\cos\theta} = \frac{u}{\sqrt{1-e^2}}, \text{ say;}$$

and if $e>1$, $\quad = \frac{1}{\sqrt{e^2-1}}\cosh^{-1}\frac{e+\cos\theta}{1+e\cos\theta} = \frac{v}{\sqrt{e^2-1}}$, say.

Taking $e<1$,

$$\frac{d\theta}{1+e\cos\theta} = \frac{du}{\sqrt{1-e^2}} \quad \text{and} \quad \frac{1}{1+e\cos\theta} = \frac{1-e\cos u}{1-e^2};$$

$$\therefore \int \frac{d\theta}{(1+e\cos\theta)^2} = \frac{1}{(1-e^2)^{\frac{3}{2}}}\int (1-e\cos u)\,du, \text{ (or by Art. 196),}$$

$$= \frac{1}{(1-e^2)^{\frac{3}{2}}}(u-e\sin u)$$

$$= \frac{1}{(1-e^2)^{\frac{3}{2}}}\left[\cos^{-1}\frac{e+\cos\theta}{1+e\cos\theta} - e\frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right].$$

The time T for a planet measured from passing Perihelion is expressed by this integral as

$$nT = (1-e^2)^{\frac{3}{2}}\int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2},$$

where n is a certain constant (see E. J. Routh, or Tait & Steele, *Dynamics of a Particle*). It follows that $nT = u - e\sin u$.

If $e>1$,

$$\frac{d\theta}{1+e\cos\theta} = \frac{dv}{\sqrt{e^2-1}} \quad \text{and} \quad \frac{1}{1+e\cos\theta} = \frac{e\cosh v-1}{e^2-1};$$

$$\therefore \int \frac{d\theta}{(1+e\cos\theta)^2} = \frac{1}{(e^2-1)^{\frac{3}{2}}}\int (e\cosh v-1)\,dv, \text{ (or by Art. 196).}$$

$$= \frac{1}{(e^2-1)^{\frac{3}{2}}}(e\sinh v - v)$$

$$= \frac{1}{(e^2-1)^{\frac{3}{2}}}\left[e\frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta} - \cosh^{-1}\frac{e+\cos\theta}{1+e\cos\theta}\right].$$

200. In practice, each example should be worked *ab initio*.

For example, suppose we require

$$\int_0^\pi \frac{dx}{(5+3\cos x)^4}.$$

Putting $5+3\cos x=\frac{1}{y}$, $3\sin x=\frac{1}{y^2}\frac{dy}{dx}$;

$$\therefore\ 9-\left(\frac{1}{y}-5\right)^2=\frac{1}{y^4}\left(\frac{dy}{dx}\right)^2;$$

$$\therefore\ \int y\,dx=+\int\frac{dy}{y\sqrt{9-\left(\frac{1}{y}-5\right)^2}}.$$

We take the + sign, because, as x increases in the first quadrant, $5+3\cos x$ decreases and y increases.

Thus,
$$\int\frac{dx}{5+3\cos x}=\int\frac{dy}{\sqrt{-1+10y-16y^2}}$$
$$=\frac{1}{4}\int\frac{dy}{\sqrt{\frac{9}{16^2}-\left(y-\frac{5}{16}\right)^2}}$$
$$=\frac{1}{4}\sin^{-1}\left(\frac{16y-5}{3}\right)+\text{const.}$$
$$=-\frac{1}{4}\sin^{-1}\frac{3+5\cos x}{5+3\cos x}+\text{const.}$$
$$=\frac{1}{4}\cos^{-1}\frac{3+5\cos x}{5+3\cos x}+\text{const.};$$

call this
$$=\frac{1}{4}u+\text{const.}$$

Then
$$\frac{dx}{5+3\cos x}=\frac{1}{4}du,\quad\text{where }\frac{3+5\cos x}{5+3\cos x}=\cos u,$$

and
$$\therefore\ \frac{1}{5+3\cos x}=\frac{5-3\cos u}{16};$$
$$\therefore\ \frac{1}{(5+3\cos x)^3}=\frac{(5-3\cos u)^3}{16^3};$$
$$\therefore\ \frac{dx}{(5+3\cos x)^4}=\frac{(5-3\cos u)^3\,du}{2^{14}}.$$

$$\int_0^\pi\frac{dx}{(5+3\cos x)^4}=\frac{1}{2^{14}}\int_0^\pi[5^3-3.5^2.3\cos u+3.5.3^2\cos^2 u-3^3\cos^3 u\ du$$

[for when $x=0$, $\cos u=1$, and when $x=\pi$, $\cos u=-1$],

$$=\frac{1}{2^{14}}.2\int_0^{\frac{\pi}{2}}(5^3+3.5.3^2\cos^2 u)\,du$$
$$=\frac{1}{2^{14}}.2\left[5^3\frac{\pi}{2}+3^3.5\frac{1}{2}\frac{\pi}{2}\right]$$
$$=\frac{\pi}{2^{14}}.5.\left(25+\frac{27}{2}\right)=\frac{5\pi}{2^{15}}\times 77$$
$$=\frac{385\pi}{2^{15}}.$$

201. The integrals

$$I_1 \equiv \int \frac{\sin^m x}{a+b\cos x}\,dx \quad \text{and} \quad I_2 \equiv \int \frac{\sin^m x}{(a+b\cos x)^2}\,dx$$

can both be integrated in finite terms when m is a positive integer.

Consider the first, viz. $\int \frac{\sin^m x}{a+b\cos x}\,dx.$

The case $m=1$ obviously gives $-\frac{1}{b}\log(a+b\cos x)$.

If m be odd, $=2k+1$, say, put $a+b\cos x=z$, and therefore $b\sin x\,dx=-dz$.

Thus,
$$\int \frac{\sin^{2k+1} x}{a+b\cos x}\,dx = -\frac{1}{b}\int \frac{\left[1-\left(\frac{z-a}{b}\right)^2\right]^k dz}{z},$$

every term of which is integrable when expanded in powers of z.

If m be even, $=2k$, say,

$$\int \frac{\sin^{2k} x}{a+b\cos x}\,dx = \int \frac{(1-\cos^2 x)^k}{a+b\cos x}\,dx,$$

and if the numerator be expanded in descending powers of $\cos x$, and then divided by $b\cos x+a$, we arrive at an expression of form

$$\int \left(\lambda_1 \cos^{2k-1} x + \lambda_2 \cos^{2k-2} x + \ldots + \lambda_{2k} + \frac{\lambda_{2k+1}}{a+b\cos x}\right) dx,$$

where the λ's are numerical coefficients.

Hence, in all cases, $\int \frac{\sin^m x}{a+b\cos x}\,dx$ can be integrated in finite terms.

The same argument applies to $\int \frac{\sin^m x}{(a+b\cos x)^2}\,dx.$

202. If $I_n \equiv \int \frac{\sin^m x}{(a+b\cos x)^n}\,dx$, there is a **reduction formula** connecting I_n with I_{n-1} and I_{n-2}. Hence all such integrations can be effected in finite terms.

To obtain this reduction formula, put

$$P = \frac{\sin^{m+1} x}{(a+b\cos x)^{n-1}}$$

[*i.e. increase the index of the numerator by unity and decrease that of the denominator by unity*].

Then

$$\frac{dP}{dx}=\frac{(m+1)\sin^m x\cos x\,(a+b\cos x)+(n-1)b\sin^m x\,(1-\cos^2 x)}{(a+b\cos x)^n}$$

$$=\frac{\sin^m x}{(a+b\cos x)^n}\left[A+B(a+b\cos x)+C(a+b\cos x)^2\right],$$

where
$$A+Ba+Ca^2=(n-1)b,$$
$$Bb+2Cab=(m+1)a,$$
$$Cb^2=(m+1)b-(n-1)b,$$

giving

$$A=-(n-1)\frac{a^2-b^2}{b},\quad B=(2n-m-3)\frac{a}{b},\quad C=\frac{m-n+2}{b}.$$

Hence

$$\frac{\sin^{m+1}x}{(a+b\cos x)^{n-1}}=-(n-1)\frac{a^2-b^2}{b}I_n+(2n-m-3)\frac{a}{b}I_{n-1}$$
$$+\frac{m-n+2}{b}I_{n-2},$$

and the reduction formula required is

$$I_n=-\frac{1}{n-1}\frac{b}{a^2-b^2}\frac{\sin^{m+1}x}{(a+b\cos x)^{n-1}}+\frac{2n-m-3}{n-1}\frac{a}{a^2-b^2}I_{n-1}$$
$$+\frac{m-n+2}{n-1}\frac{1}{a^2-b^2}I_{n-2},$$

of which the formula of Art. 186 is a particular case.

And since I_1 and I_2 have been shown integrable in finite terms when m is given, we can use the reduction formula just established to find successively I_3, I_4, etc., in terms of I_1 and I_2, and thus integrate them.

203. **Again, Integrals of form**

$$I_1'\equiv\int\frac{\sin^p\theta\cos^q\theta}{a+b\cos\theta}d\theta,\quad I_2'\equiv\int\frac{\sin^p\theta\cos^q\theta\,d\theta}{(a+b\cos\theta)^2},$$
$$I_3'\equiv\int\frac{\sin^p\theta\cos^q\theta\,d\theta}{(a+b\cos\theta)^3}$$

are always integrable in finite terms, p and q being positive integers.

For (1) if p be odd, $=2k+1$,

$$I_1'=-\int\frac{(1-c^2)^k c^q\,dc}{a+bc},\text{ where } c=\cos\theta,$$

and after expansion of the numerator in descending powers of c and division by $bc+a$, we get a series of powers of c and a remainder $\dfrac{A}{a+bc}$, and each term is integrable with respect to c.

(2) If p be even, $=2k$,

$$\sin^p\theta\cos^q\theta=(1-\cos^2\theta)^k\cos^q\theta,$$

which, when expanded in descending powers of $\cos\theta$ and divided by $b\cos\theta+a$, gives a series of powers of $\cos\theta$ with a remainder of form $\dfrac{A}{a+b\cos\theta}$, and each term is integrable with respect to θ by Arts. 117, 173.

And the same argument holds good for I_2', I_3', except that the remainders to be integrated involve such terms as

$$A\int\frac{d\cos\theta}{a+b\cos\theta}+B\int\frac{d\cos\theta}{(a+b\cos\theta)^2}+C\int\frac{d\cos\theta}{(a+b\cos\theta)^3}$$

or

$$A'\int\frac{d\theta}{a+b\cos\theta}+B'\int\frac{d\theta}{(a+b\cos\theta)^2}+C'\int\frac{d\theta}{(a+b\cos\theta)^3},$$

according as p is odd or even, and such integrations have been already considered.

204. We may then obtain a reduction formula for

$$I'_n=\int\frac{\sin^p\theta\cos^q\theta}{(a+b\cos\theta)^n}\,d\theta.$$

Let
$$P=\frac{\sin^{p+1}\theta\cos^{q+1}\theta}{(a+b\cos\theta)^{n-1}}.$$

Then

$$\frac{dP}{d\theta}=\frac{[(p+1)\sin^p\theta\cos^{q+2}\theta-(q+1)\sin^{p+2}\theta\cos^q\theta](a+b\cos\theta)+(n-1)b\sin^{p+2}\theta\cos^{q+1}\theta}{(a+b\cos\theta)^n}$$

$$=\frac{\sin^p\theta\cos^q\theta}{(a+b\cos\theta)^n}[\{-(q+1)+(p+q+2)\cos^2\theta\}(a+b\cos\theta)+(n-1)b(1-\cos^2\theta)\cos\theta]$$

$$=\frac{\sin^p\theta\cos^q\theta}{(a+b\cos\theta)^n}[A+B(a+b\cos\theta)+C(a+b\cos\theta)^2+D(a+b\cos\theta)^3],\text{ say,}$$

where
$$\left.\begin{aligned} A+Ba+Ca^2+Da^3&=-(q+1)\,a,\\ Bb+2Cab+3Da^2b&=-(q+1)\,b+(n-1)\,b,\\ Cb^2+3Dab^2&=\quad(p+q+2)\,a,\\ Db^3&=\quad(p+q+2)\,b-(n-1)\,b,\end{aligned}\right\}$$

whence
$$A=(n-1)\,a\frac{a^2-b^2}{b^2},$$
$$B=(n-q-2)-(3n-p-q-5)\frac{a^2}{b^2},$$
$$C=(3n-2p-2q-7)\,\frac{a}{b^2},$$
$$D=\frac{(p+q-n+3)}{b^2},$$

and the reduction formula is
$$\frac{\sin^{p+1}\theta\cos^{q+1}\theta}{(a+b\cos\theta)^{n-1}}=AI'_n+BI'_{n-1}+CI'_{n-2}+DI'_{n-3},$$
from which I'_n can be expressed in terms of three integrals of the next lower orders and ultimately made to depend upon I'_1, I'_2, I'_3, whose integration has been discussed.

205. **General Conclusion.**

From what has been said in Art. 204, it will now appear that any integral of form
$$\int\frac{f(\sin\theta,\cos\theta)\,d\theta}{(a+b\cos\theta)^n}$$
can be integrated when n is a positive (or negative) integer, and $f(x, y)$ is a rational integral algebraic function of $\sin\theta$, $\cos\theta$; for $f(\sin\theta, \cos\theta)$ is then the sum of a number of terms of type
$$A\,.\sin^p\theta\cos^q\theta.$$

206. HERMITE (*Proc. Lond. Math. Soc.* 1872) has shown how to integrate any expression of form
$$\frac{f(\sin\theta,\cos\theta)}{\sin(\theta-a_1)\sin(\theta-a_2)\sin(\theta-a_3)\ldots\sin(\theta-a_n)},$$
where $f(x, y)$ is any homogeneous function of x, y of $(n-1)$ dimensions.

For by the ordinary rules of partial fractions,
$$\frac{f(t,1)}{(t-a_1)(t-a_2)\ldots(t-a_n)}=\sum_{r=1}^{r=n}\frac{f(a_r,1)}{(a_r-a_1)(a_r-a_2)\ldots(a_r-a_n)}\cdot\frac{1}{t-a_r}$$
(the factor a_r-a_r being omitted in the denominator of the above coefficient).

Writing $t=\tan\theta$, $a_1=\tan\alpha_1$, $a_2=\tan\alpha_2$, etc., this becomes

$$\frac{f(\sin\theta,\ \cos\theta)}{\prod\limits_{r=1}^{r=n}\sin(\theta-\alpha_r)}=\sum\frac{A_r}{\sin(\theta-\alpha_r)},$$

where $$A_r=\frac{f(\sin\alpha_r,\ \cos\alpha_r)}{\sin(\alpha_r-\alpha_1)\sin(\alpha_r-\alpha_2)\ldots\sin(\alpha_r-\alpha_n)},$$

the factor $\sin(\alpha_r-\alpha_r)$ being omitted in the denominator.

Thus, $$\int\frac{f(\sin\theta,\ \cos\theta)}{\prod\limits_1^n\sin(\theta-\alpha_r)}\,d\theta=\sum_1^n A_r\log\tan\frac{\theta-\alpha_r}{2}.$$

207. (i) Thus, for example, we have

$$\frac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}$$
$$=\sum\frac{\sin^2 a}{\sin(a-b)\sin(a-c)}\,\frac{1}{\sin(x-a)};$$
$$\therefore\ \int\frac{\sin^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}\,dx$$
$$=\sum\frac{\sin^2 a}{\sin(a-b)\sin(a-c)}\log\tan\frac{x-a}{2}.$$

(ii) Similarly,

$$\int\frac{\cos^2 x}{\sin(x-a)\sin(x-b)\sin(x-c)}\,dx$$
$$=\sum\frac{\cos^2 a}{\sin(a-b)\sin(a-c)}\log\tan\frac{x-a}{2}.$$

(iii) Hence adding,

$$\int\frac{dx}{\sin(x-a)\sin(x-b)\sin(x-c)}$$
$$=\sum\frac{1}{\sin(a-b)\sin(a-c)}\log\tan\frac{x-a}{2},$$

(iv) or subtracting,

$$\int\frac{\cos 2x\,dx}{\sin(x-a)\sin(x-b)\sin(x-c)}$$
$$=\sum\frac{\cos 2a}{\sin(a-b)\sin(a-c)}\log\tan\frac{x-a}{2}.$$

(v) It is easy to show that

$$\frac{\sin x}{\sin(x-a)\sin(x-b)\sin(x-c)}$$
$$=\sum\frac{\sin a}{\sin(a-b)\sin(a-c)}\cot(x-a);$$
$$\therefore\ \int\frac{\sin x}{\sin(x-a)\sin(x-b)\sin(x-c)}\,dx$$
$$=\sum\frac{\sin a}{\sin(a-b)\sin(a-c)}\log\sin(x-a).$$

EXAMPLES.

1. Integrate

(i) $\displaystyle\int\frac{a\sin\theta+b\cos\theta}{c\sin\theta+e\cos\theta}d\theta.$ [a, 1883.] (ii) $\displaystyle\int\frac{d\theta}{\cos\theta-\sin\theta}.$ [I. C. S., 1880.]

(iii) $\displaystyle\int\frac{\alpha+\beta\sin\theta}{a+b\cos\theta}d\theta.$ [TRIN. H. AND MAGD.]

(iv) $\displaystyle\int\frac{dx}{\cos\alpha+\cos x}.$ [I.C.S., 1889.] (v) $\displaystyle\int\frac{dx}{a\cos x+b\sin x}.$ [COLL., 1876.]

(vi) $\displaystyle\int\frac{1-\tan\theta}{1+\tan\theta}d\theta.$ [TRIN., 1884.] (vii) $\displaystyle\int\frac{\sec\theta}{1+\operatorname{cosec}\theta}d\theta.$ [OX. I. P., 1889.]

(viii) $\displaystyle\int\frac{dx}{3(1-\sin x)-\cos x}.$ [a, 1881.]

(ix) $\displaystyle\int\frac{\sqrt{2}\,dx}{2\sqrt{2}+\cos x+\sin x}.$ [OX. I. P., 1888.] (x) $\displaystyle\int\frac{dx}{a+b\tan x}.$ [ST. JOHN'S, 1888.]

(xi) Apply the transformation $t=\tan\frac{1}{2}x$ to the integrals

$$\int\frac{4dx}{5+3\cos x},\quad\int\frac{4dx}{3+5\cos x}.$$

Hence or otherwise, evaluate these integrals to the nearest hundredth, when the limits are $x=0$ and $\frac{1}{2}\pi$. Prove in any way that the second is the greater of the two integrals, when taken between 0 and $\frac{1}{2}\pi$. [MATH. TRIP. I., 1913.]

(xii) Prove that

$$\int_0^a\frac{dx}{x+(a^2-x^2)^{\frac{1}{2}}}=\tfrac{1}{4}\pi,\quad\int_0^a\frac{a\,dx}{\{x+(a^2-x^2)^{\frac{1}{2}}\}^2}=\frac{1}{\sqrt{2}}\log_e(1+\sqrt{2}),$$

the positive sign being taken for the radical in each of the subjects of integration. [MATH. TRIP. II., 1913.]

2. Evaluate

(i) $\displaystyle\int_0^\pi\frac{dx}{4+5\sin x}.$ [I. C. S., 1889.] (ii) $\displaystyle\int_0^\pi\frac{d\theta}{a+c\cos\theta}$ $(c<a)$. [I. C. S., 1879.]

(iii) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{2+\cos x}.$ [ST. JOHN'S, 1882.] (iv) $\displaystyle\int_0^\pi\frac{dx}{1-2a\cos x+a^2}.$ [I. C. S., 1888.]

3. Show that $\displaystyle\int_0^a\frac{dx}{1-\cos\alpha\cos x}=\frac{\pi}{2}\operatorname{cosec}\alpha,$ [OX. II. P., 1889; TRIN., 1887.]

and integrate $\displaystyle\int\frac{\cos\alpha\cos x+1}{\cos\alpha+\cos x}dx.$

4. Evaluate

(i) $\displaystyle\int \frac{dx}{\alpha^2 - \beta^2 \cos^2 x}$ $(\alpha > \beta)$. [Ox. II. P., 1887.]

(ii) $\displaystyle\int \frac{dx}{\sin x \cos x + \sin x + \cos x - 1}$. [$\gamma$, 1899.]

(iii) $\displaystyle\int \frac{dx}{\sin x + \sin 2x}$. [$\beta$, 1891.]

5. Evaluate $\displaystyle\int \frac{dx}{(e^x + e^{-x})^2}$. [Ox. I. P., 1889.]

6. Integrate

(i) $\displaystyle\int \frac{dx}{(4 + 5\cos x)^2}$. [$\alpha$, 1883.]

(ii) $\displaystyle\int \frac{dx}{(a + b\cos x)^2}$. [St. John's, 1884.]

(iii) $\displaystyle\int \frac{d\theta}{(a\cos\theta + b\sin\theta)^2}$. [$\alpha$, 1881.]

(iv) $\displaystyle\int \frac{dx}{(a + b\cos x + c\sin x)^2}$. [Coll., 1892.]

(v) Employ the substitution

$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$ to evaluate the integral $\displaystyle\int_0^\theta \frac{d\theta}{(1 + e\cos\theta)^2}$.

[Math. Trip. I., 1909.]

7. Prove that

(i) $\displaystyle\int_0^\pi \frac{d\theta}{(1 + \cos\alpha\cos\theta)^2} = \pi \operatorname{cosec}^3\alpha$, [Coll., 1886; St. John's, 1886.]

(ii) $\displaystyle\int_0^\pi \frac{\sin x\, dx}{(1 + \cos\alpha\sin x)^2} = \frac{2(\sin\alpha - \alpha\cos\alpha)}{\sin^3\alpha}$, [$\beta$, 1887.]

and evaluate

$$\int_0^\pi \frac{x\sin x\, dx}{(1 + \cos\alpha\cos x)^3} \text{ and } \int_0^\pi \frac{x\sin x\, dx}{(1 + \cos\alpha\cos x)^4},$$

if α be less than $\frac{\pi}{2}$.

8. Evaluate the integral

$$\int \cos 2\theta \log(1 + \tan\theta)\, d\theta.$$

[γ, 1882.]

9. Find the values of the following integrals: $\left(\alpha < \frac{\pi}{2}\right)$

(i) $\displaystyle\int_0^\infty \frac{dx}{2\cos\alpha + e^x + e^{-x}}$, (ii) $\displaystyle\int_0^\infty \frac{dx}{2 + \cos\alpha(e^x + e^{-x})}$.

[Trin., 1882.]

10. Evaluate (i) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}$. [Ox. I. P., 1889.]

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{4 + 5\sin^2\theta}$. [I. C. S., 1885.]

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{a\sin^2\theta + b\cos^2\theta}{c\sin^2\theta + d\cos^2\theta}\,d\theta$ (c and d positive). [St. John's.]

(iv) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{(\alpha^2\cos^2\theta + \beta^2\sin^2\theta)^2}$. [Ox. II. P., 1887.]

(v) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta\,d\theta}{\sin^4\theta + \cos^4\theta}$, [I. C. S., 1891.]

and (vi) Shew that if $c > a > 0$,

$$\int_0^a \frac{\sqrt{a^2 - x^2}}{c^2 - x^2}\,dx = \pi(c - \sqrt{c^2 - a^2})/2c.$$

[Math. Trip. I., 1908.]

11. Prove that $\displaystyle\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pi}{\sqrt{a^2 - b^2}}$,

where $a > b$, and deduce or otherwise obtain the value of

$$\int_0^{\pi} \frac{d\theta}{(a + b\cos\theta)^3}.$$

[γ, 1899.]

12. Prove that if $a > b$,

$$\int_0^{\pi} \frac{dx}{(a + b\cos x)^4} = \frac{\pi}{2}\left\{\frac{5a^3}{(a^2 - b^2)^{\frac{7}{2}}} - \frac{3a}{(a^2 - b^2)^{\frac{5}{2}}}\right\}.$$

[γ, 1888.]

Evaluate $\displaystyle\int_0^{\pi} \frac{dx}{(1 + e\cos x)^4}$, where $e < 1$. [St. John's, 1892.]

13. Evaluate $\displaystyle\int \frac{dx}{a + be^x + ce^{-x}}$, where $a^2 < 4bc$. [I. C. S., 1897.]

14. Prove that if $\alpha < \dfrac{\pi}{6}$,

$$\int_0^{\alpha} \frac{\sin x}{\cos 3x}\,dx = \frac{1}{6}\log\frac{\cos^3\alpha}{\cos 3\alpha}.$$

[C. S., 1896.]

15. Prove that $\displaystyle\int_0^{2\pi} \frac{d\phi}{a + b\cos\phi + c\sin\phi} = \frac{2\pi}{\sqrt{a^2 - r^2}}$,

where $r^2 = b^2 + c^2$ and $r < a$. [C. S., 1900.]

16. Integrate

(i) $\displaystyle\int \frac{\sqrt{\tan x}}{\sin x\cos x}\,dx$. (ii) $\displaystyle\int \frac{\sec x\,dx}{a + b\tan x}$. (iii) $\displaystyle\int \frac{\sec^2 x\,dx}{a + b\tan x}$.

17. Integrate

(i) $\int \frac{d\theta}{15 \sin^2\theta - 16 \cos\theta}$. (ii) $\int_0^\pi \frac{x}{1+\sin x} dx$.

(iii) $\int \frac{\cot\theta - 3\cot 3\theta}{3\tan 3\theta - \tan\theta} d\theta$. [Ox. I. P., 1888.]

(iv) $\int \frac{\sin 2x\, dx}{(a+b\cos x)^2}$. [$a$, 1889.]

18. Integrate

(i) $\int \cos 2\theta \log \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} d\theta$. (ii) $\int \frac{\sin\theta - \cos\theta}{\sqrt{\sin 2\theta}} d\theta$.

(iii) $\int \sqrt{\frac{1-\cos\theta}{\cos\theta(1+\cos\theta)(2+\cos\theta)}} d\theta$.

19. Integrate $\int \sqrt{\frac{1+\sin x}{1-\sin x} \cdot \frac{2+\sin x}{2-\sin x}} dx$.

20. Integrate

$$\int \frac{\sin\theta - \cos\theta}{(\sin\theta + \cos\theta)\sqrt{\sin\theta\cos\theta + \sin^2\theta\cos^2\theta}} d\theta.$$

21. Integrate $\int \frac{\sin^3\theta\, d\theta}{(1+\cos^2\theta)\sqrt{1+\cos^2\theta+\cos^4\theta}}$.

22. Integrate

(i) $\int \sqrt{1+\sin x}\, dx$. (ii) $\int \frac{\sin x}{\sqrt{1+\sin x}} dx$. (iii) $\int \frac{\tan x}{\sqrt{a+b\tan^2 x}} dx$.

23. Integrate $\int \frac{\sinh x \sin x - \cosh x}{1-\cos x} dx$.

24. Integrate $\int \frac{x^2\, dx}{(x\sin x + \cos x)^2}$.

25. Integrate $\int \frac{\sec x \operatorname{cosec} x}{\log\tan x} dx$.

26. Integrate

(i) $\int \sin^{-1}\frac{2x}{1+x^2} dx$. (ii) $\int \tan^{-1}\frac{3x-x^3}{1-3x^2} dx$.

(iii) $\int \tan^{-1}\frac{-1+\sqrt{1+x^2}}{x} dx$.

27. Integrate $\int \frac{\sqrt{1+x^2}}{1-x^2} dx$.

28. Integrate $\int \frac{\sin x}{\sin 2x}dx$, $\int \frac{\sin x}{\sin 3x}dx$, $\int \frac{\sin x}{\sin 4x}dx$,

and prove that

$$5\int \frac{\sin x}{\sin 5x}dx = \sin\frac{2\pi}{5}\log\left\{\frac{\sin\left(x-\frac{2\pi}{5}\right)}{\sin\left(x+\frac{2\pi}{5}\right)}\right\} - \sin\frac{\pi}{5}\log\left\{\frac{\sin\left(x-\frac{\pi}{5}\right)}{\sin\left(x+\frac{\pi}{5}\right)}\right\}.$$

[TRIN. COLL., 1892.]

29. Integrate $\int \frac{d\theta}{\sin^2\theta - \sin^2\alpha}$ and $\int \frac{\cos\theta}{\sin^2\theta - \sin^2\alpha}d\theta$.

Show that $\frac{\sin\theta}{\sin n\theta}$ can be expressed in partial fractions of type

$$\frac{1}{\sin^2\theta - \sin^2\alpha} \quad \text{or} \quad \frac{\cos\theta}{\sin^2\theta - \sin^2\alpha},$$

according as n is an odd or an even integer and can thereby be integrated.

30. Integrate

(i) $\int_0^{\frac{\pi}{2}} \frac{\sin 3x\,dx}{(a+b\cos x)^2}$. (ii) $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x\,dx}{a+b\cos x}$.

(iii) $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x\,dx}{(a+b\cos x)^2}$. (iv) $\int_0^{\frac{\pi}{2}} \frac{\sin 3x\,dx}{(a+b\cos x)^n}$ $(n > 3)$.

31. Show how to effect the integration of

$$\int \frac{\cos^p x}{\sin 2nx}dx, \quad \int \frac{\cos^p x}{\cos nx}dx,$$

p and n being integers. [ϵ, 1883, AND COLL., 1879.]

32. Integrate $\int \cot(x-\alpha)\cot(x-\beta)\,dx$, [$\gamma$, 1891.]

and show that

$$\int \cot(x-a)\cot(x-b)\cot(x-c)\,dx = \Sigma\cot(a-b)\cot(a-c)\log\sin(x-a).$$

[TRINITY, 1891.]

33. Show that

$$\int \sin x \sec(x-\alpha)\sec(x-\beta)\,dx$$
$$= \frac{1}{\sin(\beta-\alpha)}\left[\cos\alpha\cosh^{-1}\sec(x-\alpha) - \cos\beta\cosh^{-1}\sec(x-\beta)\right].$$

[TRINITY, 1889.]

34. Prove that

$$\int_0^{\beta} x\sec x\sec(\beta - x)\,dx = \beta\,\mathrm{cosec}\,\beta\log\sec\beta.$$

[OXF. II. P., 1901.]

35. Prove that, if α and β be positive quantities,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(\alpha\cos^2 x+\beta\sin^2 x)^{n+1}}=\frac{\pi}{2}\frac{(-1)^n}{n!}\left(\frac{d}{d\alpha}+\frac{d}{d\beta}\right)^n(\alpha\beta)^{-\frac{1}{2}}.$$

[α, 1884.]

36. Prove that, if

$$f(x)=(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

and $$P=\prod_{r=1}^{r=n}(1-2\alpha_r\cos\theta+\alpha_r^2),$$

where $\alpha_1, \alpha_2, \dots \alpha_n$ denote real quantities, then

$$\int_0^\pi \frac{d\theta}{P}=\pi\sum_1^n \epsilon_r/A_r,$$

where $A_r=\alpha_r f(1/\alpha_r)f'(\alpha_r)$, and $\epsilon=-1$, or $+1$, according as α_r is numerically greater or less than 1. [ST. JOHN'S, 1886.]

37. If c be less than $a\sin\theta$, show that the coefficient of c^m in the expansion of

$$\frac{2c}{\sqrt{c^2-a^2}}\tan^{-1}\left\{\sqrt{\frac{c-a}{c+a}}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

is $$(-1)^m\frac{1}{a^m}\int\frac{d\theta}{\sin^m\theta}+A_m,$$

where A_m is independent of θ. [COLL., 1892.]

38. Show that if n be a positive integer,

$$\int\frac{\cos n\theta-\cos n\alpha}{\cos\theta-\cos\alpha}d\theta=\theta\frac{\sin n\alpha}{\sin\alpha}+\frac{2}{\sin\alpha}\sum_{r=1}^{r=n-1}\frac{\sin r\theta}{r}\sin(n-r)\alpha.$$

[HERMITE.]

39. Prove that

$$\int_0^x(1+\cos x)^n dx=\frac{1}{2^n}\cdot {}^{2n}C_n x+\frac{1}{2^{n-1}}\sum_1^n {}^{2n}C_{n-r}\frac{\sin rx}{r}.$$

40. Show that

$$\int\prod_1^n\cot(\theta-\alpha_r)d\theta=\theta\cos\frac{n\pi}{2}+\sum_1^n A_r\log\sin(\theta-\alpha_r),$$

where $$A_r=\cot(\alpha_r-\alpha_1)\cot(\alpha_r-\alpha_2)\dots\cot(\alpha_r-\alpha_n),$$

the factor $\cot(\alpha_r-\alpha_r)$ being omitted. [HERMITE.]

41. (1) Show that

$$\int\frac{dx}{a-\sin x}=\frac{2}{\sqrt{a^2-1}}\tan^{-1}\left(\frac{a\tan\frac{x}{2}-1}{\sqrt{a^2-1}}\right)\quad(a>1).$$

(2) Differentiate with regard to x,

$$\tan^{-1}\sqrt{\frac{a+1}{a-1}\cdot\frac{1-\sin x}{1+\sin x}}.$$

Deduce from (1) and (2) that

$$\tan^{-1}\sqrt{\frac{a+1}{a-1}\cdot\frac{1-\sin x}{1+\sin x}}+\tan^{-1}\frac{a\tan\frac{x}{2}-1}{\sqrt{a^2-1}}$$

is independent of x, and verify your conclusion. [C. S., 1898.]

42. Integrate $$\int\frac{dx}{1+ab-a\cos x-b\sec x},$$
where $a<1,\ b>1,\ ab\neq 1.$ [OXF. I. P., 1917.]

43. Integrate (i) $e^x(x\cos x+\sin x)$. (ii) $(x^2+1)^{-\frac{5}{2}}$.
(iii) $(x^4+2x^3+2x+1)/(x^2+x+1)^3$. [OXF. I. P., 1918.]

44. Integrate $$\int\frac{\cos^3 x\,dx}{\sin(x-\alpha)\sin(x-\beta)\sin(x-\gamma)}.$$

45. Deduce from the identity

$$\int_0^{\frac{\pi}{2}}\cos n\theta\,d\theta=\int_0^{\frac{\pi}{2}}d\theta\left\{1-\frac{n^2}{2!}\sin^2\theta-\frac{n^2(2^2-n^2)}{4!}\sin^4\theta\right.$$
$$\left.-\frac{n^2(2^2-n^2)(4^2-n^2)}{6!}\sin^6\theta-\dots\right\}$$

the expression for $\sin x$ as an infinite product. [OXF. II. P., 1887.]

46. Evaluate the integrals

(i) $\displaystyle\int(x+a\log x)^2\,dx.$ (ii) $\displaystyle\int\frac{(a+x)\,dx}{x^2+ax\log x}.$

(iii) $\displaystyle\int\frac{(1-\log x)\,dx}{(x+a\log x)^2}.$ [MATH. TRIPOS, 1885.]

47. Show that

$$\int_0^x\frac{x^2+a(a-1)}{(x\sin x+a\cos x)^2}\,dx=\frac{a\sin x-x\cos x}{x\sin x+a\cos x}.$$

[TRIN. COLL., 1891.]

48. Evaluate the indefinite integrals

(i) $$\int\frac{(\sin x+\cos x)^2}{\{(x-1)\cos x-(x+1)\sin x\}^2}\,dx.$$

(ii) $$\int\frac{x^2}{\{(x-1)\cos x-(x+1)\sin x\}^2}\,dx.$$

[COLLEGES, 1886.]

49. If $T\equiv x+\sqrt{1+x^2}$, show that

$$\int T^r dx=\tfrac{1}{2}\left\{\frac{T^{r-1}}{r-1}+\frac{T^{r+1}}{r+1}\right\}.$$

50. Integrate (i) $\int e^x \frac{x^3-x+2}{(1+x^2)^2}\,dx.$ [Ox. II. P., 1899.]

(ii) $\int \frac{\log(\cos\theta+\sqrt{\cos 2\theta})}{1-\cos^2\theta}\,d\theta.$ [Coll. α, 1891.]

51. Prove that, if n be an integer,

$$\int_0^\pi \frac{\cos nx}{1+\cos\alpha\cos x}\,dx = \pi\operatorname{cosec}\alpha(\tan\alpha-\sec\alpha)^n,$$

and deduce the value of

$$\int_0^\pi \frac{\cos nx}{(1+\cos\alpha\cos x)^2}\,dx.$$ [Colleges γ, 1891.]

52. From considering the integral

$$\int_0^\pi \frac{\cos n\theta}{a-\cos\theta}\,d\theta,$$

show that

$$1+\frac{n+2}{1}\frac{\cos^2\alpha}{2^2}+\frac{(n+3)(n+4)}{1\,.\,2}\frac{\cos^4\alpha}{2^4}+\dots$$
$$=2^n(\sec\alpha-\tan\alpha)^n\sec^n\alpha\operatorname{cosec}\alpha.$$

53. Prove that, if $0<\alpha<\frac{\pi}{2}$ and n be a positive integer,

$$\int_0^\pi \sin n\phi\tan^{-1}\left(\frac{\tan\frac{\phi}{2}}{\tan\frac{\alpha}{2}}\right)d\phi=\frac{\pi}{2n}[(\sec\alpha-\tan\alpha)^n-(-1)^n].$$

54. Show that

$$\int\frac{\sin n\theta}{\sin\theta}\frac{d\theta}{\cos n\theta-\cos n\alpha}=\frac{1}{n}\sum_{r=0}^{r=n-1}\operatorname{cosec}\beta_r\log\frac{\sin\frac{1}{2}(\beta_r+\theta)}{\sin\frac{1}{2}(\beta_r-\theta)};$$

where $$\beta_r=\alpha+\frac{2r\pi}{n}.$$

55. Discuss the integration of

$$(a)\ \int\frac{\sin p\theta}{\sin q\theta}\,d\theta,\quad (b)\ \int\sin\theta\frac{\sin p\theta}{\sin q\theta}\,d\theta,$$

where p and q are positive integers.

56. With the help of the substitution $x^{-1}=\sqrt{t^2-1}$, or otherwise prove that

$$\int_0^\infty\frac{dx}{(9+25x^2)\sqrt{1+x^2}}=\frac{1}{12}\tan^{-1}\frac{4}{3}.$$

[Math. Trip., Pt. II., 1920.]

CHAPTER VII.

FURTHER REDUCTION FORMULAE.

208. Several "**Formulae of Reduction**" have already been established, and the student will have gathered some information as to their nature, mode of construction and use.

The nature of these formulae is that a connection, in general linear, is found between two or more integrals, so that when all but one have been found, the remaining one can be inferred.

209. It will be useful to summarize those which have already occurred. They are as follows:

1. The rule for integration by parts, Art. 90, and for continued integration by parts, Art. 95.

2. Reduction formulae for $\int x^m \binom{\sin}{\cos} nx\,dx$, Art. 102.

3. Reduction formulae for $\int e^{ax} \binom{\sin}{\cos}^m bx\,dx$, Art. 104.

4. Reduction formulae for $\int x^m (\log x)^n\,dx$, Art. 106.

5. Reduction formulae for $\int \sec^n x\,dx.\ \int \operatorname{cosec}^n x\,dx$, Art. 120, etc.

$\int \tan^n x\,dx,\ \int \cot^n x\,dx$, Art. 125, etc.

6. Reduction formulae for $\int \frac{dx}{(a+b\cos x+c\sin x)^n}$, etc., Arts. 185 to 199.

7. Reduction formulae for $\int \frac{\sin^p x\cos^q x}{(a+b\cos x)^n}dx$, etc., Arts. 201 to 203.

210. General Remarks.

The subject of the present chapter will be the construction of such further reduction formulae as may be necessary for present or future uses in the book, and a general indication to the student of the mode of procedure to facilitate their speedy production. It will be noted also that two distinct modes of procedure have been exhibited:

(i) That of integration by parts, or, what comes to the same thing, a proper choice of "P," with a differentiation and subsequent arrangement of the result as a linear function of the expressions whose integrals are to be connected, as exemplified in Arts. 185 to 188.

(ii) A change of the variable, taking the integrand itself, or some function of it, or of some essential part of it, as a new variable, as exemplified in Arts. 194 to 198

We shall also complete the discussion of such integrations as are to be considered, both for the general cases when reduction formulae are required and for *the particular cases in which it is convenient to avoid their use.*

211. Integration of $\int x^{m-1}X^p\,dx$, where $X\equiv a+bx^n$.

In three cases this admits of direct integration, and no reduction formulae is required:

I. When p **is a positive integer.**

II. When $\dfrac{m}{n}$ **is an integer:** (i) Positive. (ii) Negative.

III. When $\dfrac{m}{n}+p$ **is an integer:** (i) Positive. (ii) Negative.

In other cases a *reduction formula is necessary.*

212. I. If *p be a positive integer* we can expand $(a+bx^n)^p$ in a finite series by the binomial theorem and integrate each term.

Thus

$$\int x^{m-1}(a+bx^n)^p\,dx=a^p\frac{x^m}{m}+{}^pC_1a^{p-1}b\frac{x^{m+n}}{m+n}+\dots+b^p\frac{x^{m+pn}}{m+pn}.$$

If p be *fractional or negative*, the binomial expansion is *non-terminating*, and therefore the integration after expansion would not express the result *in finite terms.* Expansion therefore in such cases *should not be resorted to if avoidable.*

213. II. *Let* $p=\frac{r}{s}$ *where* r *and* s *are integers*, and s, at least, positive (which covers *all commensurable fractional or negative values of* p).

Put $X \equiv a+bx^n=z^s$,

$$\therefore \quad bnx^{n-1}\,dx=sz^{s-1}\,dz;$$

$$\therefore \int x^{m-1}X^p\,dx=\frac{s}{bn}\int x^{m-1}z^r\frac{z^{s-1}\,dz}{x^{n-1}}=\frac{s}{bn}\int x^{m-n}z^{r+s-1}\,dz$$

$$=\frac{s}{bn}\int z^{r+s-1}\left(\frac{z^s-a}{b}\right)^{\frac{m}{n}-1}dz$$

$$=\frac{s}{nb^{\frac{m}{n}}}\int z^{r+s-1}(z^s-a)^{\frac{m}{n}-1}\,dz.$$

(i) Hence *when* $\frac{m}{n}$ *is a positive integer* >0, a finite expression may be found for the integral by *expanding this binomial*, integrating each term, and finally substituting back for z its value, viz. $(a+bx^n)^{\frac{1}{s}}$.

(ii) And *when* $\frac{m}{n}$ *is a negative integer or zero*,

$$\frac{z^{r+s-1}}{(z^s-a)^{-\frac{m}{n}+1}}$$

may be put into *partial fractions* by the rules explained in Chapter V., and the integration can then be effected in finite terms.

214. III. Again, we may write the integral

$$\int x^{m-1}(a+bx^n)^{\frac{r}{s}}dx \quad \text{as} \quad \int x^{m+\frac{rn}{s}-1}(b+ax^{-n})^{\frac{r}{s}}dx,$$

and therefore by case II. this is integrable in finite terms if $\dfrac{m+\frac{rn}{s}}{-n}$ *be an integer, positive or negative, i.e. if* $\frac{m}{n}+\frac{r}{s}$ *be an integer negative or positive, and the proper substitution is*

$b+ax^{-n}=z^s$, leading to a finite expansion if $\frac{m}{n}+\frac{r}{s}$ be a negative integer, or to partial fractions if $\frac{m}{n}+\frac{r}{s}$ be a positive integer or zero.

215. To sum up:

Case I. p a positive integer: **Expand.**

Case II. $\frac{m}{n}$ an integer: **Substitute $a+bx^n=z^s$; then expand, or partial fractions, as the case may require.**

Case III. $\frac{m}{n}+p$ an integer: **Substitute $ax^{-n}+b=z^s$; then expand, or partial fractions, as the case may require.**

216. **Illustrative Examples.**

1. p a positive integer.

Consider $I\equiv\int x^5(1+x^7)^3\,dx=\int(x^5+3x^{12}+3x^{19}+x^{26})\,dx$

$$=\frac{x^6}{6}+\frac{3x^{13}}{13}+\frac{3x^{20}}{20}+\frac{x^{27}}{27}.$$

2. $\frac{m}{n}$ a positive integer.

Consider $I\equiv\int x^{13}(1+x^7)^{\frac{3}{5}}\,dx.$ Here $\frac{m}{n}=\frac{14}{7}=2.$

Let $\qquad 1+x^7=z^5;\quad\therefore\ dx=\frac{5}{7}\frac{z^4}{x^6}\,dz.$

$$I=\int x^{13}z^3\cdot\frac{5}{7}\frac{z^4}{x^6}\,dz=\frac{5}{7}\int z^7(z^5-1)\,dz=\frac{5}{7}\left(\frac{z^{13}}{13}-\frac{z^8}{8}\right)$$

$$=\frac{5}{7}\left[\frac{1}{13}(1+x^7)^{\frac{13}{5}}-\frac{1}{8}(1+x^7)^{\frac{8}{5}}\right].$$

3. $\frac{m}{n}$ a negative integer.

Consider $I\equiv\int x^{-8}(1+x^7)^{\frac{1}{3}}\,dx.$ Here $\frac{m}{n}=-\frac{7}{7}=-1.$

Let $\qquad 1+x^7=z^3;\quad\therefore\ dx=\frac{3}{7}\frac{z^2}{x^6}\,dz.$

$$I=\int x^{-8}\cdot z\,\frac{3}{7}\frac{z^2}{x^6}\,dz=\frac{3}{7}\int\frac{z^3}{(z^3-1)^2}\,dz.$$

Following the rules of Arts. 155-156, we may express $\frac{z^3}{(z^3-1)^2}$ as

$$\frac{1}{9}\frac{1}{z-1}+\frac{1}{9}\frac{1}{(z-1)^2}-\frac{1}{18}\frac{2z+1+5}{(z+\frac{1}{2})^2+\frac{3}{4}}+\frac{1}{6}\frac{2z+1+1}{(z^2+z+1)^2},$$

whence

$$\int \frac{z^3}{(z^3-1)^2}dz = \frac{1}{9}\log(z-1) - \frac{1}{9}\cdot\frac{1}{z-1} - \frac{1}{18}\log(z^2+z+1) - \frac{5}{9\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}$$
$$-\frac{1}{6}\frac{1}{z^2+z+1}+\frac{1}{6}\int\frac{dz}{[(z+\frac{1}{2})^2+\frac{3}{4}]^2}.$$

In the last term, put $z+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$; $\therefore\ dz=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$;

$$\therefore\ \int\frac{dz}{(z^2+z+1)^2}=\frac{\sqrt{3}}{2}\int\frac{\sec^2\theta\,d\theta}{\frac{9}{16}\sec^4\theta}=\frac{8}{3\sqrt{3}}\int\cos^2\theta\,d\theta$$
$$=\frac{4}{3\sqrt{3}}\int(1+\cos 2\theta)\,d\theta=\frac{4}{3\sqrt{3}}(\theta+\sin\theta\cos\theta)$$
$$=\frac{4}{3\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}+\frac{1}{3}\frac{2z+1}{z^2+z+1}.$$

Hence $\int x^{-8}(1+x^7)^{\frac{1}{3}}\,dx$

$$=\frac{3}{7}\left[\frac{1}{9}\log(z-1)-\frac{1}{9}\frac{1}{z-1}-\frac{1}{18}\log(z^2+z+1)+\frac{1}{9}\frac{z-1}{z^2+z+1}-\frac{1}{3\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}\right],$$

where $z=\sqrt[3]{1+x^7}$.

4. $\frac{m}{n}+\frac{r}{s}$ a positive integer.

Consider $I\equiv\int x^{\frac{1}{2}}(1+x^3)^{\frac{1}{2}}\,dx$. Here $\frac{m}{n}+\frac{r}{s}=\frac{\frac{3}{2}}{3}+\frac{1}{2}=1$.

Then $$I=\int x^2(1+x^{-3})^{\frac{1}{2}}\,dx.$$

Let $$1+x^{-3}=z^2;\quad dx=-\frac{2}{3}x^4z\,dz;$$

$$\therefore\ I=-\frac{2}{3}\int x^6z^2\,dz=-\frac{2}{3}\int\frac{z^2}{(z^2-1)^2}dz,$$

which can be put into partial fractions. In this case, however, the labour can be avoided by the substitution $z=\sec\theta$, and then

$$\int\frac{z^2}{(z^2-1)^2}dz=\int\frac{\sec^2\theta\sec\theta\tan\theta\,d\theta}{\tan^4\theta}=\int\operatorname{cosec}^3\theta\,d\theta$$
$$=-\int\sqrt{1+\cot^2\theta}\,d\cot\theta$$
$$=-\left[\frac{\cot\theta\sqrt{1+\cot^2\theta}}{2}+\frac{1}{2}\log(\cot\theta+\sqrt{1+\cot^2\theta})\right];$$

$$\therefore\ I=\frac{1}{3}\left[\cot\theta\operatorname{cosec}\theta+\log(\cot\theta+\operatorname{cosec}\theta)\right],$$

where $\cos\theta=\frac{1}{z}=\frac{x^{\frac{3}{2}}}{(1+x^3)^{\frac{1}{2}}}$.

5. $\frac{m}{n}+\frac{r}{s}$ a negative integer.

Consider $I=\int x^2(1+x^5)^{-\frac{13}{5}}\,dx$. Here $\frac{m}{n}+\frac{r}{s}=\frac{3}{5}-\frac{13}{5}=-2.$

Then $$I=\int x^{-11}(1+x^{-5})^{-\frac{13}{5}}\,dx.$$

Put $$1+x^{-5}=z^5\,;\quad dx=-x^6z^4\,dz\,;$$

$$\therefore\ I=-\int x^{-5}z^{-9}\,dz$$
$$=+\int z^{-9}(1-z^5)\,dz$$
$$=-\frac{z^{-8}}{8}+\frac{z^{-3}}{3}=\frac{1}{3}\frac{x^3}{(1+x^5)^{\frac{3}{5}}}-\frac{1}{8}\frac{x^8}{(1+x^5)^{\frac{8}{5}}}$$
$$=\frac{1}{24}\frac{x^3(8+5x^5)}{(1+x^5)^{\frac{8}{5}}}.$$

217. The Six Connections Possible.

When $X\equiv a+bx^n$ and $\int x^{m-1}X^p\,dx$ is not immediately integrable by one of the foregoing rules, it may be shown that, by integration by parts, it can be connected with any of six other integrals.

Thus, for instance,

$$\int x^{m-1}X^p\,dx=\frac{x^mX^p}{m}-\int\frac{npb}{m}x^{m+n-1}X^{p-1}\,dx,$$

and by different modes of treatment we may show that the six integrals, with any one of which

$$\int x^{m-1}X^p\,dx$$

can be linearly connected, are

$$\int x^{m-1}X^{p-1}\,dx,\qquad \int x^{m-1}X^{p+1}\,dx,$$
$$\int x^{m-n-1}X^p\,dx,\qquad \int x^{m+n-1}X^p\,dx,$$
$$\int x^{m-n-1}X^{p+1}\,dx,\qquad \int x^{m+n-1}X^{p-1}\,dx,$$

that is, the index of X can be decreased or increased by 1, leaving the index of x unaltered;

the index of x can be decreased or increased by n, leaving the index of X unaltered;

the index of x can be decreased by n, and that of X increased by 1;

or, the index of x can be increased by n, and that of X decreased by 1.

That is, either index can be increased or decreased, leaving the other unaltered, that of x by n, that of X by 1;

or, the one increased and the other decreased in that way (but *not both increased or both decreased* at the same operation).

The **rule for effecting this connection** may be put into the following handy form:

Let $P=x^{\lambda+1}X^{\mu+1}$, where λ and μ are the *smaller indices of x and X respectively*, in the two expressions whose integrals are to be connected. Find $\frac{dP}{dx}$. Rearrange this if necessary *as a linear function of the expressions whose integrals are to be connected.* Integrate, and the connection is complete.

In the rearrangement it may be necessary to substitute $a+bx^n$ for X, or $\frac{X-a}{b}$ for x^n, as may be required for the particular case in hand.

The rearrangement can always be performed. It will be unnecessary to integrate by parts. The advantage derivable from the use of the rule of "The Smaller Index $+1$" will be that it will enable us to connect at once with the particular one of the six possible integrals which may appear desirable.

218. **Proof of the Rule of "The Smaller $+1$."**

For proof it is sufficient to verify the rule in each case. Thus to connect

$$\int x^{m-1}X^p\,dx \quad \text{with} \quad \int x^{m-1}X^{p-1}\,dx,$$

put $P=x^mX^p$.

$$\therefore\ \frac{dP}{dx}=mx^{m-1}X^p+x^m pX^{p-1}\frac{dX}{dx}$$

$$=mx^{m-1}X^p+pbnx^{m+n-1}X^{p-1}$$

$$=mx^{m-1}X^p+pnx^{m-1}(X-a)X^{p-1},$$

(note the rearrangement "as a linear function, etc."),

$$=(m+pn)x^{m-1}X^p-apnx^{m-1}X^{p-1}.$$

Hence, $$P=(m+pn)\int x^{m-1}X^p\,dx-apn\int x^{m-1}X^{p-1}\,dx;$$

or, $$\int x^{m-1}X^p\,dx=\frac{x^mX^p}{m+pn}+\frac{apn}{m+pn}\int x^{m-1}X^{p-1}\,dx.$$

The *advantage in this reduction* lies in the fact that the *index of the often troublesome factor X^p may be lowered* if p be positive, or *raised* if p be negative, and by *successive applications of the same formula,* if necessary, we may ultimately *reduce the integral* to one which has been previously obtained, or which can be managed with greater ease.

219. **List of the Six Connections.**

The student should verify all six connections by the above rule, and also by integration by parts.

They are as follow:

(1) $$\int x^{m-1}X^p\,dx=\frac{x^mX^p}{m+pn}+\frac{apn}{m+pn}\int x^{m-1}X^{p-1}\,dx.$$

(2) $$\int x^{m-1}X^p\,dx=-\frac{x^mX^{p+1}}{an(p+1)}+\frac{m+pn+n}{an(p+1)}\int x^{m-1}X^{p+1}\,dx.$$

(3) $$\int x^{m-1}X^p\,dx=\frac{x^{m-n}X^{p+1}}{b(m+pn)}-\frac{(m-n)a}{b(m+pn)}\int x^{m-n-1}X^p\,dx.$$

(4) $$\int x^{m-1}X^p\,dx=\frac{x^mX^{p+1}}{am}-\frac{(m+pn+n)b}{am}\int x^{m+n-1}X^p\,dx.$$

(5) $$\int x^{m-1}X^p\,dx=\frac{x^{m-n}X^{p+1}}{bn(p+1)}-\frac{m-n}{bn(p+1)}\int x^{m-n-1}X^{p+1}\,dx.$$

(6) $$\int x^{m-1}X^p\,dx=\frac{x^mX^p}{m}-\frac{bpn}{m}\int x^{m+n-1}X^{p-1}\,dx.$$

We have written $m-1$ as the index of x in the primary integral. This is merely for the convenience of making the several coefficients on the right-hand side smaller and more compact than they would be with an index m.

220. **Special Cases.**

The case where $m+pn=0$ comes under the heading $\frac{m}{n}+p=$ integer, already discussed (Art. 211), and needs no reduction formula.

The case $p=0$ integrates at once; as also the case $n=0$.

The case $p+1=0$ integrates by partial fractions.

The case $m=0$ needs no reduction formula, coming under the heading of Case II. Art. 213, (ii).

When the student is convinced of the truth of the rule *in all cases*, the *six possibilities* of connection and the *method* of connection are all that need be remembered.

That the increase or decrease in the index of x should be "n at a time," whilst that of X is only "1 at a time," is to be expected, since $X \equiv a+bx^n$.

221. An integral of form

$$\int x^{n-1}(ax^p+bx^q)^r\,dx$$

can be written as $$\int x^{n+pr-1}(a+bx^{q-p})^r\,dx,$$

or as $$\int x^{n+qr-1}(b+ax^{p-q})^r\,dx,$$

and therefore is reduced at once to the form considered.

222. Integrals of form

$$\int\frac{x^m}{(a+bx^n)^p}\,dx, \quad \text{or} \quad \int\frac{(a+bx^n)^p}{x^m}\,dx, \quad \text{or} \quad \int\frac{dx}{x^m(a+bx^n)^p},$$

are obviously included in the same rules, as there has been no limitation as to the signs of the indices in the formulae discussed.

223. Illustrative Examples.

Ex. 1. Find the value of $I\equiv\int(x^2+a^2)^{\frac{5}{2}}dx$.

We may connect with $\int(x^2+a^2)^{\frac{3}{2}}dx$, and this again with $\int(x^2+a^2)^{\frac{1}{2}}dx$, and this last is a standard form.

As the reduction is to be used *more than once*, we will connect

$$\int(x^2+a^2)^{\frac{n}{2}}dx \quad \text{with} \quad \int(x^2+a^2)^{\frac{n}{2}-1}dx.$$

Let $P=x(x^2+a^2)^{\frac{n}{2}}$.

Then $$\frac{dP}{dx}=(x^2+a^2)^{\frac{n}{2}}+nx^2(x^2+a^2)^{\frac{n}{2}-1}$$

$$=(x^2+a^2)^{\frac{n}{2}}+n(x^2+a^2-a^2)(x^2+a^2)^{\frac{n}{2}-1}$$

(note this preparatory step, which *might be performed mentally*)

$$=(n+1)(x^2+a^2)^{\frac{n}{2}}-na^2(x^2+a^2)^{\frac{n}{2}-1}$$

(which is now arranged as a *linear function of the two expressions whose integrals were to be connected*).

Integrating, $$P=(n+1)\int(x^2+a^2)^{\frac{n}{2}}dx-na^2\int(x^2+a^2)^{\frac{n}{2}-1}dx,$$

i.e. $$\int(x^2+a^2)^{\frac{n}{2}}dx=\frac{x(x^2+a^2)^{\frac{n}{2}}}{n+1}+\frac{na^2}{n+1}\int(x^2+a^2)^{\frac{n}{2}-1}dx.$$

Putting $n=5$ and then $n=3$,

$$\int(x^2+a^2)^{\frac{5}{2}}dx=\frac{x(x^2+a^2)^{\frac{5}{2}}}{6}+\frac{5a^2}{6}\int(x^2+a^2)^{\frac{3}{2}}dx$$

and $$\int(x^2+a^2)^{\frac{3}{2}}dx=\frac{x(x^2+a^2)^{\frac{3}{2}}}{4}+\frac{3a^2}{4}\int(x^2+a^2)^{\frac{1}{2}}dx$$

and $$\int(x^2+a^2)^{\frac{1}{2}}dx=\frac{x(x^2+a^2)^{\frac{1}{2}}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}.$$

Thus $$\int(x^2+a^2)^{\frac{5}{2}}dx=\frac{1}{6}x(x^2+a^2)^{\frac{5}{2}}+\frac{5}{6.4}a^2x(x^2+a^2)^{\frac{3}{2}}$$
$$+\frac{5.3}{6.4.2}a^4x(x^2+a^2)^{\frac{1}{2}}+\frac{5.3}{6.4.2}a^6\sinh^{-1}\frac{x}{a}.$$

This result might have been obtained more quickly by substituting $x=a\tan\theta$ and using the reduction formula

$$\int\sec^{n+2}\theta\,d\theta=\frac{1}{n+1}\tan\theta\sec^n\theta+\frac{n}{n+1}\int\sec^n\theta\,d\theta \text{ (Art. 122)},$$

whence we get

$$I=\int(x^2+a^2)^{\frac{5}{2}}dx=a^6\int\sec^7\theta\,d\theta$$
$$=a^6[\tfrac{1}{6}\tan\theta\sec^5\theta+\tfrac{5}{6}\{\tfrac{1}{4}\tan\theta\sec^3\theta+\tfrac{3}{4}(\tfrac{1}{2}\tan\theta\sec\theta$$
$$+\tfrac{1}{2}\log\overline{\sec\theta+\tan\theta})\}],$$

which gives the same result as before.

Ex. 2. Find the value of $I\equiv\int\frac{dx}{(x^2+a^2)^{\frac{5}{2}}}$.

First connect $\int(x^2+a^2)^{-\frac{n}{2}}dx$ with $\int(x^2+a^2)^{-\frac{n}{2}+1}dx$.

Put $P=x(x^2+a^2)^{-\frac{n}{2}+1}$.

$$\frac{dP}{dx}=(x^2+a^2)^{-\frac{n}{2}+1}-(n-2)x^2(x^2+a^2)^{-\frac{n}{2}}$$
$$=(x^2+a^2)^{-\frac{n}{2}+1}-(n-2)(x^2+a^2-a^2)(x^2+a^2)^{-\frac{n}{2}}$$
$$=(3-n)(x^2+a^2)^{-\frac{n}{2}+1}+(n-2)a^2(x^2+a^2)^{-\frac{n}{2}};$$

$$\therefore\int(x^2+a^2)^{-\frac{n}{2}}dx=\frac{x(x^2+a^2)^{-\frac{n}{2}+1}}{(n-2)a^2}+\frac{n-3}{(n-2)a^2}\int(x^2+a^2)^{-\frac{n}{2}+1}dx.$$

Putting $n=5$ and then $n=3$,

$$\int (x^2+a^2)^{-\frac{5}{2}}\,dx=\frac{1}{3}\frac{x(x^2+a^2)^{-\frac{3}{2}}}{a^2}+\frac{2}{3a^2}\int (x^2+a^2)^{-\frac{3}{2}}\,dx$$

and
$$\int (x^2+a^2)^{-\frac{3}{2}}\,dx=\frac{1}{1}\cdot\frac{x(x^2+a^2)^{-\frac{1}{2}}}{a^2}+0\,;$$

$$\therefore\ \int (x^2+a^2)^{-\frac{5}{2}}\,dx=\frac{1}{3}\frac{x(x^2+a^2)^{-\frac{3}{2}}}{a^2}+\frac{2}{3a^4}x(x^2+a^2)^{-\frac{1}{2}}.$$

This again would have been shortened by the substitution $x=a\tan\theta$, which is specially suited for functions involving $\sqrt{x^2+a^2}$.

Thus
$$\int \frac{dx}{(x^2+a^2)^{\frac{5}{2}}}=\frac{1}{a^4}\int\frac{\sec^2\theta\,d\theta}{\sec^5\theta}=\frac{1}{a^4}\int\cos^3\theta\,d\theta$$

$$=\frac{1}{a^4}\int(1-\sin^2\theta)\,d\sin\theta$$

$$=\frac{1}{a^4}\left(\sin\theta-\frac{\sin^3\theta}{3}\right),\quad\text{where }\sin\theta=\frac{x}{\sqrt{a^2+x^2}},$$

$$=\frac{1}{a^4}\left\{\frac{x}{(a^2+x^2)^{\frac{1}{2}}}-\frac{1}{3}\frac{x^3}{(a^2+x^2)^{\frac{3}{2}}}\right\},$$

which is the same as the previous result, though in a different form.

Ex. 3. Find the value of $I_n\equiv\int (x^2+a^2)^{\frac{n}{2}}\,dx$, n being a positive odd integer. Let $x(x^2+a^2)^{\frac{n}{2}}\equiv P_n$.

Since
$$I_n=\frac{P_n}{n+1}+\frac{na^2}{n+1}I_{n-2}\quad\text{(Ex. 1)},$$

$$I_{n-2}=\frac{P_{n-2}}{n-1}+\frac{n-2}{n-1}a^2I_{n-4},$$

etc.,

and
$$I_1=\int\sqrt{x^2+a^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a},$$

we have

$$I_n=\frac{P_n}{n+1}+\frac{n}{(n+1)(n-1)}a^2P_{n-2}+\frac{n(n-2)}{(n+1)(n-1)(n-3)}a^4P_{n-4}+\dots$$
$$+\frac{n(n-2)(n-4)\dots 3}{(n+1)(n-1)\dots 4.2}a^{n-1}P_1$$
$$+\frac{n(n-2)(n-4)\dots 3.1}{(n+1)(n-1)\dots 4.2}a^{n+1}\sinh^{-1}\frac{x}{a}.$$

Ex. 4. Find the value of $I_n\equiv\int\frac{dx}{(x^2+a^2)^{\frac{n}{2}}}$, n being a positive integer. Let $\frac{x}{(x^2+a^2)^{\frac{n-2}{2}}}\equiv P_n$.

Since
$$I_n=\frac{P_n}{(n-2)a^2}+\frac{n-3}{n-2}\frac{1}{a^2}I_{n-2},$$

we have
$$I_{n-2}=\frac{P_{n-2}}{(n-4)a^2}+\frac{n-5}{n-4}\frac{1}{a^2}I_{n-4}$$

etc.

When n is an odd positive integer, we ultimately arrive at I_3, and

$$I_3=\int\frac{dx}{(x^2+a^2)^{\frac{3}{2}}}=\frac{x(x^2+a^2)^{-\frac{1}{2}}}{a^2}=\frac{P_3}{a^2};$$

$$\therefore\quad I_n=\frac{1}{n-2}\frac{P_n}{a^2}+\frac{n-3}{(n-2)(n-4)}\frac{P_{n-2}}{a^4}+\frac{(n-3)(n-5)}{(n-2)(n-4)(n-6)}\frac{P_{n-4}}{a^6}+\dots$$

$$+\frac{(n-3)(n-5)\dots 2}{(n-2)(n-4)\dots 3.1}\frac{P_3}{a^{r-1}},\quad \text{where } P_n\equiv\frac{x}{(x^2+a^2)^{\frac{n-2}{2}}}$$

In the case when n is an even integer, $\equiv 2m$ say,

$$I_{2m}=\int\frac{dx}{(x^2+a^2)^m}=\frac{P_{2m}}{(2m-2)a^2}+\frac{2m-3}{2m-2}\cdot\frac{1}{a^2}I_{2(m-1)},\quad \text{where } P_{2m}=\frac{x}{(x^2+a^2)^{m-1}}$$

$$=\frac{1}{2m-2}\cdot\frac{P_{2m}}{a^2}+\frac{2m-3}{(2m-2)(2m-4)}\frac{P_{2(m-1)}}{a^4}+\frac{(2m-3)(2m-5)}{(2m-2)(2m-4)(2m-6)}\frac{P_{2(m-2)}}{a^6}$$

$$+\dots+\frac{(2m-3)(2m-5)\dots 1}{(2m-2)(2m-4)(2m-6)\dots 2}\frac{1}{a^{2m-1}}\tan^{-1}\frac{x}{a}.$$

In integration between limits 0 and ∞,

$$I_{2m}\Big]_0^\infty=\frac{(2m-3)(2m-5)\dots 1}{(2m-2)(2m-4)\dots 2}\cdot\frac{1}{a^{2m-1}}\cdot\frac{\pi}{2}.$$

M. Bertrand* shows a very ingenious deduction from this result, viz. putting $a=1$ and $x=\dfrac{z}{\sqrt{m}}$,

$$\frac{1}{\sqrt{m}}\int_0^\infty\frac{dz}{\left(1+\frac{z^2}{m}\right)^m}=\frac{(2m-3)(2m-5)\dots 1}{(2m-2)(2m-4)\dots 2}\frac{\pi}{2}.$$

Take the case when m is indefinitely increased; then

$$Lt_{m=\infty}\left(1+\frac{z^2}{m}\right)^m=e^{z^2}.$$

Hence $$\int_0^\infty e^{-z^2}dz=\frac{\pi}{2}Lt_{m=\infty}\frac{1.3.5\dots(2m-3)}{2.4.6\dots(2m-2)}\sqrt{m},$$

and by Wallis's Theorem (Hobson, *Trigonometry*, p. 331),

$$\frac{2.4.6\dots(2m-2)}{1.3.5\dots(2m-3)}\quad\text{and}\quad\sqrt{\frac{\pi}{2}(2m-1)}$$

become infinite in a ratio of equality.

Hence $$\frac{\pi}{2}Lt_{m=\infty}\frac{1.3.5\dots(2m-3)}{2.4.6\dots(2m-2)}\sqrt{m}$$

$$=\frac{\pi}{2}Lt\frac{\sqrt{m}}{\sqrt{\frac{\pi}{2}(2m-1)}}=\frac{\sqrt{\pi}}{2};$$

$$\therefore\quad\int_0^\infty e^{-z^2}dz=\tfrac{1}{2}\sqrt{\pi}.$$

Consider also $I_m=\int_0^\infty x^m\epsilon^{-x^2}dx$, m being a positive integer.

* Bertrand, *Calc. Diff.* p. 130; see also Hall, *D. and I. C.*, p. 330.

Integrating by parts,

$$I_m = -\frac{1}{2}\int_0^\infty x^{m-1}(-2xe^{-x^2})\,dx$$

$$= -\frac{1}{2}\left\{\left[x^{m-1}e^{-x^2}\right]_0^\infty - (m-1)\int_0^\infty x^{m-2}e^{-x^2}\,dx\right\}$$

$$= \frac{m-1}{2}I_{m-2} \quad (m>1).$$

Now $$I_0 = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

and $$I_1 = \int_0^\infty xe^{-x^2}\,dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^\infty = \frac{1}{2};$$

$$\therefore \int_0^\infty x^{2n}e^{-x^2}\,dx = \frac{(2n-1)(2n-3)\dots 1}{2^{n+1}}\sqrt{\pi},$$

$$\int_0^\infty x^{2n+1}e^{-x^2}\,dx = \frac{2n.(2n-2)\dots 4.2}{2^{n+1}} = \frac{n!}{2},$$ n being a positive integer

Note also that if the integration extends from $-\infty$ to $+\infty$,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

$$\int_{-\infty}^{\infty} x^{2n}e^{-x^2}\,dx = \frac{(2n-1)(2n-3)\dots 1}{2^n}\sqrt{\pi},$$

but $$\int_{-\infty}^{\infty} x^{2n+1}e^{-x^2}\,dx = 0,$$

for to any positive element of the integrand in the third integral there is always an equal negative element.

Ex. 5. Calculate the value of $\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx$, m being a positive integer.

We proceed to connect

$$\int x^m\sqrt{2ax-x^2}\,dx \text{ with } \int x^{m-1}\sqrt{2ax-x^2}\,dx,$$

i.e. $$\int x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx \text{ with } \int x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx.$$

Let $P = x^{m+\frac{1}{2}}(2a-x)^{\frac{3}{2}}$, according to the rule; then

$$\frac{dP}{dx} = (m+\tfrac{1}{2})x^{m-\frac{1}{2}}(2a-x)^{\frac{3}{2}} - \tfrac{3}{2}x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}$$

$$= (2m+1)ax^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}} - (m+2)x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}.$$

Hence

$$(m+2)\int x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx = -x^{m+\frac{1}{2}}(2a-x)^{\frac{3}{2}} + (2m+1)a\int x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx,$$

i.e.

$$\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx = -\left[\frac{x^{m-1}(2ax-x^2)^{\frac{3}{2}}}{m+2}\right]_0^{2a} + \frac{2m+1}{m+2}a\int_0^{2a} x^{m-1}\sqrt{2ax-x^2}dx$$

$$= \frac{2m+1}{m+2}a\int_0^{2a} x^{m-1}\sqrt{2ax-x^2}\,dx;$$

$\therefore$ if $I_m = \int_0^{2a} x^m \sqrt{2ax - x^2}\, dx$,

$$I_m = \frac{2m+1}{m+2} a I_{m-1} = \frac{2m+1}{m+2} \cdot \frac{2m-1}{m+1} a^2 I_{m-2}$$

$$= \frac{2m+1}{m+2} \cdot \frac{2m-1}{m+1} \cdot \frac{2m-3}{m} a^3 I_{m-3} = \text{etc.}$$

$$= \frac{2m+1}{m+2} \cdot \frac{2m-1}{m+1} \cdot \frac{2m-3}{m} \cdots \frac{5}{4} \cdot \frac{3}{3} \cdot a^m I_0.$$

Now, to find I_0 or $\int_0^{2a} \sqrt{2ax - x^2}\, dx$, put $x = a(1 - \cos\theta)$.

Then $$dx = a \sin\theta\, d\theta$$

and $$\sqrt{2ax - x^2} = a \sin\theta.$$

Also, when $x = 0$ we have $\theta = 0$; when $x = 2a$ we have $\theta = \pi$.
Hence

$$I_0 = \int_0^{\pi} a^2 \sin^2\theta\, d\theta = \frac{a^2}{2} \int_0^{\pi} (1 - \cos 2\theta)\, d\theta = \frac{a^2}{2}\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi} = \frac{\pi a^2}{2}.$$

Hence $$I_m = \frac{(2m+1)(2m-1)\ldots 3}{(m+2)(m+1)\ldots 3} a^{m+2}\frac{\pi}{2} = \frac{(2m+1)!}{m!\,(m+2)!} \cdot \frac{\pi a^{m+2}}{2^m}.$$

EXAMPLES.

Prove that

1. $\int x^{m-1}(a+bx)^p\, dx = \dfrac{x^m (a+bx)^p}{m+p} + \dfrac{ap}{m+p}\int x^{m-1}(a+bx)^{p-1}\, dx.$

2. $\int \dfrac{(a+bx)^p}{x^{m+1}}\, dx = -\dfrac{(a+bx)^{p+1}}{ma x^m} + \dfrac{(p-m+1)}{m}\dfrac{b}{a}\int \dfrac{(a+bx)^p}{x^m}\, dx.$

3. $\int \dfrac{dx}{x^m(a+bx)} = -\dfrac{1}{m-1}\cdot\dfrac{1}{a}\cdot\dfrac{1}{x^{m-1}} + \dfrac{1}{m-2}\cdot\dfrac{b}{a^2}\cdot\dfrac{1}{x^{m-2}} - \dfrac{1}{m-3}\cdot\dfrac{b^2}{a^3}\cdot\dfrac{1}{x^{m-3}}$

$$+ \ldots + (-1)^{m-1}\frac{1}{1}\cdot\frac{b^{m-2}}{a^{m-1}}\cdot\frac{1}{x} + (-1)^m \frac{b^{m-1}}{a^m}\log\frac{a+bx}{x}.$$

4. $\int \dfrac{(a+bx)^p}{x}\, dx = \dfrac{(a+bx)^p}{p} + a\int \dfrac{(a+bx)^{p-1}}{x}\, dx.$ [BERTRAND.]

5. $\int \dfrac{dx}{(a+bx^2)^{p+1}} = \dfrac{x}{2ap(a+bx^2)^p} + \dfrac{2p-1}{2ap}\int \dfrac{dx}{(a+bx^2)^p}.$ [BERTRAND.]

6. $\int \dfrac{x^n\, dx}{(a+bx^3)^{p+1}} = \dfrac{x^{n+1}}{3ap(a+bx^3)^p} - \dfrac{n-3p+1}{3ap}\int \dfrac{x^n}{(a+bx^3)^p}\, dx,$ [BERTRAND.]

and evaluate

$$\int \frac{x^7}{a+bx^3}\, dx, \quad \int \frac{x^3}{(a+bx^3)^2}\, dx, \quad \int \frac{dx}{x^3(a+bx^3)}.$$

7. $$\int x^n(a+bx^4)^p\,dx = \frac{x^{n+1}(a+bx^4)^p}{n+1} - \frac{4bp}{n+1}\int x^{n+4}(a+bx^4)^{p-1}\,dx,$$

$$\int \frac{x^n}{a+bx^4}\,dx = \frac{x^{n-3}}{(n-3)b} - \frac{a}{b}\int \frac{x^{n-4}}{a+bx^4}\,dx,$$

$$\int \frac{dx}{(a+bx^4)^{p+1}} = \frac{x}{4ap(a+bx^4)^p} + \frac{4p-1}{4ap}\int \frac{dx}{(a+bx^4)^p},$$ [BERTRAND.]

and evaluate $$\int \frac{dx}{(a+bx^4)^4}, \quad \int \frac{dx}{x^2(a+bx^4)^3}.$$

224. Reduction formulae for $\int \sin^p x \cos^q x\,dx$.

Integrals of this form also conform to the rule of "the smaller index $+1$," explained in Art. 217.

Connection can be effected with any of the following six integrals:

$$\int \sin^{p-2} x \cos^q x\,dx, \qquad \int \sin^{p+2} x \cos^q x\,dx,$$

$$\int \sin^p x \cos^{q-2} x\,dx, \qquad \int \sin^p x \cos^{q+2} x\,dx,$$

$$\int \sin^{p-2} x \cos^{q+2} x\,dx, \qquad \int \sin^{p+2} x \cos^{q-2} x\,dx,$$

by the following rule:

Put $P = \sin^{\lambda+1} x \cos^{\mu+1} x$, where λ and μ are the smaller indices of $\sin x$ and $\cos x$ respectively in the two expressions whose integrals are to be connected.

Find $\frac{dP}{dx}$, and rearrange as a *linear function of the expressions whose integrals are to be connected.* This rearrangement can always be performed.

Integrate, and the connection is effected.

Each of these connections might be effected by integration by parts, but the advantage to be gained by the present rule is the same as has been explained in Art. 217.

For example, let us connect the integrals

$$\int \sin^p x \cos^q x\,dx \quad \text{and} \quad \int \sin^{p-2} x \cos^q x\,dx.$$

Let $P = \sin^{p-1}x\cos^{q+1}x$.

$$\frac{dP}{dx} = (p-1)\sin^{p-2}x\cos^{q+2}x - (q+1)\sin^p x\cos^q x$$
$$= (p-1)\sin^{p-2}x\cos^q x(1-\sin^2 x) - (q+1)\sin^p x\cos^q x$$
$$= (p-1)\sin^{p-2}x\cos^q x - (p+q)\sin^p x\cos^q x.$$

[Note the last two lines of *rearrangement as a linear function* of $\sin^p x\cos^q x$ and $\sin^{p-2}x\cos^q x$.]

Hence

$$P = (p-1)\int\sin^{p-2}x\cos^q x\,dx - (p+q)\int\sin^p x\cos^q x\,dx$$

and

$$\int\sin^p x\cos^q x\,dx = -\frac{\sin^{p-1}x\cos^{q+1}x}{p+q} + \frac{p-1}{p+q}\int\sin^{p-2}x\cos^q x\,dx.$$

225. **List of the Six Connections.**

The student should note carefully the *possibilities* of connection for $\int\sin^p x\cos^q x\,dx$.

The indices of either $\sin x$ or $\cos x$ may be increased or diminished by 2, the other index being unaltered; or, the index of the one lowered by 2 and the other increased by 2.

Writing s for $\sin x$ and c for $\cos x$, the six connections are:

$$(1)\quad \int s^p c^q\,dx = -\frac{s^{p-1}c^{q+1}}{p+q} + \frac{p-1}{p+q}\int s^{p-2}c^q\,dx.$$

$$(2)\quad \int s^p c^q\,dx = \frac{s^{p+1}c^{q+1}}{p+1} + \frac{p+q+2}{p+1}\int s^{p+2}c^q\,dx.$$

$$(3)\quad \int s^p c^q\,dx = \frac{s^{p+1}c^{q-1}}{p+q} + \frac{q-1}{p+q}\int s^p c^{q-2}\,dx.$$

$$(4)\quad \int s^p c^q\,dx = -\frac{s^{p+1}c^{q+1}}{q+1} + \frac{p+q+2}{q+1}\int s^p c^{q+2}\,dx.$$

$$(5)\quad \int s^p c^q\,dx = -\frac{s^{p-1}c^{q+1}}{q+1} + \frac{p-1}{q+1}\int s^{p-2}c^{q+2}\,dx.$$

$$(6)\quad \int s^p c^q\,dx = \frac{s^{p+1}c^{q-1}}{p+1} + \frac{q-1}{p+1}\int s^{p+2}c^{q-2}\,dx.$$

Each of these should be verified by the student by means of the rule given, viz. "Put $P = \sin^{\lambda+1}x\cos^{\mu+1}x$, where λ, μ are, etc. ...," and also by integration by parts.

226. Special Cases.

When $p+q=0$, the integral is $\int \tan^p x\,dx$, and is integrated by the reduction formulae of Art. 125.

When $p+1=0$,

$$\int \sin^p x \cos^q x\,dx = \int \frac{\cos^q x}{\sin x}\,dx = -\int \frac{\cos^q x}{1-\cos^2 x}\,d(\cos x),$$

and then we write $\cos x = z$, and use the method of partial fractions, or proceed as in Art. 228.

When $q+1=0$,

$$\int \sin^p x \cos^q x\,dx = \int \frac{\sin^p x}{\cos x}\,dx = \int \frac{\sin^p x}{1-\sin^2 x}\,d(\sin x),$$

and then we again use partial fractions, or proceed as in Art. 228.

227. The student is again reminded that **when either p or q is odd, or when $p+q$ is a negative even integer,** *there is an easier mode of procedure* (Art. 114). Also that *in any case we have the method of multiple angles* when the indices are positive and integral; and in general this will be a more speedy method of obtaining the *indefinite* integral than the employment of a reduction formula. The results, however, will be necessarily produced in a different form by such processes.

228. We must also notice that, in the formulae of Art. 225, *either p or q, or both of them, may be negative.* Hence we now have reduction formulae for integrals such as

$$\int \frac{\sin^p x}{\cos^q x}\,dx, \quad \int \frac{\cos^q x}{\sin^p x}\,dx, \quad \text{or} \quad \int \frac{dx}{\sin^p x \cos^q x},$$

and to these the "multiple-angle method" of Art. 112 would not apply, by reason of the non-termination of the binomial expansion used for the purpose of conversion.

Thus, putting $-q$ for q in formula (5) of Art. 225,

$$\int \frac{\sin^p x}{\cos^q x}\,dx = \frac{\sin^{p-1} x}{(q-1)\cos^{q-1} x} - \frac{p-1}{q-1}\int \frac{\sin^{p-2} x}{\cos^{q-2} x}\,dx.$$

Putting $-p$ for p in formula (6),

$$\int \frac{\cos^q x}{\sin^p x}\,dx = -\frac{\cos^{q-1} x}{(p-1)\sin^{p-1} x} - \frac{q-1}{p-1}\int \frac{\cos^{q-2} x}{\sin^{p-2} x}\,dx.$$

Putting $-p$ for p and $-q$ for q in (2) and (4),

$$\int\frac{dx}{\sin^p x\cos^q x}=-\frac{1}{(p-1)\sin^{p-1}x\cos^{q-1}x}+\frac{p+q-2}{p-1}\int\frac{dx}{\sin^{p-2}x\cos^q x}$$

or

$$=\frac{1}{(q-1)\sin^{p-1}x\cos^{q-1}x}+\frac{p+q-2}{q-1}\int\frac{dx}{\sin^p x\cos^{q-2}x}.$$

etc.

If, however, $p=1$ or $q=1$ in these results, *i.e.* for integrals of form $\int\frac{\sin^p x}{\cos x}dx$ or $\int\frac{\cos^q x}{\sin x}dx$, these reductions obviously fail.

In the case $\int\frac{\sin^p x}{\cos x}dx$, we may put $q=-1$ in formula (1), Art. 225.

Then
$$\int\frac{\sin^p x}{\cos x}dx=-\frac{\sin^{p-1}x}{p-1}+\int\frac{\sin^{p-2}x}{\cos x}dx,$$
and repeating the operation, we presently arrive at $\int\frac{\sin^2 x}{\cos x}dx$ if p be an even integer, or at $\int\frac{\sin x}{\cos x}dx$ if p be odd, giving respectively $\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)-\sin x$ or $\log\sec x$ in the two cases.

Similarly, for $\int\frac{\cos^q x}{\sin x}dx$, put $p=-1$ in formula (3);

$$\therefore \int\frac{\cos^q x}{\sin x}dx=\frac{\cos^{q-1}x}{q-1}+\int\frac{\cos^{q-2}x}{\sin x}dx,$$

finally arriving at $\int\frac{\cos^2 x}{\sin x}dx$ or at $\int\frac{\cos x}{\sin x}dx$,

i.e. $\log\tan\frac{x}{2}+\cos x$ or $\log\sin x$ as the case may be.

229. The cases when p or q vanishes, *i.e.* the integrals

$$\int\sin^n x\,dx \quad\text{and}\quad \int\cos^n x\,dx$$

are of primary importance.

Connect $\int\sin^n x\,dx$ with $\int\sin^{n-2}x\,dx.$

Let $P=\sin^{n-1}x\cos x$, according to rule; then

$$\frac{dP}{dx}=(n-1)\sin^{n-2}x\cos^2 x-\sin^n x$$
$$=(n-1)\sin^{n-2}x-n\sin^n x;$$

$$\therefore \int\sin^n x\,dx=-\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int\sin^{n-2}x\,dx.$$

Similarly,

$$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

230. To calculate

$$S_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx \quad \text{and} \quad C_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$$

Since $\sin^{n-1} x \cos x$ vanishes when n is an integer, not less than 2, at both limits, $x=0$ and $x=\frac{\pi}{2}$, we have

$$S_n = \frac{n-1}{n} S_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2} S_{n-4}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4} S_{n-6} = \text{etc.}$$

If n be *even* this ultimately comes to

$$S_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,dx,$$

i.e.

$$S_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}.$$

If n be *odd* we similarly get

$$S_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin x\,dx,$$

and since

$$\int_0^{\frac{\pi}{2}} \sin x\,dx = \Big[-\cos x\Big]_0^{\frac{\pi}{2}} = 1,$$

we have

$$S_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}.$$

In a similar way it may be seen that $\int_0^{\frac{\pi}{2}} \cos^n x\,dx$ has *precisely the same value* as the above integral in each case, n odd, n even. This may be shown, too, from other considerations.

We thus have

$$\int_0^{\frac{\pi}{2}} \binom{\sin}{\cos}^n x\,dx = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}, \quad n \text{ even};$$

or

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{4}{5}\,\frac{2}{3}\,1, \quad n \text{ odd}.$$

231. The student should notice that these formulae are written down most easily by *beginning with the denominator.* We then have the ordinary sequence of the natural numbers written backwards,

$(n \text{ under } \overline{n-1}) \times (\overline{n-2} \text{ under } \overline{n-3}) \times (\overline{n-4} \text{ under } \overline{n-5}) \ldots$ etc.,

stopping at (2 under 1) if n be even, and *writing a factor* $\frac{\pi}{2}$; or stopping at (3 under 2) if n be odd, with *no extra factor.*

Thus (1) $\int_0^{\frac{\pi}{2}} \sin^{12}\theta\, d\theta = \frac{11}{12}\,\frac{9}{10}\,\frac{7}{8}\,\frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}.$

(2) $\int_0^{\frac{\pi}{2}} \sin^{11}\theta\, d\theta = \frac{10}{11}\,\frac{8}{9}\,\frac{6}{7}\,\frac{4}{5}\,\frac{2}{3}.$

(3) $\int_0^{\frac{\pi}{4}} \cos^6 2\theta \cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^6\phi\, \frac{1+\cos\phi}{2}\, \tfrac{1}{2}\, d\phi,$

where $\phi = 2\theta$

$$=\frac{1}{4}\left[\frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}+\frac{6}{7}\,\frac{4}{5}\,\frac{2}{3}\right].$$

(4) $\int_0^{\frac{\pi}{6}} \cos^7 3\theta \sin^4 6\theta\, d\theta = 2^4 \int_0^{\frac{\pi}{6}} \sin^4 3\theta \cos^{11} 3\theta\, d\theta$

$$=\frac{2^4}{3}\int_0^{\frac{\pi}{2}} \sin^4\phi \cos^{11}\phi\, d\phi, \text{ where } \phi = 3\theta,$$

$$=\frac{2^4}{3}\int_0^{\frac{\pi}{2}} (\cos^{11}\phi - 2\cos^{13}\phi + \cos^{15}\phi)\, d\phi$$

$$=\frac{2^4}{3}\,\frac{2}{3}\,\frac{4}{5}\,\frac{6}{7}\,\frac{8}{9}\,\frac{10}{11}\left(1 - 2\frac{12}{13} + \frac{12}{13}\,\frac{14}{15}\right)$$

$$= \text{etc.}$$

Examples.

1. Prove that

$$\int \sin^p\theta \cos^q\theta\, d\theta = \frac{\sin^{p+1}\theta \cos^{q-1}\theta}{p+q} - \frac{q-1}{(p+q)(p+q-2)} \sin^{p-1}\theta \cos^{q-1}\theta$$
$$+\frac{(p-1)(q-1)}{(p+q)(p+q-2)}\int \sin^{p-2}\theta \cos^{q-2}\theta\, d\theta$$

the indices *being both diminished.*

2. Prove that $\int \frac{\sin^p\theta}{\cos^q\theta}\, d\theta = \frac{\sin^{p-1}\theta}{(q-1)\cos^{q-1}\theta} - \frac{p-1}{q-1}\int \frac{\sin^{p-2}\theta}{\cos^{q-2}\theta}\, d\theta.$

3. Prove that $\int \frac{\cos^p\theta}{\sin^q\theta}\, d\theta = -\frac{\cos^{p-1}\theta}{(q-1)\sin^{q-1}\theta} - \frac{p-1}{q-1}\int \frac{\cos^{p-2}\theta}{\sin^{q-2}\theta}\, d\theta.$

4. Prove that

$$\int \frac{d\theta}{\sin^p\theta\cos^q\theta} = \frac{1}{(q-1)\sin^{p-1}\theta\cos^{q-1}\theta} + \frac{p+q-2}{q-1}\int \frac{d\theta}{\sin^p\theta\cos^{q-2}\theta}$$

$$= -\frac{1}{(p-1)\sin^{p-1}\theta\cos^{q-1}\theta} + \frac{p+q-2}{p-1}\int \frac{d\theta}{\sin^{p-2}\theta\cos^q\theta}.$$

5. $$\int \frac{\sin^p\theta}{\cos\theta}\,d\theta = -\frac{\sin^{p-1}\theta}{p-1} + \int \frac{\sin^{p-2}\theta\,d\theta}{\cos\theta}.$$

6. $$\int \frac{\cos^p\theta}{\sin\theta}\,d\theta = \frac{\cos^{p-1}\theta}{p-1} + \int \frac{\cos^{p-2}\theta}{\sin\theta}\,d\theta.$$

7. (a) $$\int \sin^{2n}\theta\,d\theta = -\frac{c}{2n}\left[s^{2n-1} + \frac{2n-1}{2n-2}s^{2n-3} + \frac{(2n-1)(2n-3)}{(2n-2)(2n-4)}s^{2n-5} + \dots + \frac{1.3\dots(2n-1)}{2.4\dots(2n-2)}s\right] + \frac{1.3\dots(2n-1)}{2.4\dots 2n}\theta.$$

(b) $$\int \sin^{2n+1}\theta\,d\theta = -\frac{c}{2n+1}\left[s^{2n} + \frac{2n}{2n-1}s^{2n-2} + \frac{2n(2n-2)}{(2n-1)(2n-3)}s^{2n-4} + \dots + \frac{2.4\dots 2n}{1.3\dots(2n-1)}\right],$$

where c and s stand respectively for $\cos\theta$ and $\sin\theta$. [BERTRAND.]

8. (a) $$\int \cos^{2n}\theta\,d\theta = \frac{s}{2n}\left[c^{2n-1} + \frac{2n-1}{2n-2}c^{2n-3} + \frac{(2n-1)(2n-3)}{(2n-2)(2n-4)}c^{2n-5} + \dots + \frac{1.3\dots(2n-1)}{2.4\dots(2n-2)}c\right] + \frac{1.3\dots(2n-1)}{2.4\dots 2n}\theta.$$

(b) $$\int \cos^{2n+1}\theta\,d\theta = \frac{s}{2n+1}\left[c^{2n} + \frac{2n}{2n-1}c^{2n-2} + \dots + \frac{2.4\dots 2n}{1.3\dots(2n-1)}\right],$$

c and s being respectively $\cos\theta$ and $\sin\theta$. [BERTRAND.]

9. Prove

(a) $$\int \operatorname{cosec}^5\theta\,d\theta = -\frac{1}{4}\frac{c}{s^4} - \frac{1.3}{2.4}\frac{c}{s^2} + \frac{1.3}{2.4}\log\tan\frac{\theta}{2}.$$

(b) $$\int \sec^5\theta\,d\theta = \frac{1}{4}\frac{s}{c^4} + \frac{1.3}{2.4}\frac{s}{c^2} + \frac{3}{8}\log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right),$$

where $c \equiv \cos\theta$, $s \equiv \sin\theta$.

10. Prove that

(a) $$\int \cos^{2n}\theta\,d\theta = \frac{1}{2^{2n-1}}\left[\frac{\sin 2n\theta}{2n} + 2n\frac{\sin(2n-2)\theta}{2n-2} + \frac{2n(2n-1)}{1.2}\frac{\sin(2n-4)\theta}{2n-4} + \dots + \frac{2n(2n-1)\dots(n+1)}{1.2\dots n}\theta\right].$$

(b) $$\int \cos^{2n+1}\theta\,d\theta = \frac{1}{2^{2n}}\left[\frac{\sin(2n+1)\theta}{2n+1} + (2n+1)\frac{\sin(2n-1)\theta}{2n-1} + \frac{(2n+1)2n}{1.2}\frac{\sin(2n-3)\theta}{2n-3} + \dots + \frac{(2n+1)2n\dots(n+2)}{1.2\dots n}\sin\theta\right].$$

(c) $$\int \sin^{2n}\theta\,d\theta = \frac{(-1)^n}{2^{2n-1}}\left[\frac{\sin 2n\theta}{2n} - 2n\frac{\sin(2n-2)\theta}{2n-2} + \frac{2n(2n-1)}{1.2}\frac{\sin(2n-4)\theta}{2n-4} - \dots + (-1)^{n-1}\frac{2n(2n-1)\dots(n+2)}{1.2\dots(n-1)}\frac{\sin 2\theta}{2} + (-1)^n\frac{2n(2n-1)\dots(n+1)}{1.2\dots n}\frac{\theta}{2}\right].$$

(d) $$\int \sin^{2n+1}\theta\,d\theta = \frac{(-1)^{n+1}}{2^{2n}}\left[\frac{\cos(2n+1)\theta}{2n+1} - (2n+1)\frac{\cos(2n-1)\theta}{2n-1} + \dots + (-1)^n\frac{(2n+1)\dots(n+2)}{1.2\dots n}\cos\theta\right].$$

[BERTRAND.]

232. INTRODUCTION OF THE GAMMA FUNCTION.

For what follows we shall require a new function $\Gamma(n+1)$, which will be *defined sufficiently for present purposes* by the equations

$$\Gamma(n+1) = n\Gamma(n), \quad \Gamma(1) = 1, \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

This will be enough to find its value whenever n is a positive integer, or of the form $\frac{2k+1}{2}$, where k is a positive integer.

For instance

$$\Gamma(6) = 5\,\Gamma(5) = 5.4\,\Gamma(4) = 5.4.3\,\Gamma(3)$$
$$= 5.4.3.2\,\Gamma(2) = 5.4.3.2.1\,\Gamma(1) = 5\,!,$$

$$\Gamma(\tfrac{11}{2}) = \tfrac{9}{2}\,\Gamma(\tfrac{9}{2}) = \tfrac{9}{2}.\tfrac{7}{2}\,\Gamma(\tfrac{7}{2}) = \tfrac{9}{2}.\tfrac{7}{2}.\tfrac{5}{2}\,\Gamma(\tfrac{5}{2}) = \tfrac{9}{2}.\tfrac{7}{2}.\tfrac{5}{2}.\tfrac{3}{2}\,\Gamma(\tfrac{3}{2})$$
$$= \tfrac{9}{2}.\tfrac{7}{2}.\tfrac{5}{2}.\tfrac{3}{2}.\tfrac{1}{2}\,\Gamma(\tfrac{1}{2}) = \tfrac{9}{2}.\tfrac{7}{2}.\tfrac{5}{2}.\tfrac{3}{2}.\tfrac{1}{2}\sqrt{\pi}.$$

This function is called a Gamma function. We shall define it more generally later and investigate its properties. For the present, it is temporarily introduced to secure facility in the rapid evaluation of a class of integrals to be discussed.

233. It will be noted that the products of the first n odd numbers $1.3.5.7\ldots(2n-1)$ and of the first n even numbers $2.4.6\ldots 2n$ can be expressed in terms of this function, for

$$\Gamma\left(\frac{2n+1}{2}\right)=\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\cdots\frac{1}{2}\sqrt{\pi}$$

and
$$\Gamma\left(\frac{2n+2}{2}\right)=\frac{2n}{2}\cdot\frac{2n-2}{2}\cdot\frac{2n-4}{2}\cdots\frac{2}{2};$$

$$\therefore\ 1.3.5\ldots(2n-1)=\frac{2^n}{\sqrt{\pi}}\cdot\Gamma\left(\frac{2n+1}{2}\right),$$

and
$$2.4.6\ldots 2n=2^n\,\Gamma\left(\frac{2n+2}{2}\right)=2^n\Gamma(n+1).$$

234. **To investigate a formula for $\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta$, p and q being positive integers.**

Let this integral be denoted by $f(p, q)$; then since

$$\int\sin^p\theta\cos^q\theta\,d\theta=-\frac{\sin^{p-1}\theta\cos^{q+1}\theta}{p+q}+\frac{p-1}{p+q}\int\sin^{p-2}\theta\cos^q\theta\,d\theta,$$

we have, if p and q be positive integers and p not less than 2,

$$f(p, q)=\frac{p-1}{p+q}f(p-2, q).$$

CASE I. Let p be even, $=2m$, and q be also even, $=2n$.

Then
$$f(2m, 2n)=\frac{2m-1}{2m+2n}f(2m-2, 2n)$$
$$=\frac{(2m-1)(2m-3)}{(2m+2n)(2m+2n-2)}f(2m-4, 2n)=\text{etc.}$$
$$=\frac{(2m-1)(2m-3)\ldots 1}{(2m+2n)(2m+2n-2)\ldots(2n+2)}f(0, 2n),$$

and
$$(0, 2n)=\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\,d\theta=\frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2};$$

$$\therefore f(2m, 2n)=\frac{[1.3.5\ldots(2m-1)][1.3.5\ldots(2n-1)]}{2.4.6\ldots(2m+2n)}\cdot\frac{\pi}{2}$$

$$=\frac{\frac{2^m}{\sqrt{\pi}}\Gamma\left(\frac{2m+1}{2}\right)\frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{2n+1}{2}\right)}{2^{m+n}\Gamma\left(\frac{2m+2n+2}{2}\right)}\cdot\frac{\pi}{2}$$

$$=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)}.$$

CASE II. Let p be even, $=2m$, and q be odd, $=2n-1$. Then

$$f(2m,\ 2n-1)=\frac{2m-1}{2m+2n-1}f(2m-2,\ 2n-1)=\text{etc.}$$

$$=\frac{(2m-1)(2m-3)\dots 1}{(2m+2n-1)(2m+2n-3)\dots(2n+1)}f(0,\ 2n-1),$$

and $$f(0,\ 2n-1)=\int_0^{\frac{\pi}{2}}\cos^{2n-1}\theta\,d\theta=\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\dots\frac{2}{3},$$

i.e. $$f(2m,\ 2n-1)=\frac{[1.3.5\dots(2m-1)][2.4.6\dots(2n-2)]}{1.3.5\dots(2m+2n-1)}$$

$$=\frac{\dfrac{2^m}{\sqrt{\pi}}\Gamma\left(\dfrac{2m+1}{2}\right)2^{n-1}\Gamma\left(\dfrac{2n}{2}\right)}{\dfrac{2^{m+n}}{\sqrt{\pi}}\Gamma\left(\dfrac{2m+2n+1}{2}\right)}$$

$$=\frac{\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{q+1}{2}\right)}{2\Gamma\left(\dfrac{p+q}{2}+1\right)}.$$

CASE III. Let p be odd, $=2m-1$, and q be even, $=2n$.
In this case we obtain similarly

$$f(2m-1,\ 2n)=\frac{[2.4.6\dots(2m-2)][1.3.5\dots(2n-1)]}{1.3.5\dots(2m+2n-1)}.$$

But this may also be deduced at once from Case II. by putting

$$\theta=\frac{\pi}{2}-\phi,$$

for then $$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\int_{\frac{\pi}{2}}^{0}\cos^p\phi\sin^q\phi\,(-1)\,d\phi$$

$$=\int_0^{\frac{\pi}{2}}\sin^q\phi\cos^p\phi\,d\phi,$$

so that $$f(p,\ q)=f(q,\ p).$$

Hence the result is again

$$\frac{\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{q+1}{2}\right)}{2\Gamma\left(\dfrac{p+q}{2}+1\right)}.$$

CASE IV. Let p be odd, $=2m-1$, and q be odd, $=2n-1$.

$$f(2m-1,\ 2n-1)=\frac{2m-2}{2m+2n-2}f(2m-3,\ 2n-1)=\text{etc.}$$

$$=\frac{(2m-2)(2m-4)\dots 2}{(2m+2n-2)(2m+2n-4)\dots(2n+2)}f(1,\ 2n-1)$$

and $$f(1,\ 2n-1)=\int_0^{\frac{\pi}{2}}\sin\theta\cos^{2n-1}\theta\,d\theta=\left[-\frac{\cos^{2n}\theta}{2n}\right]_0^{\frac{\pi}{2}}=\frac{1}{2n};$$

$$\therefore\ f(2m-1,\ 2n-1)=\frac{[2.4.6\ldots(2m-2)][2.4.6\ldots(2n-2)]}{2.4.6\ldots(2m+2n-2)}$$

$$=\frac{2^{m-1}\Gamma\left(\frac{2m}{2}\right)2^{n-1}\Gamma\left(\frac{2n}{2}\right)}{2^{m+n-1}\Gamma\left(\frac{2m+2n}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)}.$$

235. **Hence, in every case we have the same result, viz.**

$$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)},$$

and it will be noticed that the $\frac{p+q}{2}+1$ occurring in the denominator is the *sum of the* $\frac{p+1}{2}$ *and the* $\frac{q+1}{2}$ in the numerator.

236. As it has been assumed that p is not <2 we must consider the particular cases $p=1$, $p=0$ separately.

When $p=1$, $\displaystyle\int_0^{\frac{\pi}{2}}\sin\theta\cos^q\theta\,d\theta=\left[-\frac{\cos^{q+1}\theta}{q+1}\right]_0^{\frac{\pi}{2}}=\frac{1}{q+1}.$

Now $$\frac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{q+3}{2}\right)}=\frac{1}{2\frac{q+1}{2}}=\frac{1}{q+1}$$

Hence, this case conforms to the general rule.

When $p=0$, $\displaystyle\int_0^{\frac{\pi}{2}}\binom{\sin}{\cos}^n\theta\,d\theta=\frac{(n-1)(n-3)\ldots1}{n(n-2)\ldots2}\frac{\pi}{2}$ (n even)

$$\frac{(n-1)(n-3)\ldots2}{n(n-2)\ldots3}\quad (n \text{ odd}).$$

In the case n even, the above result may be written

$$\frac{\frac{2^{\frac{n}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)}{2^{\frac{n}{2}}\Gamma\left(\frac{n+2}{2}\right)}\frac{\pi}{2},\quad i.e.\quad \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)},$$

and in the case n odd, the result is

$$\frac{2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\frac{2^{\frac{n+1}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{n+2}{2}\right)}=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)}.$$

Hence these cases also conform to the general rule

$$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)},$$

which may therefore be assumed in all cases where p and q are positive integers.

237. This, then, is a very convenient formula for evaluating quickly integrals of the above form.

Thus,
$$\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^8\theta\,d\theta=\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{9}{2}\right)}{2\Gamma(8)}$$
$$=\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$
$$=\frac{5\pi}{2^{12}}.$$

If, however, the limits be other than 0 *and an integral multiple of* $\frac{\pi}{2}$, *we must find the indefinite integral either by a reduction formula or by the method of Arts.* 114-117 *before inserting the limits.*

238. **Integrals of form**

$$I_{m,p}\equiv\int x^mX^p\,dx,\ \text{where}\ X\equiv a+bx+cx^2.$$

[This form obviously includes all such cases as

$$\int\frac{dx}{(a+bx+cx^2)^p},\quad\int\frac{x^m}{(a+bx+cx^2)^p}dx,\quad\int\frac{(a+bx+cx^2)^p}{x^m}dx,$$

$$\int\frac{dx}{x^m(a+bx+cx^2)^p},\quad\int\frac{x^m\,dx}{\sqrt{a+bx+cx^2}},\quad\int\frac{(x-p)^n}{\sqrt{x^2+ax+b}}dx,\ \text{etc.}]$$

I. Consider the case when $m=0$, *i.e.* $I_{0,p}\equiv\int X^p\,dx$.

Put $P=(b+2cx)X^p$.

Then $$\frac{dP}{dx}=2cX^p+p(b+2cx)^2X^{p-1}$$
$$=2cX^p+p(b^2-4ac+4cX)X^{p-1}$$
$$=(2p+1)2cX^p+p(b^2-4ac)X^{p-1};$$
$$\therefore\ (b+2cx)X^p=(2p+1)2cI_{0,p}+p(b^2-4ac)I_{0,p-1},$$

i.e. $$\int X^p\,dx=\frac{(b+2cx)X^p}{2(2p+1)c}-\frac{p(b^2-4ac)}{2(2p+1)c}\int X^{p-1}\,dx. \qquad \text{(A)}$$

This reduction *fails when* $2p+1=0$, but in that case the integral is $\int\frac{dx}{\sqrt{a+bx+cx^2}}$, and has been considered in Art. 80. The formula (A) will finally reduce the integration of $\int X^p\,dx$ to that of something of the form $\int X^s\,dx$, where s lies between 0 and 1. If $s=0$ or $\frac{1}{2}$, the integration can be written down. Hence, in all cases where p is integral or of form $\frac{2k+1}{2}$, where k is a positive integer, the integration of $\int X^p dx$ can be effected.

If p be a negative integer or of form $-\frac{2k+1}{2}$, we can apply the same formula to lower the index in the denominator, viz.

$$\int X^{p-1}\,dx=\frac{(b+2cx)X^p}{p(b^2-4ac)}-\frac{2(2p+1)c}{p(b^2-4ac)}\int X^p\,dx,$$

or writing $-p$ for p,

$$\int\frac{dx}{X^{p+1}}=-\frac{(b+2cx)}{p(b^2-4ac)}\frac{1}{X^p}-2\frac{(2p-1)c}{p(b^2-4ac)}\int\frac{dx}{X^p}.$$

II. Next, consider the case when $m=1$, *i.e.* $I_{1,p}\equiv\int xX^p\,dx$. Put $P=X^{p+1}$.

$$\frac{dP}{dx}=(p+1)(b+2cx)X^p;$$

$$\therefore\ X^{p+1}=(p+1)b\int X^p\,dx+2(p+1)c\int xX^p\,dx;$$

$$\therefore\ I_{1,p}=\frac{X^{p+1}}{2(p+1)c}-\frac{b}{2c}I_{0,p}, \qquad \text{(B)}$$

and the last integral has been considered.

This reduction fails when $p=-1$.

But this case is $I_{1,-1} \equiv \int \frac{x\,dx}{a+bx+cx^2}$, and no reduction is required.

239. In the case when $m=-1$, *i.e.* $I_{-1,p} \equiv \int \frac{X^p}{x}dx$, put $P=X^p$.

Then
$$\frac{dP}{dx}=p(b+2cx)X^{p-1}$$
$$=\frac{p}{x}[bx+2(X-a-bx)]X^{p-1}$$
$$=p\left[-\frac{2a}{x}-b+\frac{2X}{x}\right]X^{p-1}$$
$$=-2ap\frac{X^{p-1}}{x}-pbX^{p-1}+\frac{2pX^p}{x};$$
$$\therefore \int \frac{X^p}{x}dx=\frac{X^p}{2p}+\frac{b}{2}\int X^{p-1}\,dx+a\int \frac{X^{p-1}}{x}\,dx,$$
that is
$$I_{-1,p}=\frac{X^p}{2p}+\frac{b}{2}I_{0,p-1}+aI_{-1,p-1}. \qquad \text{(C)}$$

240. In the case $I_n \equiv \int \frac{x^m}{\sqrt{a+bx+cx^2}}dx$, put
$$P=x^{m-1}\sqrt{a+bx+cx^2}.$$
$$\frac{dP}{dx}=(m-1)x^{m-2}\sqrt{a+bx+cx^2}+\frac{x^{m-1}(b+2cx)}{2\sqrt{a+bx+cx^2}}$$
$$=\frac{2(m-1)(a+bx+cx^2)+bx+2cx^2}{2\sqrt{a+bx+cx^2}}x^{m-2};$$
$$\therefore P=(m-1)aI_{m-2}+\frac{(2m-1)}{2}bI_{m-1}+mcI_m,$$
which connects I_m with I_{m-1} and I_{m-2} (unless $m=0$).

Now
$$I_1 \equiv \int \frac{x\,dx}{\sqrt{a+bx+cx^2}}$$
$$=\frac{1}{2c}\int\left(\frac{b+2cx}{\sqrt{a+bx+cx^2}}-\frac{b}{\sqrt{a+bx+cx^2}}\right)dx$$
$$=\frac{1}{c}\sqrt{a+bx+cx^2}-\frac{b}{2c}I_0,$$
and I_0 is discussed in Arts. 80, 81.

241. III. **In the general case** $I_{m,p}=\int x^m X^p\,dx$, since

$$a+bx+cx^2 \equiv X,$$

we have $\qquad x^{m-2}X^{p+1}=x^{m-2}(a+bx+cx^2)X^p,$

and therefore

$$I_{m-2,\,p+1}=aI_{m-2,\,p}+bI_{m-1,\,p}+cI_{m,\,p}. \;\ldots\ldots\ldots\ldots (\text{D})$$

Again, let $P=x^{m-1}X^{p+1}$. Then we have

$$\frac{dP}{dx}=(m-1)x^{m-2}X^{p+1}+(p+1)x^{m-1}(b+2cx)X^p$$

$$=x^{m-2}X^p[(m-1)(a+bx+cx^2)+(p+1)(bx+2cx^2)]$$

$$=(m-1)ax^{m-2}X^p+(m+p)bx^{m-1}X^p+(m+2p+1)cx^mX^p;$$

$\therefore\; x^{m-1}X^{p+1}$

$$=(m-1)aI_{m-2,\,p}+(m+p)bI_{m-1,\,p}+(m+2p+1)cI_{m,\,p}. \ldots(\text{E})$$

Eliminating $I_{m-2,\,p}$ between (D) and (E),

$$x^{m-1}X^{p+1}-(m-1)I_{m-2,\,p+1}=(p+1)bI_{m-1,\,p}+2(p+1)cI_{m,\,p}. \quad (\text{F})$$

We thus have, collecting the results,

$$\int x^{m-2}X^{p+1}\,dx$$

$$=a\int x^{m-2}X^p\,dx+b\int x^{m-1}X^p\,dx+c\int x^mX^p\,dx, \ldots\ldots\ldots\ldots\ldots\ldots (\text{D})$$

$$(m+2p+1)\,c\int x^mX^p\,dx$$

$$=x^{m-1}X^{p+1}-(m-1)a\int x^{m-2}X^p\,dx-(m+p)b\int x^{m-1}X^p\,dx, \quad (\text{E})$$

$$\int x^mX^p\,dx$$

$$=\frac{x^{m-1}X^{p+1}}{2(p+1)c}-\frac{(m-1)}{2(p+1)c}\int x^{m-2}X^{p+1}\,dx-\frac{b}{2c}\int x^{m-1}X^p\,dx;\ \ldots\ldots(\text{F})$$

or, writing $-p$ for p to adapt them to the use of cases in which the index of X is negative,

$$\int \frac{x^{m-2}}{X^{p-1}}\,dx$$

$$=a\int\frac{x^{m-2}}{X^p}\,dx+b\int\frac{x^{m-1}}{X^p}\,dx+c\int\frac{x^m}{X^p}\,dx, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{D}')$$

$$(m-2p+1)c\int\frac{x^m}{X^p}dx$$
$$=\frac{x^{m-1}}{X^{p-1}}-(m-1)a\int\frac{x^{m-2}}{X^p}dx-(m-p)b\int\frac{x^{m-1}}{X^p}dx, \quad \text{.............(E')}$$
$$\int\frac{x^m}{X^p}dx$$
$$=-\frac{1}{2(p-1)c}\frac{x^{m-1}}{X^{p-1}}+\frac{(m-1)}{2(p-1)c}\int\frac{x^{m-2}}{X^{p-1}}dx-\frac{b}{2c}\int\frac{x^{m-1}}{X^p}dx. \quad \text{....(F')}$$

242. **Remarks.**

The case of $p=-1$, in which formula (F) fails, is

$$I_{m,\,-1}=\int\frac{x^m}{a+bx+cx^2}dx.$$

But in this case we proceed to partial fractions, and no reduction is required.

Equation (D) (p positive) expresses one integral in terms of three others, with a lower power of X at the expense of introducing higher powers of x; and

Equation (D') raises the power of X in the denominators.

Equations (E) and (E') reduce to integrations with the same powers of X but lower powers of x.

Equation (F) connects with two integrals, in both of which the index of x is lowered, whilst that of X is raised in one integral and remains the same in the other.

Equation (F') plays a similar part for the negative index of X.

243. **Integrals of form**

$$\int(px+q)^m(a+bx+cx^2)^n\,dx, \quad \text{or} \quad \int\frac{(px+q)^m}{(a+bx+cx^2)^n}dx,$$

obviously come under the heading discussed, after transformation, by making $px+q=y$, which transforms $a+bx+cx^2$ to the form $A+By+Cy^2$, where

$$Ap^2=ap^2-bpq+cq^2, \quad Bp^2=bp-2cq, \quad Cp^2=c,$$

and
$$\int(px+q)^m(a+bx+cx^2)^n\,dx$$

becomes
$$\frac{1}{p}\int y^m(A+By+Cy^2)^n\,dy,$$

and similarly in other cases.

The particular cases where $b=0$ or $c=0$ come under the heading of those discussed as $\int x^{m-1}(a+bx^r)^p\,dx$ in Art. 217.

244. Integrals of form $I_n \equiv \int \dfrac{dx}{(q+px)^n\sqrt{a+bx+cx^2}}$ may be regarded as coming under the head of those discussed in Art. 241, for the substitution $q+px=y$ immediately reduces them to that form. But as this form occurs very frequently and is of considerable importance, it is desirable to consider it independently.

Let $P \equiv \dfrac{\sqrt{a+bx+cx^2}}{(q+px)^{n-1}}$.

Then

$$\frac{dP}{dx}=\frac{b+2cx}{2(q+px)^{n-1}\sqrt{a+bx+cx^2}}-\frac{(n-1)p\sqrt{a+bx+cx^2}}{(q+px)^n}$$

$$=\frac{(b+2cx)(q+px)-2(n-1)p(a+bx+cx^2)}{2(q+px)^n\sqrt{a+bx+cx^2}}$$

$$=\frac{1}{2}\frac{\lambda+\mu(q+px)+\nu(q+px)^2}{(q+px)^n\sqrt{a+bx+cx^2}}, \text{ say,}$$

where
$$\lambda+\mu q+\nu q^2=qb-2(n-1)pa,$$
$$\mu p+2\nu pq=2qc+pb-2(n-1)pb,$$
$$\nu p^2=2pc-2(n-1)pc,$$

from which we obtain

$$\lambda=-2(n-1)(ap^2-bpq+cq^2)/p,$$
$$\mu=-(2n-3)(bp-2cq)/p,$$
$$\nu=-2(n-2)c/p.$$

And $2P=\lambda I_n+\mu I_{n-1}+\nu I_{n-2}$ is the formula sought.

That is

$$2(n-1)\frac{ap^2-bpq+cq^2}{p}I_n$$
$$=-\frac{2\sqrt{a+bx+cx^2}}{(q+px)^{n-1}}-(2n-3)\frac{bp-2cq}{p}I_{n-1}-2(n-2)\frac{c}{p}I_{n-2}.$$

The case where $n=1$ is given in Art. 287, whence I_2 can be found from the present formula, in which the coefficient of I_{n-2} vanishes when $n=2$. Then I_3, I_4, ... can be successively derived.

245. The integral

$$J_n=\int\frac{Mx+N}{(px+q)^n}\frac{dx}{\sqrt{ax^2+bx+c}}$$

may be written as

$$J_n=\int\frac{\frac{M}{p}(px+q)+\left(N-\frac{Mq}{p}\right)}{(px+q)^n\sqrt{ax^2+bx+c}}dx$$

$$=\frac{M}{p}I_{n-1}+\frac{Np-Mq}{p}I_n,$$

where I_n is the integral discussed in Art. 244.

This therefore constitutes a reduction formula for J_n.

But both this integral and the more general integral

$$J_n'=\int\frac{Mx+N}{(Ax^2+Bx+C)^n}\frac{dx}{\sqrt{ax^2+bx+c}}$$

are more conveniently evaluated by differentiation with regard to one of the constants involved, q in the one case, C in the other, as explained subsequently (see Art. 364).

246. The integrable cases.

Denote $I_{m,p}\equiv\int x^m X^p\,dx$ for shortness by (m, p).

The special cases

$$(0,\ -1),\quad (0,\ -\tfrac{1}{2}),\quad (0,\ \tfrac{1}{2}),\quad (0,\ 1)$$

are all simple elementary integrals whose values have been discussed.

Formula (A), which connects $(0, p)$ and $(0, p-1)$, will therefore continue the series both ways and yield

$$(0,\ \pm\tfrac{3}{2}),\quad (0,\ \pm 2),\quad (0,\ \pm\tfrac{5}{2}),\quad (0,\ \pm 3),\quad (0,\ \pm\tfrac{7}{2}),\quad \text{etc.},$$

$$\text{i.e. } (0,\ \pm k)\quad\text{or}\quad\left(0,\ \pm\frac{2k+1}{2}\right),$$

where k is any integer.

Formula (B) connects $(1, p)$ with $(0, p)$, and therefore contributes the integrals

$$(1,\ \pm k),\quad \left(1,\ \pm\frac{2k+1}{2}\right),$$

where k is any integer.

Formula (C) connects $(-1, p)$ with $(-1, p-1)$; and $(-1,\ -\frac{1}{2})$ and $(-1,\ \pm 1)$ are simple cases already discussed;

$$\left.\begin{array}{ll}\therefore & (-1,\ -\tfrac{3}{2}),\quad (-1,\ -\tfrac{5}{2}),\quad (-1,\ -\tfrac{7}{2}),\ \text{etc.,}\\ \text{and} & (-1,\ +\tfrac{1}{2}),\quad (-1,\ +\tfrac{3}{2}),\quad (-1,\ +\tfrac{5}{2}),\ \text{etc.,}\end{array}\right\}\text{ are contributed;}$$

as also $(-1,\ \pm 2),\quad (-1,\ \pm 3),\quad (-1,\ \pm 4)$, etc.

i.e. $(-1,\ \pm k),\ \left(-1,\ \pm\frac{2k+1}{2}\right)$, are contributed where k is any integer.

Formula (D) connects $(m-2, p+1)$, $(m-2, p)$, $(m-1, p)$, (m, p)

$$\therefore \quad (0, p+1), \quad (0, p), \quad (1, p), \quad (2, p) \text{ are connected,}$$
$$(1, p+1), \quad (1, p), \quad (2, p), \quad (3, p) \text{ are connected,}$$
$$\text{etc.};$$

$$\therefore \left.\begin{matrix} (2, \pm k), & \left(2, \pm\dfrac{2k+1}{2}\right), \\ (3, \pm k), & \left(3, \pm\dfrac{2k+1}{2}\right), \\ (4, \pm k), & \left(4, \pm\dfrac{2k+1}{2}\right), \end{matrix}\right\} \text{are contributed,}$$
$$\text{etc.};$$

Formula (E) connects $(m-2, p)$, $(m-1, p)$, (m, p);

$$\text{therefore also } \left.\begin{matrix} (-2, \pm k), & \left(-2, \pm\dfrac{2k+1}{2}\right), \\ (-3, \pm k), & \left(-3, \pm\dfrac{2k+1}{2}\right), \end{matrix}\right\} \text{are contributed,}$$
$$\text{etc.}$$

Hence *all integrals of form*

$$\int x^m X^p\, dx, \quad \text{where } X \equiv a+bx+cx^2,$$

can be integrated in finite terms when p is of form $\pm k$ or $\pm\dfrac{2k+1}{2}$, and m, k are integers positive or negative.

EXAMPLES.

247. 1. Taking

$$\int \frac{dx}{X^{p+1}} = \frac{b+2cx}{pkX^p} + \frac{2(2p-1)c}{pk}\int\frac{dx}{X^p},$$

where $X \equiv a+bx+cx^2$ and $k \equiv 4ac-b^2$,

prove
$$\int\frac{dx}{X^2} = \frac{b+2cx}{kX} + \frac{2c}{k}\int\frac{dx}{X},$$

$$\int\frac{dx}{X^3} = \frac{(b+2cx)}{k}\left(\frac{1}{2X^2} + \frac{3c}{kX}\right) + \frac{6c^2}{k^2}\int\frac{dx}{X},$$

$$\int\frac{dx}{X^4} = \frac{b+2cx}{k}\left(\frac{1}{3X^3} + \frac{5c}{3kX^2} + \frac{10c^2}{k^2X}\right) + \frac{20c^3}{k^3}\int\frac{dx}{X}.$$

[BERTRAND.]

2. Show that if $I_n \equiv \int \frac{x^n}{X}dx$, then $cI_n + bI_{n-1} + aI_{n-2} = \frac{x^{n-1}}{n-1}$, and prove

$$\int\frac{x\,dx}{X} = \frac{1}{2c}\log X - \frac{b}{2c}\int\frac{dx}{X}.$$

Deduce $$\int\frac{x^2\,dx}{X}=\frac{x}{c}-\frac{b}{2c^2}\log X+\frac{b^2-2ac}{2c^2}\int\frac{dx}{X},$$
$$\int\frac{x^3\,dx}{X}=\frac{x^2}{2c}-\frac{bx}{c^2}+\frac{b^2-ac}{2c^3}\log X+\frac{3ac-b^2}{2c^3}\,b\int\frac{dx}{X}.$$

[BERTRAND.]

3. Prove
$$\int\frac{dx}{xX}=\frac{1}{2a}\log\frac{x^2}{X}-\frac{b}{2a}\int\frac{dx}{X},$$
and deduce
$$\int\frac{dx}{x^3X}=\frac{b}{a^2x}-\frac{1}{2ax^2}+\frac{b^2-ac}{2a^3}\log\frac{x^2}{X}+\frac{b(3ac-b^2)}{2a^3}\int\frac{dx}{X}.$$

[BERTRAND.]

(The value of $\int\frac{dx}{X}$ occurring in each of these results is given in Art. 80.)

4. If $X\equiv a+bx+cx^2$, prove that
$$\int\frac{x^{m+1}}{X^{p+1}}\,dx=-\frac{x^m}{2cpX^p}+\frac{m}{2cp}\int\frac{x^{m-1}\,dx}{X^p}-\frac{b}{2c}\int\frac{x^m}{X^{p+1}}\,dx.$$

[BERTRAND.]

5. Prove that if $X\equiv x^2+ax+a^2$,
$$\int X^{\frac{n}{2}}\,dx=\frac{2x+a}{2(n+1)}X^{\frac{n}{2}}+\frac{3na^2}{4(n+1)}\int X^{\frac{n}{2}-1}\,dx.$$

[ST. JOHN'S, 1889.]

6. Prove that if $X\equiv x^2+x+1$,

$$(a)\quad \int\frac{dx}{x^4X}=\frac{1}{2}\log\frac{x^2}{X}-\frac{1}{3x^3}+\frac{1}{2x^2}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$
$$(b)\quad \int\frac{dx}{X^{p+1}}=\frac{1+2x}{3pX^p}+\frac{2(2p-1)}{3p}\int\frac{dx}{X^p}.$$

7. Show that if p be a positive integer and $X\equiv x^2+x+1$,

$$(a)\quad \int\frac{dx}{(x^2+x+1)^{p+1}}$$
$$=\frac{(1+2x)}{3}\left[\frac{1}{pX^p}+\frac{(2p-1)}{p(p-1)}\left(\frac{2}{3}\right)\frac{1}{X^{p-1}}+\frac{(2p-1)(2p-3)}{p(p-1)(p-2)}\left(\frac{2}{3}\right)^2\frac{1}{X^{p-2}}\right.$$
$$\left.+\dots+\frac{(2p-1)(2p-3)\dots 3.1}{p(p-1)\dots 2.1}\left(\frac{2}{3}\right)^{p-1}\frac{1}{X}\right]$$
$$+\frac{(2p-1)(2p-3)\dots 3.1}{p(p-1)(p-2)\dots 2.1}\,\frac{2}{\sqrt{3}}\left(\frac{2}{3}\right)^p\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

$$(b)\quad \int\frac{dx}{(x^2+x+1)^{\frac{2n+1}{2}}},$$ n being a positive integer,
$$=\frac{1+2x}{3}\left[\frac{1}{2n-1}\,\frac{2}{X^{\frac{2n-1}{2}}}+\frac{(n-1)}{(2n-1)(2n-3)}\left(\frac{2}{3}\right)\frac{2^3}{X^{\frac{2n-3}{2}}}\right.$$
$$+\frac{(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}\left(\frac{2}{3}\right)^2\frac{2^5}{X^{\frac{2n-5}{2}}}$$
$$\left.+\dots+\frac{(n-1)!}{(2n-1)(2n-3)\dots 1}\left(\frac{2}{3}\right)^{n-1}\frac{2^{2n-1}}{X^{\frac{1}{2}}}\right].$$

248. **Reduction of** $I_n \equiv \int \frac{x^n}{\sqrt{a+bx^2+cx^4}}\,dx.$

Let $X = a+bx^2+cx^4$, and put $P = x^{n-3}\sqrt{X}$.

Then $\frac{dP}{dx} = (n-3)x^{n-4}\sqrt{X} + x^{n-3}\frac{(bx+2cx^3)}{\sqrt{X}}$

$$= \frac{(n-3)x^{n-4}(a+bx^2+cx^4)+bx^{n-2}+2cx^n}{\sqrt{X}}$$

$$= \frac{(n-1)cx^n+(n-2)bx^{n-2}+(n-3)ax^{n-4}}{\sqrt{X}};$$

$$\therefore\ x^{n-3}\sqrt{X} = (n-1)cI_n + (n-2)bI_{n-2} + (n-3)aI_{n-4}.$$

249. **Integrations of**

(i) $\int \cos px \cos^n qx\,dx$, (ii) $\int \cos px \sin^n qx\,dx$,

(iii) $\int \sin px \cos^n qx\,dx$, (iv) $\int \sin px \sin^n qx\,dx$,

including $\int \frac{\cos px}{\cos^n qx}\,dx$, etc.

There are two classes of reduction formulae for such integrals.

We may connect

$$\int \cos px \cos^n qx\,dx \text{ with } \int \cos px \cos^{n-2} qx\,dx,$$

or we may connect

$$\int \cos px \cos^n qx\,dx \text{ with } \int \cos(p-q)x \cos^{n-1} qx\,dx,$$

and the like with the other three cases.

250. First, we consider the **former class of reduction.**

(i) Let $I_n \equiv \int \cos px \cos^n qx\,dx.$

Then

$$I_n = \frac{\sin px}{p}\cos^n qx + \frac{nq}{p}\int \sin px \cos^{n-1} qx \sin qx\,dx$$

$$= \frac{\sin px}{p}\cos^n qx + \frac{nq}{p}\Big[-\frac{\cos px}{p}\cos^{n-1} qx \sin qx$$

$$+\int \frac{\cos px}{p}\{-(n-1)q\cos^{n-2}qx(1-\cos^2 qx)+q\cos^n qx\}\,dx\Big]$$

$$=\frac{\sin px}{p}\cos^n qx-\frac{nq}{p^2}\cos px\cos^{n-1}qx\sin qx$$

$$+\frac{nq^2}{p^2}\int\cos px\{-(n-1)\cos^{n-2}qx+n\cos^n qx\}\,dx;$$

$$\therefore\ \left(1-\frac{n^2q^2}{p^2}\right)I_n=\frac{\cos^{n-1}qx}{p^2}(p\sin px\cos qx-nq\cos px\sin qx)$$
$$-\frac{n(n-1)q^2}{p^2}I_{n-2};$$

$$\therefore\ I_n=\cos^{n-1}qx\,\frac{p\sin px\cos qx-nq\cos px\sin qx}{p^2-n^2q^2}$$
$$-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

Now $$\frac{d}{dx}\frac{\cos^n qx}{\cos px}=\cos^{n-1}qx\,\frac{p\sin px\cos qx-nq\cos px\sin qx}{\cos^2 px}.$$

Hence the reduction formula may be written more compactly as

$$I_n=\frac{\cos^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\cos^n qx}{\cos px}-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

By successive reduction, the power-factor $\cos^n qx$ may be reduced either to $\cos qx$ or to unity, when n is a positive integer, and the integration can then be completed.

If n be negative $(=-m)$, we can, by solving for I_{n-2}, express the same formula as

$$I_{-m-2}=\frac{\cos^2 px}{m(m+1)q^2}\frac{d}{dx}\frac{\sec^m qx}{\cos px}-\frac{p^2-m^2q^2}{m(m+1)q^2}I_{-m},$$

and therefore a reduction formula for $\int\frac{\cos px}{\cos^m qx}dx$ is also furnished.

Similar work and remarks apply to the other three cases, (ii), (iii) and (iv), but it is desirable to consider them in detail.

251. (ii) Let $I_n=\int\cos px\sin^n qx\,dx$.

Then

$$I_n=\frac{\sin px}{p}\sin^n qx-\frac{nq}{p}\int\sin px\sin^{n-1}qx\cos qx\,dx$$

$$=\frac{\sin px}{p}\sin^n qx-\frac{nq}{p}\left[-\frac{\cos px}{p}\sin^{n-1}qx\cos qx\right.$$

$$\left.+\int\frac{\cos px}{p}\{(n-1)q\sin^{n-2}qx(1-\sin^2 qx)-q\sin^n qx\}\,dx\right]$$

$$=\frac{\sin px}{p}\sin^n qx+\frac{nq}{p^2}\cos px\sin^{n-1}qx\cos qx$$

$$-\frac{nq^2}{p^2}\int\cos px\{(n-1)\sin^{n-2}qx-n\sin^n qx\}\,dx;$$

$$\therefore\ \left(1-\frac{n^2q^2}{p^2}\right)I_n=\frac{\sin^{n-1}qx}{p^2}(p\sin px\sin qx+nq\cos px\cos qx)$$

$$-n(n-1)\frac{q^2}{p^2}I_{n-2};$$

$$\therefore\ I_n=\sin^{n-1}qx\frac{(p\sin px\sin qx+nq\cos px\cos qx)}{p^2-n^2q^2}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2},$$

i.e.
$$I_n=\frac{\cos^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\sin^n qx}{\cos px}-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

252. (iii) Let $I_n\equiv\int\sin px\cos^n qx\,dx.$

Then

$$I_n=-\frac{\cos px}{p}\cos^n qx-\frac{nq}{p}\int\cos px\cos^{n-1}qx\sin qx\,dx$$

$$=-\frac{\cos px}{p}\cos^n qx-\frac{nq}{p}\left[\frac{\sin px}{p}\cos^{n-1}qx\sin qx\right.$$

$$\left.-\int\frac{\sin px}{p}\{-(n-1)q\cos^{n-2}qx(1-\cos^2 qx)+q\cos^n qx\}\,dx\right];$$

$$\therefore\ \left(1-\frac{n^2q^2}{p^2}\right)I_n=-\frac{\cos^{n-1}qx}{p^2}(p\cos px\cos qx+nq\sin px\sin qx)$$

$$-n(n-1)\frac{q^2}{p^2}I_{n-2};$$

$$\therefore\ I_n=-\cos^{n-1}qx\frac{(p\cos px\cos qx+nq\sin px\sin qx)}{p^2-n^2q^2}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2},$$

i.e.
$$I_n=\frac{\sin^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\cos^n qx}{\sin px}-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

253. (iv) Let $I_n=\int \sin px \sin^n qx\, dx$.

Then

$$I_n=-\frac{\cos px}{p}\sin^n qx+\frac{nq}{p}\int \cos px \sin^{n-1}qx \cos qx\, dx$$

$$=-\frac{\cos px}{p}\sin^n qx+\frac{nq}{p}\left[\frac{\sin px}{p}\sin^{n-1}qx\cos qx\right.$$

$$\left.-\int\frac{\sin px}{p}\{(n-1)q\sin^{n-2}qx(1-\sin^2 qx)-q\sin^n qx\}\,dx\right];$$

$$\therefore\ \left(1-\frac{n^2q^2}{p^2}\right)I_n=-\frac{\sin^{n-1}qx}{p^2}(p\cos px\sin qx-nq\sin px\cos qx)$$
$$-n(n-1)\frac{q^2}{p^2}I_{n-2};$$

$$\therefore\ I_n=-\sin^{n-1}qx\,\frac{p\cos px\sin qx-nq\sin px\cos qx}{p^2-n^2q^2}$$
$$-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

i.e.
$$I_n=\frac{\sin^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\sin^n qx}{\sin px}-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

254. The four results are therefore

$$\int\cos px\cos^n qx\,dx=\frac{\cos^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\cos^n qx}{\cos px}$$
$$-\frac{n(n-1)q^2}{p^2-n^2q^2}\int\cos px\cos^{n-2}qx\,dx.$$

$$\int\cos px\sin^n qx\,dx=\frac{\cos^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\sin^n qx}{\cos px}$$
$$-\frac{n(n-1)q^2}{p^2-n^2q^2}\int\cos px\sin^{n-2}qx\,dx.$$

$$\int\sin px\cos^n qx\,dx=\frac{\sin^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\cos^n qx}{\sin px}$$
$$-\frac{n(n-1)q^2}{p^2-n^2q^2}\int\sin px\cos^{n-2}qx\,dx.$$

$$\int\sin px\sin^n qx\,dx=\frac{\sin^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\sin^n qx}{\sin px}$$
$$-\frac{n(n-1)q^2}{p^2-n^2q^2}\int\sin px\sin^{n-2}qx\,dx.$$

That is, if A stands for the first factor and P the second, or power-factor, *i.e.* $I_n=\int AP\,dx$, we have, *in all cases*,

$$(p^2-n^2q^2)\,I_n=A^2\frac{d}{dx}\left(\frac{P}{A}\right)-n(n-1)\,q^2I_{n-2}$$

or
$$(p^2-n^2q^2)\,I_n=A\frac{dP}{dx}-P\frac{dA}{dx}-n(n-1)\,q^2I_{n-2}.$$

Writing $-m$ for n, for the cases where n is negative, we may write this as

$$m(m+1)q^2I_{-m-2}=A\frac{dP}{dx}-P\frac{dA}{dx}-(p^2-m^2q^2)\,I_{-m}.$$

255. Such formulae are more particularly useful for negative indices of the power factor. For if the integral sought be, say,

$$\int\cos 4x\sin^5 3x\,dx,$$

the "multiple angle" process for $\sin^5 3x$ will be more convenient than a reduction.

Thus, $$\sin^5 3x=\frac{1}{2^4}(\sin 15x-5\sin 9x+10\sin 3x);$$

$$\therefore\ \cos 4x\sin^5 3x=\frac{1}{2^5}[(\sin 19x+\sin 11x)-5(\sin 13x+\sin 5x)$$
$$+10(\sin 7x-\sin x)],$$

and the integral is

$$-\frac{1}{2^5}\left[\frac{\cos 19x}{19}+\frac{\cos 11x}{11}-\frac{5\cos 13x}{13}-\frac{5\cos 5x}{5}+\frac{10\cos 7x}{7}-\frac{10\cos x}{1}\right].$$

But to integrate $\int\frac{\cos 4x}{\sin^5 3x}dx$, this process is useless. Therefore we change n to $-n$ in the second of the formulae of Art. 254.

Then
$$\int\frac{\cos px}{\sin^{n+2}qx}\,dx=\frac{\cos^2 px}{n(n+1)q^2}\frac{d}{dx}\frac{\sec px}{\sin^n qx}-\frac{p^2-n^2q^2}{n(n+1)q^2}\int\frac{\cos px}{\sin^n qx}\,dx;$$

whence

$$\int\frac{\cos 4x}{\sin^5 3x}\,dx=\frac{\cos^2 4x}{3\,.\,4\,.\,3^2}\frac{d}{dx}\frac{\sec 4x}{\sin^3 3x}+\frac{13\,.\,5}{3\,.\,4\,.\,3^2}\int\frac{\cos 4x}{\sin^3 3x}\,dx$$

and
$$\int\frac{\cos 4x}{\sin^3 3x}\,dx=\frac{\cos^2 4x}{1\,.\,2\,.\,3^2}\frac{d}{dx}\frac{\sec 4x}{\sin 3x}-\frac{7\,.\,1}{1\,.\,2\,.\,3^2}\int\frac{\cos 4x}{\sin 3x}\,dx;$$

whilst

$$\int\frac{\cos 4x}{\sin 3x}\,dx=\frac{1}{2}\int\left(\frac{1}{\sin x}-\frac{1}{\sin 3x}-4\sin x\right)dx$$
$$=\frac{1}{2}\log\tan\frac{x}{2}-\frac{1}{6}\log\tan\frac{3x}{2}+2\cos x;$$

hence

$$\int \frac{\cos 4x}{\sin^5 3x}\,dx = \frac{\cos^2 4x}{3\,.\,4\,.\,3^2}\frac{d}{dx}\frac{\sec 4x}{\sin^3 3x} + \frac{13\,.\,5}{3\,.\,4\,.\,3^3}\left\{\frac{\cos^2 4x}{1\,.\,2\,.\,3^2}\frac{d}{dx}\frac{\sec 4x}{\sin 3x}\right.$$

$$\left. - \frac{7\,.\,1}{1\,.\,2\,.\,3^2}\left(\frac{1}{2}\log\tan\frac{x}{2} - \frac{1}{6}\log\tan\frac{3x}{2} + 2\cos x\right)\right\}$$

$$= \text{etc.}$$

256. For the **second mode of reduction**, mentioned above in Art. 249, we may connect $I_{p,n}$, that is $\int \cos px \cos^n qx\,dx$ or one of the other cases with

$$I_{p-q,n-1} \quad \text{or with} \quad I_{p-2q,n-2},$$

To shorten the expressions we shall use the notation

$$c_p \text{ for } \cos px, \quad s_p \text{ for } \sin px, \quad \text{etc.}$$

The mode of procedure is the same in all cases, viz.:

Put P = the power factor × the complementary function of the other factor. Differentiate and rearrange.

(i) $$I_{p,n} = \int c_p c_q^n\,dx.$$

Let $$P_1 = s_p c_q^n.$$

Then $$\frac{dP_1}{dx} = pc_p c_q^n - nqs_p c_q^{n-1} s_q$$

$$= c_q^{n-1}[(p+nq)\,c_p c_q - nqc_{p-q}];$$

$$\therefore\ P_1 = (p+nq)\int c_p c_q^n\,dx - nq\int c_{p-q} c_q^{n-1}\,dx.$$

(ii) $$I_{p,n} = \int c_p s_q^n\,dx.$$

Let $$P_2 = s_p s_q^n.$$

Then $$\frac{dP_2}{dx} = pc_p s_q^n + nqs_p s_q^{n-1} c_q$$

$$= s_q^{n-1}[(p+nq)\,c_p s_q + nqs_{p-q}];$$

$$\therefore\ P_2 = (p+nq)\int c_p s_q^n\,dx + nq\int s_{p-q} s_q^{n-1}\,dx.$$

(iii) $$I_{p,n} = \int s_p c_q^n\,dx.$$

Let $$P_3 = c_p c_q^n.$$

Then $$\frac{dP_3}{dx} = -ps_p c_q^n - nqc_p c_q^{n-1} s_q$$

$$= -c_q^{n-1}[(p+nq)\,s_p c_q - nqs_{p-q}];$$

$$\therefore\ P_3 = -(p+nq)\int s_p c_q^n\,dx + nq\int s_{p-q} c_q^{n-1}\,dx$$

(iv) $$I_{p,n}=\int s_p s_q^n\,dx.$$

Let $$P_4=c_p s_q^n.$$

Then $$\frac{dP_4}{dx}=-ps_p s_q^n+nqc_p s_q^{n-1}c_q$$

$$=-s_q^{n-1}[(p+nq)s_p s_q-nqc_{p-q}];$$

$$\therefore\ P_4=-(p+nq)\int s_p s_q^n\,dx+nq\int c_{p-q}s_q^{n-1}\,dx.$$

We thus obtain the four results:

$$(1)\ (p+nq)\int c_p c_q^n\,dx=\quad s_p c_q^n+nq\int c_{p-q}c_q^{n-1}\,dx.$$

$$(2)\ (p+nq)\int c_p s_q^n\,dx=\quad s_p s_q^n-nq\int s_{p-q}s_q^{n-1}\,dx.$$

$$(3)\ (p+nq)\int s_p c_q^n\,dx=-c_p c_q^n+nq\int s_{p-q}c_q^{n-1}\,dx.$$

$$(4)\ (p+nq)\int s_p s_q^n\,dx=-c_p s_q^n+nq\int c_{p-q}s_q^{n-1}\,dx.$$

Thus an integral of the first kind connects directly with a lower order integral of the first kind;
an integral of the second kind connects directly with a lower order integral of the fourth kind;
an integral of the third kind connects directly with a lower order integral of the third kind;
an integral of the fourth kind connects directly with a lower order integral of the second kind.

Thus to connect an integral of the second or fourth kind with one of its own kind, a second operation is necessary.

For example,

$$(p+nq)\int c_p s_q^n\,dx=s_p s_q^n-nq\int s_{p-q}s_q^{n-1}dx$$

$$=s_p s_q^n-\frac{nq}{p-q+(n-1)q}\left[-c_{p-q}s_q^{n-1}+(n-1)q\int c_{p-2q}s_q^{n-2}\,dx\right],$$

which connects $\int c_p s_q^n\,dx$ with $\int c_{p-2q}s_q^{n-2}\,dx$,

and similarly for $\int s_p s_q^n\,dx$ with $\int s_{p-2q}s_q^{n-2}\,dx$.

257. Avoidance of a Reduction Formula.

For integrals of the classes under discussion, viz.

$$\int \cos px \cos^n qx\, dx, \text{ etc.,}$$

it is often convenient to avoid a reduction formula altogether so long as n is a positive integer, when we shall require to put the power-factor ($\cos^n qx$ in this case) into cosines or sines of multiples of qx, as seen in the example in Art. 255.

Proceeding as in Art. 112, the formulae required are:

$$2^n \cos^n\theta = \left(y+\frac{1}{y}\right)^n = \text{etc.}$$

$$= 2[\cos n\theta + {}^nC_1 \cos(n-2)\theta + {}^nC_2 \cos(n-4)\theta + \ldots + K],$$

where $$K = \frac{1}{2}\frac{n!}{\left(\frac{n}{2}!\right)^2} \text{ if } n \text{ be even, } \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

or $$= \frac{n!}{\frac{n-1}{2}!\,\frac{n+1}{2}!}\cos\theta \text{ if } n \text{ be odd, } \qquad \ldots\ldots\ldots\text{(B)}$$

$$2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - {}^nC_1 \cos(n-2)\theta$$
$$+\ldots+(-1)^{\frac{n}{2}}\frac{n!}{2\left(\frac{n}{2}!\right)^2} \text{ if } n \text{ be even; } \quad \text{(C)}$$

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - {}^nC_1 \sin(n-2)\theta$$
$$+\ldots+(-1)^{\frac{n-1}{2}}\frac{n!}{\frac{n-1}{2}!\,\frac{n+1}{2}!}\sin\theta$$

if n be odd. (D)

Then taking $\theta = qx$,

$$2^n \cos^n qx \cos px = \text{a series of form } 2\Sigma K_r \cos rx \cos px, \text{ say,}$$
$$= \sum K_r (\cos\overline{r+p}\,x + \cos\overline{r-p}\,x),$$

and $$\int \cos px \cos^n qx\, dx = \frac{1}{2^n}\left(\sum K_r \frac{\sin\overline{r+p}\,x}{r+p} + \sum K_r \frac{\sin\overline{r-p}\,x}{r-p}\right),$$

taking due account of the final terms.

Similarly we may proceed in the other cases.

The formulae (A), (B), (C), (D) can be readily reproduced as explained previously in Art. 112 for any particular value of n for which they may be required.

Ex. $\int \sin 2x \sin^6 5x\, dx.$

$$2^6 \iota^6 \sin^6\theta = \left(y - \frac{1}{y}\right)^6 = y^6 + \frac{1}{y^6} - 6\left(y^4 + \frac{1}{y^4}\right) + 15\left(y^2 + \frac{1}{y^2}\right) - 20$$

$$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20\,;$$

$\therefore$ taking $\theta = 5x$,

$$\sin^6 5x \sin 2x = -\frac{1}{2^6}[2\sin 2x\cos 30x - 12\sin 2x\cos 20x$$
$$+ 30\sin 2x\cos 10x - 20\sin 2x]$$
$$= -\frac{1}{2^6}[\sin 32x - \sin 28x - 6(\sin 22x - \sin 18x)$$
$$+ 15(\sin 12x - \sin 8x) - 20\sin 2x],$$

and
$$\int \sin 2x \sin^6 5x\, dx = \frac{1}{2^6}\left[\frac{\cos 32x}{32} - \frac{\cos 28x}{28} - 6\left(\frac{\cos 22x}{22} - \frac{\cos 18x}{18}\right)\right.$$
$$\left. + 15\left(\frac{\cos 12x}{12} - \frac{\cos 8x}{8}\right) - 20\frac{\cos 2x}{2}\right].$$

258. **The Integrals**

$$(1)\ \int \frac{\cos nx}{\cos^p x}\,dx. \qquad (2)\ \int \frac{\sin nx}{\cos^p x}\,dx.$$

$$(3)\ \int \frac{\cos nx}{\sin^p x}\,dx. \qquad (4)\ \int \frac{\sin nx}{\sin^p x}\,dx.$$

In case (1),

$$I_{n,p} \equiv \int \frac{\cos nx}{\cos^p x}\,dx = \int \frac{2\cos x\cos(n-1)x - \cos(n-2)x}{\cos^p x}\,dx$$
$$= 2I_{n-1,p-1} - I_{n-2,p}.$$

In case (2),

$$I_{n,p} \equiv \int \frac{\sin nx}{\cos^p x}\,dx = \int \frac{2\cos x\sin(n-1)x - \sin(n-2)x}{\cos^p x}\,dx$$
$$= 2I_{n-1,p-1} - I_{n-2,p}.$$

For cases (3) and (4), let

$$I_{n,p} \equiv \int \frac{\cos nx}{\sin^p x}\,dx, \quad J_{n,p} \equiv \int \frac{\sin nx}{\sin^p x}\,dx.$$

In case (3),

$$I_{n,p} \equiv \int \frac{\cos nx}{\sin^p x}\,dx = \int \frac{-2\sin x\sin(n-1)x + \cos(n-2)x}{\sin^p x}\,dx$$
$$= -2J_{n-1,p-1} + I_{n-2,p}.$$

In case (4),

$$J_{n,p} \equiv \int \frac{\sin nx}{\sin^p x}\,dx = \int \frac{2\sin x\cos(n-1)x + \sin(n-2)x}{\sin^p x}\,dx$$
$$= 2I_{n-1,p-1} + J_{n-2,p}.$$

The cases (1) and (2), therefore, reduce to lower order integrals of the same form.

The cases (3) and (4) reduce to lower order integrals, but in each case the forms are partly interchanged.

It may be worth noting that in the form $\int \frac{\cos nx}{\cos^p x} dx$ we might as an alternative method express $\cos nx$ as a series of powers of $\cos x$ and integrate each term by methods already discussed.

If n be odd $\int \frac{\sin nx}{\sin^p x} dx$ may be treated similarly by expressing $\sin nx$ as a series of powers of $\sin x$ and integrating each term.

If n be even $\sin nx$ contains a factor $\cos x$, and the integral is immediately obtainable; *e.g.*

$$\int \frac{\sin 4x}{\sin^5 x} dx = \int \frac{4 \sin x(1-2\sin^2 x)}{\sin^5 x} \cos x \, dx$$

$$= \int (4s^{-4} - 8s^{-2}) \, ds = -\frac{4}{3} \frac{1}{\sin^3 x} + \frac{8}{\sin x}.$$

Similar remarks apply in the other cases.

259. Ex. 1. $\int \frac{\cos 5x}{\cos^3 x} dx = 2\int \frac{\cos 4x}{\cos^2 x} dx - \int \frac{\cos 3x}{\cos^3 x} dx$

$$= 2\left[2\int \frac{\cos 3x}{\cos x} dx - \int \frac{\cos 2x}{\cos^2 x} dx\right] - \int \frac{\cos 3x}{\cos^3 x} dx$$

$$= 4\int (4\cos^2 x - 3)\, dx - 2\int (2 - \sec^2 x)\, dx$$

$$- \int (4 - 3\sec^2 x)\, dx$$

$$= 8\left(x + \frac{\sin 2x}{2}\right) - 12x - 4x + 2\tan x - 4x + 3\tan x$$

$$= 4\sin 2x - 12x + 5\tan x;$$

or otherwise, and more readily, *without a reduction*,

$$\int \frac{\cos 5x}{\cos^3 x} dx = \int \frac{16\cos^5 x - 20\cos^3 x + 5\cos x}{\cos^3 x} dx$$

$$= \int \{8(1+\cos 2x) - 20 + 5\sec^2 x\}\, dx$$

$$= 4\sin 2x - 12x + 5\tan x, \text{ as before.}$$

Ex. 2. $$\int \frac{\cos 5x}{\sin^3 x}\,dx = -2\int \frac{\sin 4x}{\sin^2 x}\,dx + \int \frac{\cos 3x}{\sin^3 x}\,dx$$

$$= -2\left[2\int \frac{\cos 3x}{\sin x}\,dx + \int \frac{\sin 2x}{\sin^2 x}\,dx\right]$$

$$+\left[-2\int \frac{\sin 2x}{\sin^2 x}\,dx + \int \frac{\cos x}{\sin^3 x}\,dx\right]$$

$$= -4\int (\cot x - 4\sin x\cos x)\,dx - 8\int \cot x\,dx$$

$$+\int \frac{\cos x}{\sin^3 x}\,dx$$

$$= 16\frac{\sin^2 x}{2} - 12\log\sin x - \frac{1}{2\sin^2 x}$$

$$= 8\sin^2 x - 12\log\sin x - \tfrac{1}{2}\operatorname{cosec}^2 x\,;$$

or otherwise, and more readily, *without a reduction*,

$$\int \frac{\cos 5x}{\sin^3 x}\,dx = \int \frac{1 - 12\sin^2 x + 16\sin^4 x}{\sin^3 x}\,d\sin x$$

$$= -\tfrac{1}{2}\operatorname{cosec}^2 x - 12\log\sin x + 8\sin^2 x, \text{ as before.}$$

260. Integrals $I_n \equiv \int \frac{\sin^n px}{\cos px}\,dx, \quad J_n \equiv \int \frac{\cos^n px}{\sin px}\,dx.$

$$I_n \equiv \int \frac{\sin^n px}{\cos px}\,dx = \int \frac{\sin^{n-2} px(1-\cos^2 px)}{\cos px}\,dx$$

$$= -\int \cos px \sin^{n-2} px\,dx + I_{n-2},$$

$$\therefore\ I_n = -\frac{\sin^{n-1} px}{(n-1)p} + I_{n-2}.$$

$$J_n = \int \frac{\cos^n px}{\sin px}\,dx = \int \frac{\cos^{n-2} px(1-\sin^2 px)}{\sin px}\,dx$$

$$= -\int \sin px \cos^{n-2} px\,dx + J_{n-2}\,;$$

$$\therefore\ J_n = \frac{\cos^{n-1} px}{(n-1)p} + J_{n-2}.$$

Also since

$$I_1 = \int \tan px\,dx = \frac{1}{p}\log\sec px,$$

$$I_2 = \int (\sec px - \cos px)\,dx = -\frac{\sin px}{p} + \frac{1}{p}\log\tan\left(\frac{px}{2} + \frac{\pi}{4}\right),$$

$$J_1 = \int \cot px\,dx = \frac{1}{p}\log\sin px,$$

$$J_2 = \int (\operatorname{cosec} px - \sin px)\,dx = \frac{\cos px}{p} + \frac{1}{p}\log\tan\frac{px}{2},$$

we have

$$p\int\frac{\sin^{2n}px}{\cos px}dx = -\frac{\sin^{2n-1}px}{2n-1}-\frac{\sin^{2n-3}px}{2n-3}-\dots$$
$$-\frac{\sin^3 px}{3}-\frac{\sin px}{1}+\log\tan\left(\frac{px}{2}+\frac{\pi}{4}\right),$$

$$p\int\frac{\sin^{2n+1}px}{\cos px}dx = -\frac{\sin^{2n}px}{2n}-\frac{\sin^{2n-2}px}{2n-2}-\dots$$
$$-\frac{\sin^4 px}{4}-\frac{\sin^2 px}{2}+\log\sec px,$$

$$p\int\frac{\cos^{2n}px}{\sin px}dx = \frac{\cos^{2n-1}px}{2n-1}+\frac{\cos^{2n-3}px}{2n-3}+\dots$$
$$+\frac{\cos^3 px}{3}+\frac{\cos px}{1}+\log\tan\frac{px}{2},$$

$$p\int\frac{\cos^{2n+1}px}{\sin px}dx = \frac{\cos^{2n}px}{2n}+\frac{\cos^{2n-2}px}{2n-2}+\dots$$
$$+\frac{\cos^4 px}{4}+\frac{\cos^2 px}{2}+\log\sin px.$$

261. **Integration of**

$$\int\frac{\cos px}{\cos qx}dx,\quad \int\frac{\cos px}{\sin qx}dx,\quad \int\frac{\sin px}{\cos qx}dx,\quad \int\frac{\sin px}{\sin qx}dx.$$

(i) We may regard p, q as integral and prime to each other

For if p, q be fractional, $=\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ respectively, let

$$\frac{r_1}{s_1} \text{ and } \frac{r_2}{s_2}$$

be reduced to the forms $\quad \frac{R_1}{S},\ \frac{R_2}{S},$

where S is the L.C.M. of s_1 and s_2 and R_1, R_2 are integers.

Let $x=Sy$. Then

$$\int\frac{\begin{matrix}\cos\\ \sin\end{matrix}(px)}{\begin{matrix}\cos\\ \sin\end{matrix}(qx)}dx=\int\frac{\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{R_1}{S}\right)x}{\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{R_2}{S}\right)x}dx=S\int\frac{\begin{matrix}\cos\\ \sin\end{matrix}(R_1y)}{\begin{matrix}\cos\\ \sin\end{matrix}(R_2y)}dy.$$

Hence we only need to consider the case where p and q are integers.

The signs of p and q are also immaterial to the discussion.

Again, if p and q were not prime to each other, let G be the G.C.M., and let $p = Gp'$, $q = Gq'$, and let $x = \frac{y}{G}$. Then

$$\int \frac{\begin{matrix}\cos\\ \sin\end{matrix}(Gp'x)}{\begin{matrix}\cos\\ \sin\end{matrix}(Gq'x)}\,dx = \frac{1}{G}\int \frac{\begin{matrix}\cos\\ \sin\end{matrix}(p'y)}{\begin{matrix}\cos\\ \sin\end{matrix}(q'y)}\,dy,$$

where p' and q' are prime to each other.

Therefore we shall need only to consider the case where p, q are *positive integers, prime to each other*

(ii) Supposing $p > q$.

Since
$$\begin{aligned}
\cos px + \cos(p-2q)x &= 2\cos(p-q)x\cos qx,\\
\cos px - \cos(p-2q)x &= -2\sin(p-q)x\sin qx,\\
\sin px + \sin(p-2q)x &= 2\sin(p-q)x\cos qx,\\
\sin px - \sin(p-2q)x &= 2\cos(p-q)x\sin qx,
\end{aligned}$$
we have
$$\begin{aligned}
\int \frac{\cos px}{\cos qx}\,dx &= 2\frac{\sin(p-q)x}{p-q} - \int \frac{\cos(p-2q)x}{\cos qx}\,dx\\
\int \frac{\cos px}{\sin qx}\,dx &= 2\frac{\cos(p-q)x}{p-q} + \int \frac{\cos(p-2q)x}{\sin qx}\,dx,\\
\int \frac{\sin px}{\cos qx}\,dx &= -2\frac{\cos(p-q)x}{p-q} - \int \frac{\sin(p-2q)x}{\sin qx}\,dx,\\
\int \frac{\sin px}{\sin qx}\,dx &= 2\frac{\sin(p-q)x}{p-q} + \int \frac{\sin(p-2q)x}{\sin qx}\,dx.
\end{aligned}$$

Hence, by a sufficient number of reductions of this kind, we can reduce the integration of

$$\int \frac{\begin{matrix}\cos\\ \sin\end{matrix}(px)}{\begin{matrix}\cos\\ \sin\end{matrix}(qx)}\,dx$$

to that of another integral of the same form, say

$$\int \frac{\begin{matrix}\cos\\ \sin\end{matrix}Px}{\begin{matrix}\cos\\ \sin\end{matrix}qx}\,dx,$$

where P lies between q and $-q$.

Hence we shall introduce no limitation upon our method in the discussion of such integrals in assuming $p < q$.

(iii) We take, then, p and q positive, integral, prime to each other, and $p < q$. The case p and q, both even, need not be considered, being a reducible case as shown.

Now

$$\left.\begin{aligned}
&\text{if } n \text{ be even,} \\
&\cos nx = \prod_1^{\frac{n}{2}} \frac{\sin^2\alpha_r - \sin^2 x}{\sin^2\alpha_r}, \\
&\text{if } n \text{ be odd,} \\
&\cos nx = \cos x \prod_1^{\frac{n-1}{2}} \frac{\sin^2\alpha_r - \sin^2 x}{\sin^2\alpha_r}.
\end{aligned}\right\} \text{ where } \alpha_r = (2r-1)\frac{\pi}{2n}.$$

$$\left.\begin{aligned}
&\text{if } n \text{ be even,} \\
&\sin nx = n \sin x \cos x \prod_1^{\frac{n-2}{2}} \frac{\sin^2\alpha_r - \sin^2 x}{\sin^2\alpha_r}, \\
&\text{if } n \text{ be odd,} \\
&\sin nx = n \sin x \prod_1^{\frac{n-1}{2}} \frac{\sin^2\alpha_r - \sin^2 x}{\sin^2\alpha_r},
\end{aligned}\right\} \text{ where } \alpha_r = \frac{r\pi}{n}.$$

And where necessary a factor $\sin^2\alpha_r - \sin^2 x$ can be written as $\cos^2 x - \cos^2\alpha_r$. (See Hobson, *Trigonometry*, p. 114.)

Factorizing both numerator and denominator of

$$\frac{\begin{matrix}\cos\\ \sin\end{matrix}(px)}{\begin{matrix}\cos\\ \sin\end{matrix}(qx)},$$

the number of factors in the numerator is less than that in the denominator, and in all cases the integrand can be thrown into partial fractions by the ordinary rules (factors not repeated) and expressed in one of the forms,

$$\sum\frac{A}{\sin^2\alpha - \sin^2 x}, \quad \sum\frac{A\cos x}{\sin^2\alpha - \sin^2 x}, \quad \sum\frac{A\sin x}{\cos^2 x - \cos^2\alpha},$$

and the particular fractions

$$\frac{A\sin x}{\cos^2 x - 1}, \quad \frac{A\cos x}{1 - \sin^2 x}, \quad \frac{A}{\cos^2 x} \text{ or } \frac{A}{\sin^2 x}$$

may occur.

(iv) Finally,

$$\int \frac{\sin a \cos a \, dx}{\sin^2 a - \sin^2 x} = \tanh^{-1}\left(\frac{\tan x}{\tan a}\right),$$

$$\int \frac{\sin a \cos x \, dx}{\sin^2 a - \sin^2 x} = \tanh^{-1}\left(\frac{\sin x}{\sin a}\right),$$

$$\int \frac{\sin x \cos a \, dx}{\cos^2 x - \cos^2 a} = \coth^{-1}\left(\frac{\cos x}{\cos a}\right),$$

and

$$\int \frac{\sin x}{\cos^2 x - 1} dx = -\log \tan \frac{x}{2}, \quad \int \frac{\cos x}{1 - \sin^2 x} dx = \log \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

Hence in all such cases the integration can be performed.

It is not *essential* that the numerator $\frac{\cos}{\sin}(px)$ should be factorized. It might be expanded in powers of $\cos x$ or $\sin x$, as the case may be. But the *factorization is convenient, presents no difficulty, and as a rule is simpler in application*, as it indicates in factorized form the values of the constants occurring in the partial fractions.

262. Ex. Find the integral $I = \int \frac{\cos \frac{2}{3}x}{\cos \frac{1}{2}x} dx$.

Let $x = 6y$. Then

$$I = 6\int \frac{\cos 4y}{\cos 3y} dy \quad \text{and} \quad \int \frac{\cos 4y}{\cos 3y} dy = 2\sin y - \int \frac{\cos 2y}{\cos 3y} dy,$$

by the first reduction formula, (Art 261, ii).

Also $$\int \frac{\cos 2y}{\cos 3y} dy = \int \frac{\dfrac{\sin^2\frac{\pi}{4} - \sin^2 y}{\sin^2\frac{\pi}{4}}}{\cos y \dfrac{\sin^2\frac{\pi}{6} - \sin^2 y}{\sin^2\frac{\pi}{6}}} dy$$

$$= \frac{\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{4}} \int \frac{\sin^2\frac{\pi}{4} - \sin^2 y}{\left(\sin^2\frac{\pi}{2} - \sin^2 y\right)\left(\sin^2\frac{\pi}{6} - \sin^2 y\right)} \cos y \, dy$$

$$= \frac{\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{4}} \int \left[\frac{\sin^2\frac{\pi}{4} - \sin^2\frac{\pi}{2}}{\sin^2\frac{\pi}{6} - \sin^2\frac{\pi}{2}} \frac{\cos y}{1 - \sin^2 y} \right.$$

$$\left. + \frac{\sin^2\frac{\pi}{4} - \sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{2} - \sin^2\frac{\pi}{6}} \frac{\cos y}{\sin^2\frac{\pi}{6} - \sin^2 y}\right] dy$$

$$=\frac{\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{4}}\left[\frac{\sin^2\frac{\pi}{4}-\sin^2\frac{\pi}{2}}{\sin^2\frac{\pi}{6}-\sin^2\frac{\pi}{2}}\log\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)\right.$$

$$\left.+\frac{\sin^2\frac{\pi}{4}-\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{2}-\sin^2\frac{\pi}{6}}\operatorname{cosec}\frac{\pi}{6}\tanh^{-1}\frac{\sin y}{\sin\frac{\pi}{6}}\right]$$

So far, obvious arithmetical simplification is postponed, so that the general process may be exhibited and made clear.

Simplifying the arithmetic, we shall finally get

$$\int\frac{\cos\frac{2}{3}x}{\cos\frac{1}{2}x}dx=12\sin\frac{x}{6}-2\log\tan\left(\frac{x}{12}+\frac{\pi}{4}\right)-2\tanh^{-1}\left(2\sin\frac{x}{6}\right).$$

263. **Integrals of form**

$$\int\frac{\cos^n px}{\cos x}dx,\quad \int\frac{\cos^n px}{\sin x}dx,\quad \int\frac{\sin^n px}{\cos x}dx,\quad \int\frac{\sin^n px}{\sin x}dx,$$

where p and n are integers, n being positive.

These are generally integrated as follows:

First put the power factor in the numerator into the form of a series of cosines or sines of multiples of px, say

$$\Sigma A_r \frac{\cos}{\sin}(rpx).$$

We are then to integrate each term, viz. expressions of type

$$\int\frac{\frac{\cos}{\sin}(rpx)}{\frac{\cos}{\sin}(x)}dx$$

by a reduction formula, a case of Art. 261 (ii), viz.:

$$\int\frac{\cos kx}{\cos x}dx=\quad 2\frac{\sin(k-1)x}{k-1}-\int\frac{\cos(k-2)x}{\cos x}dx,$$

$$\int\frac{\cos kx}{\sin x}dx=\quad 2\frac{\cos(k-1)x}{k-1}+\int\frac{\cos(k-2)x}{\sin x}dx,$$

$$\int\frac{\sin kx}{\cos x}dx=-2\frac{\cos(k-1)x}{k-1}-\int\frac{\sin(k-2)x}{\cos x}dx,$$

$$\int\frac{\sin kx}{\sin x}dx=\quad 2\frac{\sin(k-1)x}{k-1}+\int\frac{\sin(k-2)x}{\sin x}dx,$$

which obviously follow from the trigonometrical formulae

$$\cos kx+\cos(k-2)x=2\cos x\cos(k-1)x,$$

etc.

Ex. Consider $\int \frac{\cos^5 3x}{\cos x} dx$.

We have, taking $y = e^{3\iota x}$,

$$2^5 \cos^5 3x = \left(y + \frac{1}{y}\right)^5 = \text{etc.} = 2\cos 15x + 10\cos 9x + 20\cos 3x$$

$$\therefore \frac{\cos^5 3x}{\cos x} = \frac{1}{2^4}\left(\frac{\cos 15x}{\cos x} + \frac{5\cos 9x}{\cos x} + \frac{10\cos 3x}{\cos x}\right).$$

But

$$\int \frac{\cos 15x}{\cos x} dx = \frac{2\sin 14x}{14} - \frac{2\sin 12x}{12} + \frac{2\sin 10x}{10}$$
$$\qquad - \frac{2\sin 8x}{8} + \frac{2\sin 6x}{6} - \frac{2\sin 4x}{4} + \frac{2\sin 2x}{2} - x,$$

$$\int \frac{\cos 9x}{\cos x} dx = \frac{2\sin 8x}{8} - \frac{2\sin 6x}{6} + \frac{2\sin 4x}{4} - \frac{2\sin 2x}{2} + x,$$

$$\int \frac{\cos 3x}{\cos x} dx = \frac{2\sin 2x}{2} - x;$$

$$\therefore \int \frac{\cos^5 3x}{\cos x} dx = \frac{1}{2^4}\left[\frac{2\sin 14x}{14} - \frac{2\sin 12x}{12} + \frac{2\sin 10x}{10} + \frac{8\sin 8x}{8}\right.$$
$$\qquad \left. - \frac{8\sin 6x}{6} + \frac{8\sin 4x}{4} + \frac{12\sin 2x}{2} - 6x\right].$$

264. Integrals of form

$$\int \frac{\cos^n px}{\cos qx} dx, \quad \int \frac{\cos^n px}{\sin qx} dx, \quad \int \frac{\sin^n px}{\cos qx} dx, \quad \int \frac{\sin^n px}{\sin qx} dx.$$

These are dealt with in a similar manner to those of the previous article.

First expressing the power factor as

$$\Sigma A_r \frac{\cos}{\sin}(rpx),$$

we reduce the integration in each case to that of a series of terms of type

$$\int \frac{\frac{\cos}{\sin}(p'x)}{\frac{\cos}{\sin}(qx)} dx,$$

and proceed as explained in Art. 261.

Ex. Integrate $I = \int \frac{\cos^5 5x}{\cos 4x} dx$.

We have, taking $y = e^{5\iota x}$,

$$2^5 \cos^5 5x = \left(y + \frac{1}{y}\right)^5 = \text{etc.} = 2\cos 25x + 10\cos 15x + 20\cos 5x;$$

$$\therefore I = \frac{1}{2^4}\left[\int \frac{\cos 25x}{\cos 4x} dx + 5\int \frac{\cos 15x}{\cos 4x} dx + 10\int \frac{\cos 5x}{\cos 4x} dx\right].$$

The reduction formula

$$\int \frac{\cos px}{\cos qx}\,dx = \frac{2\sin(p-q)x}{p-q} - \int \frac{\cos(p-2q)x}{\cos qx}\,dx$$

gives

$$\int \frac{\cos 25x}{\cos 4x}\,dx = \frac{2\sin 21x}{21} - \int \frac{\cos 17x}{\cos 4x}\,dx$$

$$= \frac{2\sin 21x}{21} - \frac{2\sin 13x}{13} + \int \frac{\cos 9x}{\cos 4x}\,dx$$

$$= \frac{2\sin 21x}{21} - \frac{2\sin 13x}{13} + \frac{2\sin 5x}{5} - \int \frac{\cos x}{\cos 4x}\,dx,$$

$$\int \frac{\cos 15x}{\cos 4x}\,dx = \frac{2\sin 11x}{11} - \frac{2\sin 3x}{3} + \int \frac{\cos(-x)}{\cos 4x}\,dx,$$

and $$\int \frac{\cos 5x}{\cos 4x}\,dx = \frac{2\sin x}{1} - \int \frac{\cos(-3x)}{\cos 4x}\,dx.$$

Hence

$$I = \frac{1}{2^4}\left[\frac{2\sin 21x}{21} - \frac{2\sin 13x}{13} + \frac{10\sin 11x}{11} + \frac{2\sin 5x}{5} - \frac{10\sin 3x}{3} + \frac{20\sin x}{1} - K\right],$$

where

$$K = \int \frac{\cos x - 5\cos x + 10\cos 3x}{\cos 4x}\,dx = \int \frac{40\cos^3 x - 34\cos x}{\cos 4x}\,dx$$

$$= 2\int \cos x \frac{3 - 20\sin^2 x}{\left(\sin^2\frac{\pi}{8} - \sin^2 x\right)\left(\sin^2\frac{3\pi}{8} - \sin^2 x\right)} \cdot \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}\,dx$$

$$= \frac{2\sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}}{\sin^2\frac{3\pi}{8} - \sin^2\frac{\pi}{8}} \int \cos x \left(\frac{3 - 20\sin^2\frac{\pi}{8}}{\sin^2\frac{\pi}{8} - \sin^2 x} - \frac{3 - 20\sin^2\frac{3\pi}{8}}{\sin^2\frac{3\pi}{8} - \sin^2 x}\right) dx$$

$$= \frac{1}{2\sqrt{2}}\left[\left(3 - 20\sin^2\frac{\pi}{8}\right)\operatorname{cosec}\frac{\pi}{8}\tanh^{-1}\frac{\sin x}{\sin\frac{\pi}{8}} - \left(3 - 20\sin^2\frac{3\pi}{8}\right)\operatorname{cosec}\frac{3\pi}{8}\tanh^{-1}\frac{\sin x}{\sin\frac{3\pi}{8}}\right];$$

$$\therefore \int \frac{\cos^5 5x}{\cos 4x}\,dx$$

$$= \frac{1}{8}\Bigg[\frac{1}{21}\sin 21x - \frac{1}{13}\sin 13x + \frac{5}{11}\sin 11x + \frac{1}{5}\sin 5x - \frac{5}{3}\sin 3x + 10\sin x$$
$$- \frac{1}{4\sqrt{2}}\left\{\left(3 - 20\sin^2\frac{\pi}{8}\right)\operatorname{cosec}\frac{\pi}{8}\tanh^{-1}\frac{\sin x}{\sin\frac{\pi}{8}} - \left(3 - 20\sin^2\frac{3\pi}{8}\right)\operatorname{cosec}\frac{3\pi}{8}\tanh^{-1}\frac{\sin x}{\sin\frac{3\pi}{8}}\right\}\Bigg].$$

265. **Integrals of form**

$$\int \frac{\sin^p x}{x^q}\,dx,\quad \int \frac{\cos^p x}{x^q}\,dx,\quad \int \frac{x^q}{\sin^p x}\,dx,\quad \int \frac{x^q}{\cos^p x}\,dx.$$

$$I_{p,q} \equiv \int \frac{\sin^p x}{x^q}\,dx = -\frac{\sin^p x}{(q-1)x^{q-1}} + \frac{p}{q-1}\int \frac{\sin^{p-1}x\cos x}{x^{q-1}}\,dx$$

$$= -\frac{\sin^p x}{(q-1)x^{q-1}} + \frac{p}{q-1}\left[\left\{-\frac{\sin^{p-1}x\cos x}{(q-2)x^{q-2}}\right\}\right.$$

$$\left. + \frac{1}{q-2}\int \frac{(p-1)\sin^{p-2}x(1-\sin^2 x)-\sin^p x}{x^{q-2}}\,dx\right]$$

$$= -\frac{\sin^{p-1}x}{(q-1)(q-2)x^{q-1}}[(q-2)\sin x + px\cos x]$$

$$+\frac{p}{(q-1)(q-2)}[(p-1)I_{p-2,q-2} - pI_{p,q-2}].$$

Therefore, provided $q \neq 1$ or 2,

$$(q-1)(q-2)I_{p,q} - p(p-1)I_{p-2,q-2} + p^2 I_{p,q-2}$$

$$= -\frac{\sin^{p-1}x}{x^{q-1}}[(q-2)\sin x + px\cos x]. \quad \text{(A)}$$

This formula will be found useful in evaluating certain definite integrals of the form $\int_0^\infty \frac{\sin^p x}{x^q}\,dx$, in the case where $p \nless q$ and where p and q are either both odd or both even integers > 2; for in this case the right-hand side vanishes at both limits. We thus have

$$(q-1)(q-2)\int_0^\infty \frac{\sin^p x}{x^q}\,dx + p^2\int_0^\infty \frac{\sin^p x}{x^{q-2}}\,dx - p(p-1)\int_0^\infty \frac{\sin^{p-2}x}{x^{q-2}}\,dx = 0,$$

where $p \nless q > 2$ (see Chap. XXVI.)

266. In the same way, in the second case, supposing $q \neq 1$ or 2,

$$I_{p,q} \equiv \int \frac{\cos^p x}{x^q}\,dx = -\frac{\cos^p x}{(q-1)x^{q-1}} - \frac{p}{q-1}\int \frac{\cos^{p-1}x\sin x}{x^{q-1}}\,dx$$

$$= -\frac{\cos^p x}{(q-1)x^{q-1}} - \frac{p}{q-1}\left[\left\{-\frac{\cos^{p-1}x\sin x}{(q-2)x^{q-2}}\right\}\right.$$

$$\left. + \frac{1}{q-2}\int \frac{\cos^p x - (p-1)\cos^{p-2}x(1-\cos^2 x)}{x^{q-2}}\,dx\right]$$

$$= -\frac{\cos^{p-1}x}{(q-1)(q-2)x^{q-1}}[(q-2)\cos x - px\sin x]$$

$$-\frac{p}{(q-1)(q-2)}[pI_{p,q-2} - (p-1)I_{p-2,q-2}];$$

$$\therefore\ (q-1)(q-2)I_{p,q} - p(p-1)I_{p-2,q-2} + p^2 I_{p,q-2}$$

$$= -\frac{\cos^{p-1}x}{x^{q-1}}[(q-2)\cos x - px\sin x]. \quad \text{(B)}$$

267. Again, in the third case,

$$I_{p,q}=\int x^q \operatorname{cosec}^p x\,dx=\frac{x^{q+1}}{q+1}\operatorname{cosec}^p x+\frac{p}{q+1}\int x^{q+1}\operatorname{cosec}^p x\cot x\,dx$$

$$=\frac{x^{q+1}}{q+1}\operatorname{cosec}^p x+\frac{p}{q+1}\left[\frac{x^{q+2}}{q+2}\operatorname{cosec}^p x\cot x\right.$$

$$\left.+\frac{1}{q+2}\int x^{q+2}(p\operatorname{cosec}^p x\cot^2 x+\operatorname{cosec}^{p+2}x)\,dx\right]$$

$$=\frac{x^{q+1}\operatorname{cosec}^{p+1}x}{(q+1)(q+2)}[(q+2)\sin x+px\cos x]$$

$$+\frac{p}{(q+1)(q+2)}\int x^{q+2}[(p+1)\operatorname{cosec}^{p+2}x-p\operatorname{cosec}^p x]\,dx;$$

$$\therefore\ (q+1)(q+2)I_{p,q}-p(p+1)I_{p+2,q+2}+p^2 I_{p,q+2}$$

$$=x^{q+1}\operatorname{cosec}^{p+1}[(q+2)\sin x+px\cos x].\quad\text{(C)}$$

268. And finally, in the fourth case,

$$I_{p,q}=\int x^q\sec^p x\,dx=\frac{x^{q+1}}{q+1}\sec^p x-\frac{p}{q+1}\int x^{q+1}\sec^p x\tan x\,dx$$

$$=\frac{x^{q+1}}{q+1}\sec^p x-\frac{p}{q+1}\left[\frac{x^{q+2}}{q+2}\sec^p x\tan x\right.$$

$$\left.-\frac{1}{q+2}\int x^{q+2}(p\sec^p x\tan^2 x+\sec^{p+2}x)\,dx\right]$$

$$=\frac{x^{q+1}\sec^{p+1}x}{(q+1)(q+2)}[(q+2)\cos x-px\sin x]$$

$$+\frac{p}{(q+1)(q+2)}\int x^{q+2}[(p+1)\sec^{p+2}x-p\sec^p x]\,dx;$$

$$\therefore\ (q+1)(q+2)I_{p,q}-p(p+1)I_{p+2,q+2}+p^2 I_{p,q+2}$$

$$=x^{q+1}\sec^{p+1}x\,[(q+2)\cos x-px\sin x].\quad\text{(D)}$$

It will be seen that formulae (C) and (D) could have been derived from (A) and (B) by changing the signs of p and q.

269. **Integrals of form** $I_n\equiv\int\frac{x}{\cos^n x}dx$ **or** $\int x\sec^n x\,dx$, included as the case $q=1$ in Art. 265, may be treated thus:

$$I_n=\int\frac{x}{\cos^n x}dx=\int\cos x\cdot\frac{x}{\cos^{n+1}x}dx$$

$$=\sin x\cdot\frac{x}{\cos^{n+1}x}-\int\sin x\frac{\cos x+(n+1)x\sin x}{\cos^{n+2}x}dx$$

$$=x\frac{\sin x}{\cos^{n+1}x}-\int\frac{\sin x}{\cos^{n+1}x}dx-(n+1)\int x\frac{1-\cos^2 x}{\cos^{n+2}x}dx$$

$$=x\frac{\sin x}{\cos^{n+1}x}-\frac{1}{n}\frac{1}{\cos^n x}-(n+1)(I_{n+2}-I_n).$$

Therefore

$$(n+1)\,I_{n+2}=\frac{nx\sin x-\cos x}{n\cos^{n+1}x}+nI_n$$

or
$$I_{n+2}=\frac{nx\sin x-\cos x}{n(n+1)\cos^{n+1}x}+\frac{n}{n+1}I_n;$$

or, changing n to $n-2$,

$$I_n=\frac{(n-2)x\sin x-\cos x}{(n-1)(n-2)\cos^{n-1}x}+\frac{n-2}{n-1}I_{n-2}. \quad\text{...........(1)}$$

Now,

$$I_2=\int x\sec^2 x\,dx=x\tan x+\log\cos x$$

and

$$I_1=\int x\sec x\,dx=x\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)-\int\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)dx.$$

Thus, I_4, I_6, ... can be readily written down.

But I_3, I_5, I_7, ... ultimately connect with $\int\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$, which is not expressible in finite terms as an indefinite integral.

270. **Similarly, if** $I_n=\int\frac{x}{\sin^n x}dx$ or $\int x\,\text{cosec}^n x\,dx$, we have

$$I_n=\int\sin x\,.\,\frac{x}{\sin^{n+1}x}dx$$

$$=(-\cos x)\frac{x}{\sin^{n+1}x}+\int\cos x\frac{\sin x-(n+1)x\cos x}{\sin^{n+2}x}dx$$

$$=(-\cos x)\frac{x}{\sin^{n+1}x}+\int\frac{\cos x}{\sin^{n+1}x}dx-(n+1)\int x\frac{1-\sin^2 x}{\sin^{n+2}x}dx$$

$$=-x\frac{\cos x}{\sin^{n+1}x}-\frac{1}{n}\frac{1}{\sin^n x}-(n+1)(I_{n+2}-I_n);$$

$$\therefore\ (n+1)I_{n+2}=-\frac{nx\cos x+\sin x}{n\sin^{n+1}x}+nI_n;$$

or, changing n to $n-2$,

$$I_n=-\frac{(n-2)x\cos x+\sin x}{(n-1)(n-2)\sin^{n-1}x}+\frac{n-2}{n-1}I_{n-2}. \quad\text{.........(2)}$$

Noting that

$$I_2=\int x\,\text{cosec}^2 x\,dx=-x\cot x+\log\sin x$$

and
$$I_1=\int x\,\text{cosec}\,x\,dx=x\log\tan\frac{x}{2}-\int\log\tan\frac{x}{2}dx,$$

it is clear that I_4, I_6, ... can be successively written down, but that I_3, I_5, ..., which connect with $\int \log \tan \frac{x}{2}\,dx$, cannot be expressed in finite terms as an indefinite integral.

It is also obvious that these formulae (1) and (2) could be reproduced by taking

$$P=\frac{(n-2)x\sin x-\cos x}{\cos^{n-1}x} \quad \text{and} \quad P=\frac{(n-2)x\cos x+\sin x}{\sin^{n-1}x}$$

respectively, differentiating, and rearranging the terms.

271. Reduction formula for

$$I_n=\int\frac{x^{2n}}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,dx,$$

n being integral.

Let $R=(1-x^2)(1-k^2x^2)$, and put $P=x^{2n-3}\sqrt{R}$.

Then $\dfrac{dP}{dx}=(2n-3)x^{2n-4}\sqrt{R}+x^{2n-3}\dfrac{-(1+k^2)x+2k^2x^3}{\sqrt{R}}$

$$=[(2n-3)\{1-(1+k^2)x^2+k^2x^4\}-(1+k^2)x^2+2k^2x^4]\frac{x^{2n-4}}{\sqrt{R}}$$

$$=[(2n-3)-2(n-1)(1+k^2)x^2+(2n-1)k^2x^4]\frac{x^{2n-4}}{\sqrt{R}}$$

$$=(2n-3)\frac{x^{2n-4}}{\sqrt{R}}-2(n-1)(1+k^2)\frac{x^{2n-2}}{\sqrt{R}}+(2n-1)k^2\frac{x^{2n}}{\sqrt{R}}.$$

Hence $P=(2n-3)I_{n-2}-2(n-1)(1+k^2)I_{n-1}+(2n-1)k^2I_n$,

i.e. $$I_n=\frac{x^{2n-3}\sqrt{R}}{(2n-1)k^2}+2\frac{n-1}{2n-1}\frac{(1+k^2)}{k^2}I_{n-1}-\frac{2n-3}{2n-1}\cdot\frac{1}{k^2}I_{n-2}.$$

[Serret, p. 44, Tom. ii., *Calc. Diff. et Intégral.*]

By successive reduction I_n may be made to depend upon I_0 and I_1 by putting in succession $n=2, 3, 4, \ldots$; and I_0, I_1, which are respectively

$$\int\frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \quad \text{and} \quad \int\frac{x^2\,dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}},$$

are the integrals known as Legendrian Elliptic Integrals, and discussed later.

When $n=1$, $\qquad k^2I_1=x^{-1}\sqrt{R}+I_{-1}$,

i.e. $$I_{-1}=k^2I_1-\frac{\sqrt{R}}{x}.$$

When $n=0$, $k^2I_0=-x^{-3}\sqrt{R}+2(1+k^2)I_{-1}-3I_{-2}$, and putting successively $n=-1, -2$, etc., we can calculate I_{-2}, I_{-3}, etc., in terms of I_0 and I_1.

272. Obviously, if we put $x=\sin\theta$,

$$I_n=\int\frac{\sin^{2n}\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

and the same reduction formula applies.

Thus $I_n=\int\frac{\sin^{2n}\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}}$ and $I_{-n}=\int\frac{\operatorname{cosec}^{2n}\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}}$

can both be connected linearly with

$$\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\quad\text{and}\quad\int\frac{\sin^2\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}};$$

and the latter being

$$\frac{1}{k^2}\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}-\frac{1}{k^2}\int\sqrt{1-k^2\sin^2\theta}\,d\theta,$$

we have connected each of I_n and I_{-n} with

$$\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\quad\text{and}\quad\int\sqrt{1-k^2\sin^2\theta}\,d\theta$$

273. Instead of starting with $P=x^{2n-3}\sqrt{R}$, we might have proceeded to form the connection required by means of integration by parts, which presents no difficulty.

Thus $$R=(1-x^2)(1-k^2x^2),$$

$$\frac{dR}{dx}=-2(1+k^2)x+4k^2x^3.$$

Multiply by $\frac{x^{2n-3}}{2\sqrt{R}}$ and integrate

$$\int\frac{x^{2n-3}}{2\sqrt{R}}\frac{dR}{dx}dx=-(1+k^2)I_{n-1}+2k^2I_n.$$

But the left side $=x^{2n-3}\sqrt{R}-(2n-3)\int x^{2n-4}\sqrt{R}\,dx$

$$=x^{2n-3}\sqrt{R}-(2n-3)\int\frac{x^{2n-4}(1-x^2)(1-k^2x^2)}{\sqrt{R}}dx;$$

$$\therefore\ -(1+k^2)I_{n-1}+2k^2I_n$$
$$=x^{2n-3}\sqrt{R}-(2n-3)[I_{n-2}-(1+k^2)I_{n-1}+k^2I_n],$$

i.e. $x^{2n-3}\sqrt{R}=(2n-1)k^2I_n-2(n-1)(1+k^2)I_{n-1}+(2n-3)I_{n-2}$,

the result already obtained.

274. **Reduction formula for**

$$I_n=\int\frac{dx}{(1+ax^2)^n\sqrt{(1-x^2)(1-k^2x^2)}},$$

i.e.
$$\int\frac{d\theta}{(1+a\sin^2\theta)^n\sqrt{1-k^2\sin^2\theta}},$$

where $x=\sin\theta$.

Let $R=(1-x^2)(1-k^2x^2)$, as before; then

$$\frac{1}{2}\frac{dR}{dx}=-(1+k^2)x+2k^2x^3.$$

Put $P=\dfrac{x\sqrt{R}}{(1+ax^2)^{n-1}}$.

Then

$$\frac{dP}{dx}=\frac{\sqrt{R}+\dfrac{x}{2\sqrt{R}}\dfrac{dR}{dx}}{(1+ax^2)^{n-1}}-\frac{2(n-1)ax^2\sqrt{R}}{(1+ax^2)^n}$$

$$=\frac{(1+ax^2)[(1-x^2)(1-k^2x^2)-(1+k^2)x^2+2k^2x^4]-2(n-1)ax^2(1-x^2)(1-k^2x^2)}{(1+ax^2)^n\sqrt{R}}$$

$$=\frac{(1+ax^2)[1-2(1+k^2)x^2+3k^2x^4]-2(n-1)ax^2(1-x^2)(1-k^2x^2)}{(1+ax^2)^n\sqrt{R}}$$

$$=\frac{A+B(1+ax^2)+C(1+ax^2)^2+D(1+ax^2)^3}{(1+ax^2)^n\sqrt{R}},\ \text{say},$$

where
$$A+B+C+D=1,$$
$$aB+2aC+3aD=a-2(1+k^2)-2(n-1)a,$$
$$a^2C+3a^2D=3k^2-2a(1+k^2)+2(n-1)a(1+k^2),$$
$$a^3D=3k^2a-2(n-1)ak^2;$$

whence we obtain

$$a^2A=\quad(2n-2)(a+1)(a+k^2),$$
$$a^2B=-(2n-3)[a(a+2)+(2a+3)k^2],$$
$$a^2C=\quad(2n-4)[a+(a+3)k^2],$$
$$a^2D=-(2n-5)k^2.$$

Then $$P = AI_n + BI_{n-1} + CI_{n-2} + DI_{n-3},$$

and I_n is connected with three integrals of the same form, but lower order. Also, the formula is true whether n is positive or negative.

Now $$I_0 = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

and is Legendre's first elliptic integral (Chaps. XI. and XXXI.).

$$I_1 = \int \frac{dx}{(1+ax^2)\sqrt{(1-x^2)(1-k^2x^2)}},$$

and is Legendre's third elliptic integral; and

$$I_{-1} = \int \frac{1+ax^2}{\sqrt{(1-x^2)(1-k^2x^2)}}dx$$

$$= \left(1+\frac{a}{k^2}\right)\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} - \frac{a}{k^2}\int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx,$$

and these are, respectively, Legendre's first and second elliptic integrals.

These integrals I_0, I_1, I_{-1} are therefore all known. Their properties will be discussed in the proper place. We thus have a means of connecting I_n with them for any integral value of n, positive or negative.

The same reduction formula obviously must hold for

$$\int \frac{d\theta}{(1+a\sin^2\theta)^n\sqrt{1-k^2\sin^2\theta}},$$

which is only another form of the same integral.

EXAMPLES.

1. If $X \equiv ax^n + b$, obtain reduction formulae for the integral $u_{p,q} \equiv \int \frac{x^p}{X^q}dx$ of the forms,

$$\text{(i)}\quad Au_{p,q} + Bu_{p-n,q} + R = 0,$$
$$\text{(ii)}\quad A'u_{p,q} + B'u_{p,q-1} + R' = 0,$$

where A, B, A', B' are constants and R, R' are algebraic functions of x.

[MATH. TRIP., 1896]

2. Prove that

(a) $$\int \cos^{2n}\phi\, d\phi = \frac{1}{2n}\tan\phi\cos^{2n}\phi + \left(1-\frac{1}{2n}\right)\int \cos^{2n-2}\phi\, d\phi,$$

[TRINITY, 1891.]

(b) $$\int \sec^{2n+1}\phi\, d\phi = \frac{1}{2n}\tan\phi\sec^{2n-1}\phi + \left(1-\frac{1}{2n}\right)\int \sec^{2n-1}\phi\, d\phi,$$

[I. C. S., 1886.]

3. Prove that

$$\int (a^2+x^2)^{\frac{2n+1}{2}}dx = \frac{x}{2n+2}(a^2+x^2)^{\frac{2n+1}{2}} + \frac{2n+1}{2n+2}a^2\int (a^2+x^2)^{\frac{2n-1}{2}}dx.$$

[I. C. S., 1886.]

4. Investigate a formula of reduction applicable to

$$\int x^m(1+x^2)^{\frac{n}{2}}dx,$$

where m and n are positive integers, and complete the integration if $m=5$, $n=7$. [ST. JOHN'S COLL., 1881.]

5. If $\phi(n) = a^{2n-1}\int_0^\infty \frac{dx}{(a^2+x^2)^n}$, prove that $\phi(n) = \frac{2n-3}{2n-2}\phi(n-1)$.

[R. P.]

6. Investigate formulae of reduction for

(a) $\int \frac{dx}{(x^2+a^2)^{\frac{n}{2}}}$.

(b) $\int x^m(a+bx)^{p+\frac{1}{2}}dx$.

(c) $\int \frac{x^m}{(a^2+x^2)^{\frac{1}{2}}}dx$.

(d) $\int \frac{x^m}{(a^3+x^3)^{\frac{n}{3}}}dx$.

(e) $\int \frac{x^m}{(x^3-1)^{\frac{1}{3}}}dx$.

(f) $\int x^{2n}(x^2+a^2)^{\frac{2p+1}{2}}dx$,

and obtain the value of $\int x^8(x^3-1)^{-\frac{1}{3}}dx$. [COLLEGES, CAMB.]

7. Investigate a formula of reduction for

$$\int \frac{x^{2n+1}}{(1-x^2)^{\frac{1}{2}}}dx,$$

and by means of this integral show that

$$\frac{1}{2n+2} + \frac{1}{2}\cdot\frac{1}{2n+4} + \frac{1.3}{2.4}\cdot\frac{1}{2n+6} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{2n+8} + \dots \textit{ ad inf.}$$
$$= \frac{2.4.6\dots 2n}{3.5.7\dots(2n+1)}.$$

Sum also the series

$$\frac{1}{2n+1} + \frac{1}{2}\cdot\frac{1}{2n+3} + \frac{1.3}{2.4}\cdot\frac{1}{2n+5} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{2n+7} + \dots \textit{ ad inf.}$$

[MATH. TRIP., 1897.]

8. Find a reduction formula for

$$\int_1^x \frac{x^n\,dx}{\sqrt{x-1}}.$$

Show that

$$\frac{2.4.6\ldots 2n}{3.5.7\ldots(2n+1)}\left[1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\ldots+\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}x^n\right]$$
$$=1+\frac{a_1}{3}(x-1)+\frac{a_2}{5}(x-1)^2+\ldots+\frac{1}{2n+1}(x-1)^n,$$

where a_1, a_2, ... are the binomial coefficients. [St. John's, 1886.]

9. Prove that if $\quad u_n \equiv \int_0^{\frac{\pi}{4}} \sin^{2n} x\,dx,$

then $\quad u_n = \left(1-\frac{1}{2n}\right)u_{n-1} - \frac{1}{n\,2^{n+1}},$

and deduce

$$\int_0^{\frac{\pi}{4}} \sin^{2n} x\,dx = -\frac{1}{2^{n+1}}\left\{\frac{1}{n}+\frac{2n-1}{n(n-1)}+\frac{(2n-1)(2n-3)}{n(n-1)(n-2)}+\ldots\right\}$$
$$+\frac{(2n-1)(2n-3)\ldots 3}{2n(2n-2)\ldots 4.2}\cdot\frac{\pi}{4}.$$

[Math. Trip., 1878.]

10. Prove that

$$\int_0^1 x^{4m+1}\sqrt{\frac{1-x^2}{1+x^2}}\,dx = \frac{1.3.5\ldots(2m-1)}{2.4.6\ldots 2m}\cdot\frac{\pi}{4} - \frac{2.4.6\ldots 2m}{3.5.7\ldots 2m+1}\cdot\frac{1}{2}.$$

11. Find a reduction formula for

$$\int e^{ax}\cos^n x\,dx,$$

where n is a positive integer, and evaluate

$$\int e^{ax}\cos^4 x\,dx.$$

[Oxford, 1889.]

12. Find formulae of reduction for

$$(1)\quad \int x^n \sin x\,dx,$$

$$(2)\quad \int e^{ax}\sin^n x\,dx.$$

Deduce from the latter a formula of reduction for

$$\int \cos ax \sin^n x\,dx.$$

[Colleges γ, 1890.]

13. Show that

$$(m+n)(m+n-2)\int \sin^m\theta\cos^n\theta\,d\theta$$
$$=(m-1)\sin^{m+1}\theta\cos^{n-1}\theta-(n-1)\sin^{m-1}\theta\cos^{n+1}\theta$$
$$+(m-1)(n-1)\int \sin^{m-2}\theta\cos^{n-2}\theta\,d\theta.$$

[TRIN. COLL. CAMB., 1889.]

14. Show that

$$2^m\int \cos mx\cos^m x\,dx$$
$$=C+x+m\cdot\frac{\sin 2x}{2}+\frac{m(m-1)}{1\cdot 2}\cdot\frac{\sin 4x}{4}+\ldots+\frac{\sin 2mx}{2m},$$

where m is a positive integer. [COLLEGES α, 1885.]

15. Show that

$$\int_0^{\frac{\pi}{2}} \sin^{2m}\theta\cos^{2m-1}\theta\,d\theta=\frac{(2m-2)(2m-4)\ldots 4\cdot 2}{(4m-1)(4m-3)\ldots(2m+1)},$$

m being a positive integer. [OXFORD, 1889.]

16. Evaluate the integral $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-nx}\cos^m x\,dx,$

m being a positive integer. [COLL., 1886.]

17. Prove that if

$$I_{m,n}\equiv\int \cos^m x\sin nx\,dx,$$

$$(m+n)\,I_{m,n}=-\cos^m x\cos nx+mI_{m-1,n-1}$$

and
$$\Big[I_{m,m}\Big]_0^{\frac{\pi}{2}}=\frac{1}{2^{m+1}}\left(\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\ldots+\frac{2^m}{m}\right).$$

[BERTRAND.]

18. If
$$u_{m,n}\equiv\int_0^{\frac{\pi}{2}} \cos^m x\sin nx\,dx,$$

prove that
$$u_{m,n}=\frac{1}{m+n}+\frac{m}{m+n}\cdot u_{m-1,n-1}.$$

Hence find the value, when m is a positive integer, of

$$\int_0^{\frac{\pi}{2}} \cos^m x\sin 2mx\,dx.$$

[γ, 1887.]

19. If
$$I_{m,n}\equiv\int \cos^m x\cos nx\,dx,$$

prove that
$$I_{m,n}=-\frac{\cos^2 nx}{m^2-n^2}\frac{d}{dx}\left(\frac{\cos^m x}{\cos nx}\right)+\frac{m(m-1)}{m^2-n^2}I_{m-2,n},$$

and show that

$$\left[J_{m,n}\right]_0^{\frac{\pi}{2}}=\frac{m(m-1)}{m^2-n^2}\left[J_{m-2,n}\right]_0^{\frac{\pi}{2}}.$$

20. Prove that $$\int_0^{\frac{\pi}{2}}\cos^n x\cos nx\,dx=\frac{\pi}{2^{n+1}},$$
n being a positive integer. [BERTRAND.]

21. If m and n be positive integers, and if $m+n$ be even, prove that

$$\int_0^{\frac{\pi}{2}}\cos^m\theta\cos n\theta\,d\theta=\frac{\pi}{2^{m+1}}\frac{m\,!}{\frac{m+n}{2}\,!\;\frac{m-n}{2}\,!}.$$

[COLLEGES, 1882.]

22. If $\int_0^{\frac{\pi}{2}}\cos^m x\cos nx\,dx$ be denoted by $f(m, n)$, show that

$$f(m, n)=\frac{m}{m-n}f(m-1, n+1)=\frac{m}{m+n}f(m-1, n-1).$$

[OXFORD, 1890.]

23. Prove that if n be a positive integer, greater than unity,

$$\int_0^{\frac{\pi}{2}}\cos^{n-2}x\sin nx\,dx=\frac{1}{n-1}.$$ [OXFORD I. P., 1889.]

24. If $u_{m,n}=\int x^m\operatorname{cosec}^n x\,dx$, prove that

$$(n-1)(n-2)u_{m,n}=(n-2)^2u_{m,n-2}+m(m-1)u_{m-2,n-2}$$
$$-x^{m-1}\{m\sin x+(n-2)x\cos x\}\operatorname{cosec}^{n-1}x.$$

[MATH. TRIP., 1896.]

25. If $$\int_0^\infty e^{-x}x^{n-1}\log x\,dx\equiv\phi(n),$$

prove that $$\phi(n+2)-(2n+1)\phi(n+1)+n^2\phi(n)=0.$$

[R. P., ST. JOHN'S COLL., 1881.]

26. Show that if $$U_n\equiv\int_0^1\frac{x^n e^x\,dx}{\sqrt{1-x}},$$

$$2U_{n+1}+U_n(2n-1)-2nU_{n-1}=0.$$ [COLLEGES β, 1887.]

27. Prove that if $\phi(m)\equiv\int x^m(x^3+3ax+c)^{-\frac{1}{2}}\,dx$,

then $$(2m-1)\phi(m)+3a(2m-3)\phi(m-2)+(2m-4)c\,\phi(m-3)$$
$$=2x^{m-2}(x^3+3ax+c)^{\frac{1}{2}}.$$ [TRINITY, 1886.]

28. If $$u_m\equiv\int_0^{\frac{\pi}{2}}e^{-nx}\cos^m x\,dx,$$

prove that $$(m^2+n^2)u_m=m(m-1)u_{m-2}+n.$$ [OXFORD I. P., 1900.]

29. Prove that if $$I_m = \int_0^{\frac{\pi}{2}} \frac{\sin^m x\,dx}{(1-k^2\sin^2 x)^{\frac{1}{2}}},$$
then $$(m-1)k^2 I_m - (m-2)(1+k^2)I_{m-2} + (m-3)I_{m-4} = 0.$$
[TRINITY, 1890.]

30. Obtain a reduction formula for the integral
$$I_n = \int (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)^{-n}\,d\theta$$
in the form
$$2(n+1)(ab-h^2)I_{n+2} - (2n+1)(a+b)I_{n+1} + 2nI_n = -\frac{1}{2n}\frac{d^2 I_n}{d\theta^2}.$$
[MATH. TRIP., 1898.]

31. Show that $$\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-e^2\sin^2\phi)^3} = \frac{8-8e^2+3e^4}{(1-e^2)^{\frac{5}{2}}}\cdot\frac{\pi}{16},$$
e being less than unity. [ST. JOHN'S COLL., 1885.]

32. If $$I_n = \int \frac{\sin nx}{\sin x}\,dx,$$
prove that $$(n-1)(I_n - I_{n-2}) = 2\sin(n-1)x,$$
and hence that
$$\int_0^{\frac{\pi}{2}} \frac{\sin nx}{\sin x} = \frac{\pi}{2}, \text{ if } n \text{ be odd},$$
$$= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + (-1)^{\frac{n}{2}+1}\frac{1}{n-1}\right), \text{ if } n \text{ be even}.$$

33. If $X \equiv a + bx^n + cx^{2n}$ and $I_{m,p} \equiv \int x^m X^p\,dx$,

prove the existence of reduction formulae of the nature of

(i) $x^{m+1}X^{p+1} = A_1 I_{m,p} + B_1 I_{m+n,p} + C_1 I_{m+2n,p}$;

(ii) $x^{m-2n+1}X^{p+1} = A_2 I_{m,p} + B_2 I_{m-n,p} + C_2 I_{m-2n,p}$;

(iii) $x^{m+1}X^p = A_3 I_{m,p} + B_3 I_{m+n,p-1} + C_3 I_{m+2n,p-1}$;

(iv) $x^{m+1}X^p = A_4 I_{m,p} + B_4 I_{m,p-1} + C_4 I_{m+n,p-1}$;

(v) $x^{m-n+1}X^{p+1} = A_5 I_{m,p} + B_5 I_{m-n,p} + C_5 I_{m-n,p+1}$;

and find the values of the fifteen constants.

34. Show that $$\int \frac{dx}{(a+bx^2+cx^4)^p}$$
can be reduced to the integration of
$$\int \frac{dx}{a+bx^2+cx^4} \text{ and } \int \frac{x^2\,dx}{a+bx^2+cx^4} \quad (b>0,\ b^2>4ac),$$
and integrate these expressions; p being integral. [BERTRAND.]

35. Show that, if $u_{m,n} \equiv \int \frac{x^m}{(\log x)^n} dx$,

$$(n-1) u_{m,n} = -\frac{x^{m+1}}{(\log x)^{n-1}} + (m+1) u_{m,n-1}.$$

[OXFORD, I. P., 1889.]

36. Find reduction formulae for

$$(\alpha) \int \tanh^n x \, dx;$$

$$(\beta) \int \frac{x}{\sin^n x} dx;$$

$$(\gamma) \int \frac{dx}{(a + be^x + ce^{-x})^n}.$$

37. If $I_m \equiv \int \frac{x^m dx}{\sqrt{X}}$, where $X \equiv ax^2 + 2bx + c$, show that

$$amI_m + (2m-1) bI_{m-1} + (m-1) cI_{m-2} = x^{m-1}\sqrt{X}.$$ [β. 1891.]

38. Establish a reduction formula for

$$I_n \equiv \int \frac{dx}{(Ax^2 + B)^n \sqrt{X}},$$

where $X \equiv ax^4 + bx^2 + c$, in the form

$$\frac{x\sqrt{X}}{(Ax^2 + B)^{n-1}} = \lambda I_n + \mu I_{n-1} + \nu I_{n-2} + \xi I_{n-3},$$

showing that

$$\left.\begin{aligned} \lambda &= (2n-2)\frac{B}{A^2}(A^2c - ABb + B^2a), \\ \mu &= -(2n-3)\frac{1}{A^2}(A^2c - 2ABb + 3B^2a), \\ \nu &= -(2n-4)\frac{1}{A^2}(Ab - 3Ba), \\ \xi &= -(2n-5)\frac{1}{A^2}a. \end{aligned}\right\}$$

39. Show that, if

$$I_{m,n} \equiv \int_0^\pi \sin^m \theta \cos n\theta \, d\theta, \quad J_{m,n} \equiv \int_0^\pi \sin^m \theta \sin n\theta \, d\theta,$$

then

$$J_{m-1,n+1} = \left(1 - \frac{n}{m}\right) I_{m,n}, \quad I_{m-1,n+1} = -\left(1 - \frac{n}{m}\right) J_{m,n},$$

where m is a positive integer; and point out how these results can be used to find the values of $I_{m,n}$ and $J_{m,n}$. [C. S., 1896.]

40. If T be a function of x such that

$$\left(\frac{dT}{dx}\right)^2 = A + 3BT + 3CT^2 + DT^3,$$

prove that

$$\frac{d}{dx}\left(\frac{1}{T^{n-1}}\frac{dT}{dx}\right) = -\frac{(n-1)A}{T^n} - \frac{3}{2}(2n-3)\frac{B}{T^{n-1}} - 3\frac{(n-2)C}{T^{n-2}} - \frac{(2n-5)D}{2T^{n-3}}.$$

Apply the result to investigate a reduction formula for

$$\int \frac{dx}{T^n}.$$

By a consideration of the case where $C=0$, $D=0$ (or in any other way), obtain a reduction formula for

$$\int \frac{dx}{(a+2bx+cx^2)^n}.$$ [I. C. S., 1897.]

41. Prove that

$$\int_0^\infty e^{-x^2}x^{2n}\,dx = \sqrt{\pi}\,\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\int_0^\infty e^{-x^2}x^{2n+1}\,dx,$$

where n is a positive integer. [COLLEGES α, 1890.]

42. If $$u_n \equiv \int_a^b x^n\{(b-x)(x-a)\}^{-\frac{1}{2}}dx,$$

show that $2nu_n = (2n-1)(a+b)u_{n-1} - 2(n-1)abu_{n-2}$.

[OXFORD I. PUB., 1912.]

43. By applying the substitution $x = a\cos^2\theta + b\sin^2\theta$ (or otherwise), prove that the definite integral

$$\int_a^b \frac{x^n\,dx}{\sqrt{(x-a)(b-x)}}$$

is a rational integral function of a and b when n is an integer; and evaluate it when $n=3$. [OXF. I. P., 1913.]

44. If $$u_n \equiv \int \frac{\cos 2nx}{\sin^2 x}\,dx,$$

obtain a formula of reduction connecting u_n and u_{n-1}.

Hence, or otherwise, evaluate

$$\int_x^{\frac{\pi}{2}} \frac{\cos 2nx}{\sin^2 x}\,dx,$$

where n is a positive integer and $\frac{\pi}{2} > x > 0$.

Consider the case when the lower limit is negative.

[OXF. I. P., 1915.]

45. By multiplying the inequality $1 \geqslant 2\sin x - \sin^2 x$ by $\sin^{2n-1}x$ and by $\sin^{2n}x$, and integrating between 0 and $\frac{1}{2}\pi$, show that

$$\left\{\frac{(4n+3)(2n+1)}{4n+4}\cdot\frac{\pi}{2}\right\}^{\frac{1}{2}} > \frac{2.4\ldots 2n}{1.3\ldots(2n-1)} > \left\{\frac{2n(2n+1)}{4n+1}\cdot\pi\right\}^{\frac{1}{2}}.$$

[MATH. TRIP. I., 1915.]

46. The expression

$$\frac{1-a}{(1-a\sin^2\theta)^{\frac{3}{2}}} - (1-a\sin^2\theta)^{\frac{1}{2}},$$

where $1 > a > 0$, is expanded in ascending powers of a, and the coefficient of a^n is denoted by u_n. Prove that

$$\int_0^{\frac{\pi}{2}} u_n d\theta = 0.$$

[MATH. TRIP. I., 1916.]

47. If $\quad s_n \equiv \int \frac{\sin(2n-1)x}{\sin x} dx, \quad v_n \equiv \int \frac{\sin^2 nx}{\sin^2 x} dx,$

prove the reduction formulae

$$n(s_{n+1} - s_n) = \sin 2nx, \quad v_{n+1} - v_n = s_{n+1};$$

and show that if v_n be taken between the limits 0 and $\frac{1}{2}\pi$, its value is $\frac{1}{2}n\pi$, where n is an integer. [MATH. TRIP. I., 1914.]

48. If $\lambda^2 \equiv \cos^2\phi/a^2 + \sin^2\phi/b^2$, find $\int_0^{\frac{\pi}{2}} \frac{d\phi}{\lambda^2}$,

and prove that $\int_0^{\frac{\pi}{2}} \frac{d\phi}{\lambda^6} = \pi ab\{3(a^4+b^4)+2a^2b^2\}/16,$

and that $\int_0^1 \frac{3\mu^2-1}{2} \frac{d\mu}{(1-e^2\mu^2)^{\frac{5}{2}}} = \frac{e^2}{3(1-e^2)^{\frac{5}{2}}}.$ [ϵ, 1883.]

49. If $U_n \equiv \int \sin^m x(a+b\cos x)^{-n} dx$, prove that U_n can be calculated by means of a reduction formula of the nature

$$AU_n + BU_{n-1} + CU_{n-2} = \sin^{m+1}x(a+b\cos x)^{-n+1};$$

and determine the constants A, B, C.

50. Prove that $\int_c^\infty \frac{dx}{(a^2-2cx+x^2)^{n+1}} = \frac{2n!}{n!\,n!}\frac{\pi}{\lambda^{n+\frac{1}{2}}},$

where λ denotes $4(a^2-c^2)$ and is supposed positive. [TRIN., 1887.]

CHAPTER VIII.

FORM $\int F(x, \sqrt{R})dx$, WHERE R IS QUADRATIC.

275. The integration of expressions of the type

$$\int \frac{dx}{X\sqrt{Y}}$$

can be effected in all cases for which X and Y are rational integral algebraic expressions of degree not greater than the second.

There are **four Cases**:

I. X and Y, both linear.
II. X quadratic, Y linear. } Put $\sqrt{Y} = y$.

III. X linear, Y quadratic. Put $X = \frac{1}{y}$.

IV. X quadratic, Y quadratic. Put $\frac{Y}{X} = y$ or y^2.

The general substitution $\frac{Y}{X} = y$ or y^2 will effect the integration in all cases. But the simpler substitutions noted, viz.:

$\sqrt{Y} = y$ in Cases I. and II.

and $X = \frac{1}{y}$ in Case III., are better.

Case IV., in which we employ the substitution

$$\frac{Y}{X} = y \text{ or } y^2,$$

is much more troublesome, but *includes the previous ones.*

We shall, in all cases, assume the radical $\sqrt{Y}$ to be real.

276. Case I. **X linear, Y linear.**

Let $I = \int \frac{dx}{(ax+b)\sqrt{px+q}}$.

Putting $\sqrt{Y} \equiv \sqrt{px+q} = y,$

$$\frac{p\,dx}{2\sqrt{px+q}} = dy$$

and $$ax+b = \frac{a}{p}(y^2-q)+b.$$

Thus I becomes $2\int \frac{dy}{ay^2-aq+bp}$,

which being of the standard form

$$\frac{2}{a}\int \frac{dy}{y^2 \pm \lambda^2}, \text{ where } \lambda^2 = \frac{bp \sim aq}{a},$$

is immediately integrable, viz.:

$$= \frac{2}{a\lambda} \tan^{-1} \frac{y}{\lambda} \quad \text{or} \quad -\frac{2}{a\lambda} \coth^{-1} \frac{y}{\lambda},$$

according as $\frac{bp-aq}{a}$ is positive or negative,

i.e. $$= \frac{2}{\sqrt{a(bp-aq)}} \tan^{-1} \frac{\sqrt{a(px+q)}}{\sqrt{bp-aq}}$$

or $$= -\frac{2}{\sqrt{a(aq-bp)}} \coth^{-1} \frac{\sqrt{a(px+q)}}{\sqrt{aq-bp}},$$

i.e. $$= \frac{2}{\sqrt{a(bp-aq)}} \sin^{-1} \sqrt{\frac{a}{p}} \sqrt{\frac{px+q}{ax+b}}$$

or $$= -\frac{2}{\sqrt{a(aq-bp)}} \cosh^{-1} \sqrt{\frac{a}{p}} \sqrt{\frac{px+q}{ax+b}},$$

with other forms, the real one to be chosen in each case.

277. Another Method.

The last form shows how the factors of the integrand are involved in the result of integration, and indicates that the substitution $\frac{px+q}{ax+b} = y^2$ mentioned above as the *general* substitution would have led directly to this result.

If we elect to proceed in this way, viz. putting $\frac{px+q}{ax+b} = y^2$, we have

$$\left(\frac{p}{px+q} - \frac{a}{ax+b}\right) dx = 2\frac{dy}{y};$$

$$\therefore \frac{dx}{(ax+b)(px+q)} = \frac{2}{bp-aq}\frac{dy}{y} \quad \text{or} \quad -\frac{2}{aq-bp}\frac{dy}{y}.$$

Now $x=\dfrac{by^2-q}{p-ay^2}$ and $px+q=\dfrac{(bp-aq)y^2}{p-ay^2}$ or $\dfrac{(aq-bp)y^2}{ay^2-p}$.

When $bp-aq$ is positive,

$$\int\frac{dx}{(ax+b)\sqrt{px+q}}=\frac{2}{\sqrt{bp-aq}}\int\frac{dy}{\sqrt{p-ay^2}}$$

$$=\frac{2}{\sqrt{a(bp-aq)}}\int\frac{dy}{\sqrt{\frac{p}{a}-y^2}}$$

$$=\frac{2}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\,y,$$

or other forms.

When $bp-aq$ is negative,

$$\int\frac{dx}{(ax+b)\sqrt{px+q}}=-\frac{2}{\sqrt{aq-bp}}\int\frac{dy}{\sqrt{ay^2-p}}$$

$$=-\frac{2}{\sqrt{a(aq-bp)}}\int\frac{dy}{\sqrt{y^2-\frac{p}{a}}}$$

$$=-\frac{2}{\sqrt{a(aq-bp)}}\cosh^{-1}\sqrt{\frac{a}{p}}\,y$$

or

$$=-\frac{2}{\sqrt{a(aq-bp)}}\sinh^{-1}\sqrt{-\frac{a}{p}}\,y,$$

or other forms, the real form to be chosen in each case.

278. **Illustrative Examples.**

Ex. 1. Integrate $$I=\int\frac{dx}{(2x+3)\sqrt{4x+5}}.$$

Put $\sqrt{4x+5}=y$; $\therefore\ \dfrac{2dx}{\sqrt{4x+5}}=dy.$

Also $2x+3=\dfrac{y^2+1}{2}$;

$$\therefore\ I=\int\frac{dy}{y^2+1}=\tan^{-1}\sqrt{4x+5}.$$

Again, if we put $\dfrac{4x+5}{2x+3}=z$, *i.e.* $x=\dfrac{3z-5}{2(2-z)}$ and $dx=\dfrac{dz}{2(2-z)^2}$,

$$I=\frac{1}{2}\int\frac{dy}{\sqrt{z(2-z)}}=\sin^{-1}\sqrt{\frac{z}{2}}\quad\text{(Art. 87)},$$

i.e. $$=\sin^{-1}\frac{1}{\sqrt{2}}\sqrt{\frac{4x+5}{2x+3}},$$

which is the same as before, but exhibits the result as a function of both the factors of the integrand.

Ex. 2. Integrate $$I \equiv \int \frac{dx}{(1-x)\sqrt{2-x}}.$$

Let $$\sqrt{2-x} = y\,; \quad \therefore \quad \frac{dx}{\sqrt{2-x}} = -2dy\,;$$

$$\therefore \quad I = -2\int \frac{dy}{y^2-1} = -\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = -\log\frac{y-1}{y+1}$$

$$= \log\frac{y+1}{y-1} = \log\frac{\sqrt{2-x}+1}{\sqrt{2-x}-1}, \text{ or other forms.}$$

279. An Extension.

The same substitution, viz. $\sqrt{Y} = y$, will suffice for the integration of
$$\int \frac{\phi(x)\,dx}{X\sqrt{Y}},$$
where X, Y are both linear and $\phi(x)$ is any rational integral algebraic function of x.

For if $Y \equiv px + q = y^2$, then $x = \dfrac{y^2-q}{p}$, and $p\,dx = 2y\,dy$;

thus $$\int \frac{\phi(x)\,dx}{(ax+b)\sqrt{px+q}} = 2\int \frac{\phi\left(\dfrac{y^2-q}{p}\right) dy}{ay^2 - aq + bp};$$

and if $\phi\left(\dfrac{y^2-q}{p}\right)$ be expanded in descending powers of y^2 and then divided by $ay^2 + (bp - aq)$ till the remainder is independent of y, we have to integrate with regard to y an expression of form
$$A_0 y^{2n-2} + A_1 y^{2n-4} + A_2 y^{2n-6} + \ldots + A_{n-1} + \frac{B}{ay^2 + (bp - aq)},$$
n being the degree of $\phi(x)$ in x; and each term is at once integrable, after which operation y is to be written back as $\sqrt{px+q}$.

280. Ex. Integrate $$I \equiv \int \frac{x^4\,dx}{(x-1)\sqrt{x+2}}.$$

Writing $\sqrt{x+2} = y$, we have $\dfrac{dx}{\sqrt{x+2}} = 2dy$ and $x = y^2 - 2$, so that
$$\frac{x^4}{x-1} = \frac{(y^2-2)^4}{y^2-3} = y^6 - 5y^4 + 9y^2 - 5 + \frac{1}{y^2-3} \text{ by division.}$$

Thus

$$\frac{I}{2}=\int\left(y^6-5y^4+9y^2-5+\frac{1}{y^2-3}\right)dy$$

$$=\frac{y^7}{7}-y^5+3y^3-5y-\frac{1}{\sqrt{3}}\coth^{-1}\frac{y}{\sqrt{3}}$$

$$=\frac{(x+2)^{\frac{1}{2}}}{7}\{(x+2)^3-7(x+2)^2+21(x+2)-35\}-\frac{1}{\sqrt{3}}\coth^{-1}\sqrt{\frac{x+2}{3}},$$

i.e. $$I=\frac{2\sqrt{x+2}}{7}(x^3-x^2+5x-13)+\frac{1}{\sqrt{3}}\log\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$$

if the logarithmic form be preferred.

281. Forms reducible to Case I.

The student should note the variety of forms reducible to the case considered, viz. X linear, Y linear, by a proper substitution. For example,

(1) $\int\frac{\sin\theta\,d\theta}{(a\cos\theta+b)\sqrt{p\cos\theta+q}}$, put $\cos\theta=x$, *i.e.* $\theta=\cos^{-1}x$.

(2) $\int\frac{\sqrt{\operatorname{cosec}\theta}\,d\theta}{(a\cos\theta+b\sin\theta)\sqrt{p\cos\theta+q\sin\theta}}$, put $\cot\theta=x$, *i.e.* $\theta=\cot^{-1}x$.

(3) $\int\frac{\cos\theta\,d\theta}{(a\cos^2\theta+b\sin^2\theta)\sqrt{p\cos^2\theta+q\sin^2\theta}}$, put $\cot^2\theta=x$, *i.e.* $\theta=\cot^{-1}\sqrt{x}$.

(4) $\int\frac{L\cos\theta+M\sin\theta\,d\theta}{(a\cos^2\theta+b\sin^2\theta)\sqrt{p\cos^2\theta+q\sin^2\theta}}$, separate into two integrals, put $\cot^2\theta=x$ in one and $\tan^2\theta=y$ in the other.

(5) $\int\frac{e^{\frac{x}{2}}\,dx}{(ae^x+be^{-x})\sqrt{pe^x+qe^{-x}}}$, put $e^{2x}=y$, *i.e.* $x=\frac{1}{2}\log y$.

(6) $\int\frac{dx}{x\log(ax^b)\sqrt{\log(cx^d)}}$, put $\log x=y$, *i.e.* $x=e^y$.

(7) $\int\frac{dx}{(ax+b)\sqrt{x(px+q)}}$, put $x=\frac{1}{y}$.

etc.

282. Ex. Integrate

$$I=\int\frac{1+2\cos^4\theta}{(1+12\cos^2\theta)\sqrt{1+3\cos^2\theta}}\frac{\sin\theta}{\cos^4\theta}d\theta.$$

Put $\tan^2\theta=y$; $\therefore\ 2\tan\theta\sec^2\theta\,d\theta=dy$;

$$\therefore\ I=\int\frac{\sec^4\theta+2}{(\sec^2\theta+12)\sqrt{\sec^2\theta+3}}\tan\theta\sec^2\theta\,d\theta=\frac{1}{2}\int\frac{(y+1)^2+2}{(y+13)\sqrt{y+4}}dy.$$

Now put $\sqrt{y+4}=z$; $\therefore\ y=z^2-4,\quad dy=2z\,dz$;

$$\therefore\ I=\int\frac{z^4-6z^2+11}{z^2+9}\,dz=\int\left(z^2-15+\frac{146}{z^2+9}\right)dz$$

$$=\frac{z^3}{3}-15z+\frac{146}{3}\tan^{-1}\frac{z}{3},\quad \text{where } z=\sqrt{y+4}=\sqrt{\tan^2\theta+4}\,;$$

$$\therefore\ I=\frac{(\tan^2\theta+4)^{\frac{3}{2}}}{3}-15\,(\tan^2\theta+4)^{\frac{1}{2}}+\frac{146}{3}\tan^{-1}\frac{\sqrt{\tan^2\theta+4}}{3}$$

or
$$\frac{(\sec^2\theta+3)^{\frac{3}{2}}}{3}-15\,(\sec^2\theta+3)^{\frac{1}{2}}+\frac{146}{3}\tan^{-1}\frac{\sqrt{\sec^2\theta+3}}{3}.$$

283. Case II. **X quadratic, Y linear.**

Let $I\equiv\int\dfrac{Mx+N}{(ax^2+bx+c)\sqrt{px+q}}\,dx$, M and N being constants. The terms $Mx+N$ now existent in the numerator do not introduce any difficulty and make the result more general.

The same substitution being made, viz. $\sqrt{Y}=y$, we put

$$\sqrt{px+q}=y\,;\quad\therefore\quad \frac{p}{2\sqrt{px+q}}\,dx=dy\,.$$

ax^2+bx+c reduces to the form Ay^4+By^2+C

and $Mx+N$ reduces to the form $M'y^2+N'$

Thus I takes the form $\dfrac{2}{p}\displaystyle\int\frac{M'y^2+N'}{Ay^4+By^2+C}\,dy.$

Now $\dfrac{M'y^2+N'}{Ay^4+By^2+C}$ can be thrown into partial fractions of the form
$$\frac{\lambda y+\mu}{\alpha y^2+\beta y+\gamma}+\frac{\lambda' y+\mu'}{\alpha' y^2+\beta' y+\gamma'},$$
and each portion is integrable by earlier rules (Arts. 155 and 156).

284. **Extension.**

Further, it is evident that the same substitution will effect the integration $\displaystyle\int\frac{\phi(x)dx}{(ax^2+bx+c)\sqrt{px+q}}$, where $\phi(x)$ is any rational integral algebraic function of x. For when $px+q=y^2$,

i.e.
$$x=\frac{y^2-q}{p}\quad\text{and}\quad \frac{dx}{\sqrt{px+q}}=\frac{2}{p}\,dy,$$

$\dfrac{\phi(x)}{ax^2+bx+c}$ reduces to the form

$$\frac{A_0y^{2n}+A_1y^{2n-2}+A_2y^{2n-4}+\ldots+A_n}{Ay^4+By^2+C}$$

where n is the degree of $\phi(x)$ in x; and therefore, by division and our rules for partial fractions the integrand may be expressed as

$$P_0y^{2n-4}+P_1y^{2n-6}+\dots P_{n-2}+\frac{\lambda y+\mu}{ay^2+\beta y+\gamma}+\frac{\lambda' y+\mu'}{a'y^2+\beta' y+\gamma'},$$

and each term is integrable.

Again, $$I\equiv\int\frac{\phi(x)}{\chi(x)}\frac{dx}{\sqrt{px+q}},$$

where ϕ and χ are any rational integral algebraic functions of x, may now be seen to be integrable by the same substitution, for it becomes

$$I=\frac{2}{p}\int\frac{\phi\left(\dfrac{y^2-q}{p}\right)}{\chi\left(\dfrac{y^2-q}{p}\right)}dy,$$

and the new integrand can be expressed by partial fraction methods (Art. 152, etc.) in the form

$$\Sigma Py^r+\sum\frac{Q}{y-a}+\sum\frac{R}{(y-\beta)^s}+\sum\frac{\lambda y+\mu}{Ay^2+By+C}$$
$$+\sum\frac{\lambda' y+\mu'}{(A'y^2+B'y+C')^t},$$

and integrals of the expressions of the first four terms can be obtained by the rules given before, and the integral of the last by aid of the reduction formula established in Art. 238.

285. Ex. 1. Integrate $I\equiv\int\frac{x+2}{(x^2+3x+3)\sqrt{x+1}}dx.$

Putting $\sqrt{x+1}=y$, we have $\frac{dx}{\sqrt{x+1}}=2\,dy$,

and $$I=2\int\frac{y^2+1}{y^4+y^2+1}dy=\int\left(\frac{1}{y^2+y+1}+\frac{1}{y^2-y+1}\right)dy$$
$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2y-1}{\sqrt{3}}=-\frac{2}{\sqrt{3}}\tan^{-1}\sqrt{3}\frac{\sqrt{x+1}}{x}$$
$$=\frac{2}{\sqrt{3}}\cos^{-1}\sqrt{3}\sqrt{\frac{x+1}{x^2+3x+3}}+\text{const.}$$

Ex. 2. Integrate $$I\equiv\int\frac{x^2-5x-37}{x^2-7x-30}\frac{dx}{\sqrt{x-1}}.$$

Put $\sqrt{x-1}=y$; $\therefore\ \frac{dx}{\sqrt{x-1}}=2dy$ and $x=1+y^2$;

$$\therefore\ I=2\int\frac{(1+y^2)^2-5(1+y^2)-37}{(1+y^2)^2-7(1+y^2)-30}dy$$
$$=2\int\frac{y^4-3y^2-41}{y^4-5y^2-36}dy$$
$$=2\int\frac{y^4-3y^2-41}{(y^2+4)(y^2-9)}dy$$
$$=2\int\left(1+\frac{1}{y^2+4}+\frac{1}{y^2-9}\right)dy$$
$$=2y+\tan^{-1}\frac{y}{2}-\frac{2}{3}\coth^{-1}\frac{y}{3}$$
$$=2\sqrt{x-1}+\tan^{-1}\frac{\sqrt{x-1}}{2}-\frac{2}{3}\coth^{-1}\frac{\sqrt{x-1}}{3}.$$

286. **Forms reducible to Case II.**

The student should again note the variety of forms which may be brought under the foregoing rule by suitable substitutions and integrated.

Thus

(1) $\displaystyle\int\frac{\sqrt{\sin\theta}\,d\theta}{(a\cos^2\theta+b\sin\theta\cos\theta+c\sin^2\theta)\sqrt{p\cos\theta+q\sin\theta}}$, put $\theta=\cot^{-1}x$.

(2) $\displaystyle\int\frac{A\sqrt{\sin\theta}+B\sqrt{\cos\theta}}{a\cos^2\theta+b\sin\theta\cos\theta+c\sin^2\theta}\,\frac{d\theta}{\sqrt{p\cos\theta+q\sin\theta}}$, separate into two integrals. Use $\theta=\cot^{-1}x$ in the one and $\theta=\tan^{-1}y$ in the other.

(3) $\displaystyle\int\frac{A\sqrt{\sinh x}+B\sqrt{\cosh x}}{a\cosh^2x+b\sinh x\cosh x+c\sinh^2x}\,\frac{dx}{\sqrt{p\cosh x+q\sinh x}}$, similarly.

(4) $\displaystyle\int\frac{A\sqrt{\sin x}+B\sqrt{\cos x}}{\lambda+\mu\cos 2x+\nu\sin 2x}\,\frac{dx}{\sqrt{p\cos x+q\sin x}}$, from (2).

(5) $\displaystyle\int\frac{A\sqrt{\sin x}+B\sqrt{\cos x}}{a+b\cos(2x+a)}\,\frac{dx}{\sqrt{\cos(x+\beta)}}$, from (4).

287. Case III. **X linear, Y quadratic.**

The proper substitution is now $X=\dfrac{1}{y}$.

Let $$I\equiv\int\frac{dx}{(ax+b)\sqrt{px^2+qx+r}}.$$

Putting $ax+b=\dfrac{1}{y}$, we have, by logarithmic differentiation,

$$\frac{dx}{ax+b}=-\frac{1}{a}\frac{dy}{y}$$

and $$px^2+qx+r=\frac{p}{a^2}\left(\frac{1}{y}-b\right)^2+\frac{q}{a}\left(\frac{1}{y}-b\right)+r,$$

i.e. of form $$\frac{Ay^2+2By+C}{y^2}.$$

Hence the integral has been reduced to the known form

$$-\frac{1}{a}\int\frac{dy}{\sqrt{Ay^2+2By+C}},$$

which has been discussed in Art. 80.

Ex. Integrate

$$I\equiv\int\frac{dx}{(x-1)\sqrt{x^2-4x+2}}.$$

Let $x-1=\frac{1}{y}$, and therefore $\frac{dx}{x-1}=-\frac{dy}{y}$.

Hence $$I=-\int\frac{dy}{y\sqrt{\left(1+\frac{1}{y}\right)^2-4\left(1+\frac{1}{y}\right)+2}}$$

$$=-\int\frac{dy}{\sqrt{1-2y-y^2}}=-\int\frac{dy}{\sqrt{2-(y+1)^2}}=\cos^{-1}\frac{y+1}{\sqrt{2}}$$

$$=\cos^{-1}\frac{x}{(x-1)\sqrt{2}}.$$

288. **Forms reducible to Case III.**

Again we note the varieties of integrals which may be reduced to the present form by a suitable transformation, for instance:

(1) $\int\frac{\sin\theta\,d\theta}{(a\cos\theta+b)\sqrt{p\cos^2\theta+q\cos\theta+r}}$, put $\cos\theta=x$.

(2) $\int\frac{d\theta}{(a\cos\theta+b\sin\theta)\sqrt{p\cos^2\theta+q\sin\theta\cos\theta+r\sin^2\theta}}$, put $\cot\theta=x$.

(3) $\int\frac{\sin\theta\cos\theta\,d\theta}{(a\cos^2\theta+b\sin^2\theta)\sqrt{p\cos^4\theta+q\sin^2\theta\cos^2\theta+q\sin^4\theta}}$, put $\cot^2\theta=x$.

(4) $\int\frac{dx}{x(ax^n+bx^{-n})\sqrt{px^{2n}+q+rx^{-2n}}}$, put $x^{2n}=y$.

(5) $\int\frac{d\theta}{\cos(\theta+a)\sqrt{a+b\cos 2(\theta+\beta)}}$ from (2)

etc.

289. Remarks.

It will now appear that any integration of the form

$$\int \frac{\phi(x)\,dx}{(ax+b)\sqrt{px^2+qx+r}}$$

can be effected, $\phi(x)$ being any rational integral algebraic function of x. For by division we can express $\frac{\phi(x)}{ax+b}$ in the form

$$A_1x^{n-1}+A_2x^{n-2}+A_3x^{n-3}+\ldots+A_{n-1}x+A_n+\frac{M}{ax+b},$$

where n is the degree of $\phi(x)$ in x,

$$A_1x^{n-1}+\ldots+A_{n-1}x+A_n$$

is the quotient, and M the remainder independent of x.

We have thus reduced the process to the integration of a number of terms of the class

$$\int \frac{Ex^m\,dx}{\sqrt{px^2+qx+r}}$$

and one of the class

$$\int \frac{M\,dx}{(ax+b)\sqrt{px^2+qx+r}}.$$

The latter has been discussed in Art. 287, and integrals of the former class may be obtained by the reduction formula of Art. 240, viz.

$$x^{m-1}\sqrt{px^2+qx+r}=(m-1)rI_{m-2}+\frac{2m-1}{2}qI_{m-1}+mpI_m,$$

I_m standing for $\displaystyle\int \frac{x^m\,dx}{\sqrt{px^2+qx+r}}$;

that is,

$$I_m=x^{m-1}\frac{\sqrt{px^2+qx+r}}{mp}-\frac{2m-1}{2m}\frac{q}{p}I_{m-1}-\frac{m-1}{m}\frac{r}{p}I_{m-2}.$$

Ex. Integrate $\displaystyle I\equiv\int \frac{x^5+x^2+2}{(x+1)\sqrt{x^2+1}}\,dx.$

By division $\displaystyle \frac{x^5+x^2+2}{x+1}=x^4-x^3+x^2+\frac{2}{x+1};$

$$\therefore\ I=\int\left(\frac{x^4}{\sqrt{x^2+1}}-\frac{x^3}{\sqrt{x^2+1}}+\frac{x^2}{\sqrt{x^2+1}}+\frac{2}{(x+1)\sqrt{x^2+1}}\right)dx.$$

Let $\displaystyle I_n=\int\frac{x^n\,dx}{\sqrt{x^2+1}}.$

Then

$$I_0=\sinh^{-1}x,$$

$$I_1=\sqrt{x^2+1},$$

$$I_2=\frac{x\sqrt{x^2+1}}{2}-\frac{1}{2}\sinh^{-1}x, \quad \text{by the reduction formula } (m=2),$$

$$I_3=\frac{x^2\sqrt{x^2+1}}{3}-\frac{2}{3}I_1=\frac{x^2\sqrt{x^2+1}}{3}-\frac{2}{3}\sqrt{x^2+1},$$

$$I_4=\frac{x^3\sqrt{x^2+1}}{4}-\frac{3}{4}I_2=\frac{x^3\sqrt{x^2+1}}{4}-\frac{3}{4}\cdot\frac{1}{2}x\sqrt{x^2+1}+\frac{3}{4}\cdot\frac{1}{2}\sinh^{-1}x;$$

and for the last part of the integral, viz. $2\int\frac{dx}{(x+1)\sqrt{x^2+1}}$, put $x+1=\frac{1}{y}$;

$$\therefore\ \frac{dx}{x+1}=-\frac{dy}{y};$$

$$\therefore\ \int\frac{dx}{(x+1)\sqrt{x^2+1}}=-\int\frac{1}{\sqrt{\left(\frac{1}{y}-1\right)^2+1}}\frac{dy}{y}=-\int\frac{dy}{\sqrt{2y^2-2y+1}}$$

$$=-\frac{1}{\sqrt{2}}\int\frac{dy}{\sqrt{(y-\frac{1}{2})^2+\frac{1}{4}}}=-\frac{1}{\sqrt{2}}\sinh^{-1}(2y-1)$$

$$=-\frac{1}{\sqrt{2}}\sinh^{-1}\frac{1-x}{1+x}.$$

Thus
$$I=I_4-I_3+I_2+2\int\frac{dx}{(x+1)\sqrt{x^2+1}}$$

$$=\frac{x^3\sqrt{x^2+1}}{4}-\frac{x^2\sqrt{x^2+1}}{3}+\frac{1}{8}x\sqrt{x^2+1}+\frac{2}{3}\sqrt{x^2+1}$$

$$-\frac{1}{8}\sinh^{-1}x-\sqrt{2}\sinh^{-1}\frac{1-x}{1+x}.$$

290. Extension.

Further, we are now in a position to effect any integration of the form

$$\int\frac{\phi(x)}{\chi(x)}\frac{dx}{\sqrt{px^2+qx+r}},$$

where $\phi(x)$, $\chi(x)$ are rational integral algebraic functions of x, and all the factors of $\chi(x)$ are real and linear.

For putting $\frac{\phi(x)}{\chi(x)}$ into partial fractions, as described in Arts. 140 to 146,

$$\frac{\phi(x)}{\chi(x)}\equiv\Sigma Ax^m+\sum\frac{B}{x-b}+\sum\frac{C}{(x-c)^n}.$$

Hence the integration can be performed when we can integrate

$$\int\frac{x^m dx}{\sqrt{px^2+qx+r}},\quad \int\frac{dx}{(x-b)\sqrt{px^2+qx+r}},\quad \int\frac{dx}{(x-c)^n\sqrt{px^2+qx+r}}.$$

The first species of integral is reduced as already explained by the formula of Art. 240.

The second species was discussed in Art. 287.

The third species can be reduced as explained in Art. 244, or obtained from

$$\int\frac{dx}{(x-c)\sqrt{px^2+qx+r}}$$

by $n-1$ differentiations with regard to c, as will be explained later.

EXAMPLES.

Integrate the following expressions :

1. $\dfrac{1}{x\sqrt{x+1}},\quad \dfrac{1}{(x-1)\sqrt{x+2}},\quad \dfrac{x+1}{(x-1)\sqrt{x+2}},\quad \dfrac{x^2+x+1}{(x+2)\sqrt{x-1}}.$

2. $\dfrac{1}{(x^2+1)\sqrt{x}},\quad \dfrac{1}{(x^2+2x+2)\sqrt{x+1}},$

$\dfrac{x}{(x^2+2x+2)\sqrt{x+1}},\quad \dfrac{x^2+1}{(x^2+2x+2)\sqrt{x+1}}.$

3. $\dfrac{1}{x\sqrt{x^2+1}},\quad \dfrac{1}{(x+1)\sqrt{x^2+1}},\quad \dfrac{x}{(x+1)\sqrt{x^2+1}},\quad \dfrac{x^2+x+1}{(x+1)\sqrt{x^2+2x+3}}.$

4. Prove that

$$\int\frac{dx}{(x+c)\sqrt{x}}=\frac{2}{\sqrt{c}}\tan^{-1}\sqrt{\frac{x}{c}}\quad\text{or}\quad\frac{1}{\sqrt{-c}}\log_e\frac{\sqrt{x}-\sqrt{-c}}{\sqrt{x}+\sqrt{-c}},$$

according as c is positive or negative. [C. S., 1904.]

5. Integrate $\dfrac{\sqrt{\sin\theta}}{\cos\theta(\cos\theta+\sin\theta)\sqrt{2\cos\theta+3\sin\theta}}.$

6. Integrate $\dfrac{\sqrt{\sin\theta}+\sqrt{\cos\theta}}{(a^2\cos^2\theta-b^2\sin^2\theta)\sqrt{\cos\left(\theta-\frac{\pi}{4}\right)}}\quad (a>b>0).$

7. Integrate $\dfrac{\tan 2\theta}{\sqrt{\cos^6\theta+\sin^6\theta}}.$ 8. Integrate $\dfrac{(x^3+1)^2}{(x-1)\sqrt{x^2+1}}.$

9. Integrate

$$\frac{1}{\sqrt{(a-x)(x-b)}},\quad \frac{1}{(a-x)\sqrt{x-b}},\quad \frac{1}{(x-b)\sqrt{a-x}},\quad \frac{1}{(1\pm x)\sqrt{1\pm x^2}}.$$

[ST. JOHN'S, 1890.]

10. Integrate $\dfrac{1}{x\sqrt{(x-a)(x-b)}}$. [COLLEGES, 1876.]

11. Show that

$$\int \frac{x-3}{x\sqrt{x^2-3x+2}}\,dx=\cosh^{-1}(2x-3)+\frac{3}{\sqrt{2}}\cosh^{-1}\left(\frac{4-3x}{x}\right).$$

[ST. JOHN'S, 1883.]

12. Integrate $\displaystyle\int \frac{x-\alpha}{x-\beta}\,\frac{dx}{\sqrt{x^2+2px+q}}$ $(p^2>q)$. [α, 1887.]

13. Integrate

(i) $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+1}}$, [COLL., 1879.] (ii) $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}}$, [$\epsilon$, 1883.]

(iii) $\displaystyle\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$. [I. C. S., 1888.]

14. Integrate

(i) $\displaystyle\int \frac{dx}{(1-x)\sqrt{1-x^2}}$ (in two ways). [TRINITY, 1890.]

(ii) $\displaystyle\int \frac{dx}{(1-2x)\sqrt{1+4x}}$. [COLL., 1892.]

15. Integrate

$$\int \frac{dx}{(x-\lambda)\sqrt{x-\mu}},\quad \int \frac{x^3\,dx}{x^2+\lambda^2},\quad \int \frac{dx}{(x+1)(x+2)(x+3)}.$$

[MATH. TRIP., PT. I., 1920.]

291. CASE IV. **X quadratic, Y quadratic.**

The integral is now of the form

$$\int \frac{Mx+N}{(a_1x^2+2b_1x+c_1)\sqrt{a_2x^2+2b_2x+c_2}}\,dx,$$

where a linear factor has been inserted in the numerator, as in Case II., for the same reason. [See also Art. 1894, Vol. II.]

Before beginning the integration we make the following preliminary remarks:

292. (1). The numerator of the subject of integration is for the present linear. We shall consider later, as in previous cases, a numerator which is any rational integral algebraic function of x, viz. $\phi(x)$.

293. (2). The cases $b_1^2 \geqq a_1c_1$ and $b_2^2=a_2c_2$ are excluded. For (a) if $b_2^2=a_2c_2$, the expression $\sqrt{a_2x^2+2b_2x+c_2}$ becomes rational as regards x, and such integrations have been already considered.

(β) If $b_1^2 \gtreqless a_1c_1$, $\dfrac{Mx+N}{a_1x^2+2b_1x+c_1}$

would be resolvable into partial fractions either of the form

$$\frac{A}{fx+g}+\frac{B}{hx+k} \text{ (if } b_1^2 > a_1c_1)$$

or

$$\frac{A}{fx+g}+\frac{B}{(fx+g)^2} \text{ (if } b_1^2 = a_1c_1),$$

and the forms of integral resulting have already been considered in Articles 287 to 290.

294. (3). a_1 may be regarded as positive without loss of generality, for in any case in which this is not so, we may change all the signs of the factor $a_1x^2+2b_1x+c_1$, and finally change back the sign of the result when the integration has been effected.

Hence we assume: (1) a_1 positive, (2) $a_1c_1-b_1^2$ positive.

295. (4). We shall assume the subject of integration real. If $b_2^2 > a_2c_2$, the expression $a_2x^2+2b_2x+c_2$ has real factors, and may be written

$$\equiv a_2(x-\lambda_1)(x-\lambda_2), \text{ say, } \quad \text{where } \lambda_1 < \lambda_2.$$

In order that the radical should be real, we must therefore confine both the limits of integration to lie

$$\left.\begin{array}{l}\text{either between } -\infty \text{ and } \lambda_1 \\ \text{or between } \lambda_2 \text{ and } +\infty,\end{array}\right\} \text{ when } a_2 \text{ is positive,}$$

$$\text{or between } \lambda_1 \text{ and } \lambda_2, \quad \text{when } a_2 \text{ is negative.}$$

If $b_2^2 < a_2c_2$, the factors of $a_2x^2+2b_2x+c_2$ are unreal, and the condition a_2 *positive* is all that is necessary that the radical may be real for all values of x. The limits of integration in this case may therefore be any real quantities whatever.

296. Reduction to a Canonical Form.

(5). Lemma. Any three expressions of the forms

$$Mx+N, \quad a_1x^2+2b_1x+c_1, \quad a_2x^2+2b_2x+c_2$$

can be in general simultaneously thrown into the forms

$$P\xi_1+Q\xi_2, \quad p_1\xi_1^2+q_1\xi_2^2, \quad p_2\xi_1^2+q_2\xi_2^2,$$

where ξ_1, ξ_2 are linear expressions of forms $x-x_1$, $x-x_2$ respectively.

In order to do this it is necessary to determine the eight quantities (x_1, x_2), (P, Q), (p_1, q_1), (p_2, q_2); and we have eight linear equations to find them, viz.

$$\begin{array}{lll} p_1 + q_1 = a_1, & p_2 + q_2 = a_2, & P + Q = M, \\ p_1x_1 + q_1x_2 = -b_1, & p_2x_1 + q_2x_2 = -b_2, & Px_1 + Qx_2 = -N, \\ p_1x_1^2 + q_1x_2^2 = c_1, & p_2x_1^2 + q_2x_2^2 = c_2. & \end{array}$$

It follows that

$$\begin{vmatrix} 1, & 1, & a_1 \\ x_1, & x_2, & -b_1 \\ x_1^2, & x_2^2, & c_1 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 1, & 1, & a_2 \\ x_1, & x_2, & -b_2 \\ x_1^2, & x_2^2, & c_2 \end{vmatrix} = 0.$$

Also, as the consideration of the cases in which X or Y are perfect squares is to be rejected, we may assume x_1 not equal to x_2.

The determinants give at once on division by $x_2 - x_1$,

$$\left.\begin{array}{l} a_1x_1x_2 + b_1(x_1+x_2) + c_1 = 0, \\ a_2x_1x_2 + b_2(x_1+x_2) + c_2 = 0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

i.e.
$$\frac{x_1x_2}{b_1c_2 - b_2c_1} = \frac{x_1+x_2}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1};$$

whence x_1 and x_2 are determined, being the roots of

$$(a_1b_2 - a_2b_1)\rho^2 - (c_1a_2 - c_2a_1)\rho + (b_1c_2 - b_2c_1) = 0, \quad \ldots\ldots(2)$$

i.e.
$$C\rho^2 - B\rho + A = 0,$$

where A, B, C are the co-factors of a, b, c, in

$$\Delta \equiv \begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} \equiv aA + bB + cC.$$

That is, ρ is given by
$$\begin{vmatrix} 1, & -\rho, & \rho^2 \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix} = 0.$$

The remaining six quantities are found at once by solving the equations

$$\left.\begin{array}{l} p + q = a_1, \\ px_1 + qx_2 = -b_1, \end{array}\right\} \text{ or } \left.\begin{array}{r} a_2, \\ -b_2, \end{array}\right\} \text{ or } \left.\begin{array}{r} M, \\ -N, \end{array}\right\} \quad \ldots\ldots\ldots(3)$$

which give

(p_1, q_1), (p_2, q_2), (P, Q) respectively.

297. It may be remarked that the equations (1) may be reproduced immediately from the functions

$$a_1x^2+2b_1x+c_1, \quad a_2x^2+2b_2x+c_2$$

by the simple rule:

"For x^2 write x_1x_2; for $2x$ write (x_1+x_2) and leave the coefficients unaltered."

In the case when $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ equation (2) has one root infinite. Now therefore the general theorem of our Lemma fails, and the case must receive separate consideration.

298. (6). In this case, viz. $\frac{a_1}{a_2}=\frac{b_1}{b_2}$, the three expressions are:

$$Mx+N, \quad a_1\left(x+\frac{b_1}{a_1}\right)^2+c_1-\frac{b_1^2}{a_1}, \quad a_2\left(x+\frac{b_2}{a_2}\right)^2+c_2-\frac{b_2^2}{a_2};$$

and putting $$x+\frac{b_1}{a_1}=\xi=x+\frac{b_2}{a_2},$$

they are $$M\left(\xi-\frac{b_1}{a_1}\right)+N, \quad a_1\xi^2+c_1-\frac{b_1^2}{a_1}, \quad a_2\xi^2+c_2-\frac{b_2^2}{a_2},$$

and therefore are simultaneously reducible to the forms

$$P\xi+Q, \quad p_1\xi^2+q_1, \quad p_2\xi^2+q_2,$$

i.e. the same as if we put $\xi_2=1$ in the former transformation.

299. (7). When $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, the two quadratic functions are the same function, and the integral takes the form

$$I\equiv\int\frac{Mx+N}{(ax^2+2bx+c)^{\frac{3}{2}}}\,dx,$$

and a reduction formula may be used to connect with

$$\int\frac{M'x+N'}{(ax^2+2bx+c)^{\frac{1}{2}}}\,dx;$$

which has been considered before (Art. 85). Or the integral I may be deduced from the latter integral by differentiation with regard to c.

300. Illustrative Examples.

Ex. 1. In the case

$$\frac{3x-11}{(7x^2-46x+103)\sqrt{11x^2-70x+155}},$$

$$\left.\begin{aligned}7x_1x_2-23(x_1+x_2)+103=0,\\ 11x_1x_2-35(x_1+x_2)+155=0;\end{aligned}\right\}\quad \text{whence}\quad \begin{aligned}x_1+x_2=6,\\ x_1x_2=5,\end{aligned}$$

and therefore $x_1=1$, $x_2=5$ (the order is immaterial).

Also $\left.\begin{array}{l} p+q=7, \\ p+5q=23, \end{array}\right\}$ or $\left.\begin{array}{l} =11, \\ =35, \end{array}\right\}$ or $\left.\begin{array}{l} =3, \\ =11, \end{array}\right\}$

giving $\left.\begin{array}{l} p_1=3, \\ q_1=4, \end{array}\right\}$ or $\left.\begin{array}{l} p_2=5, \\ q_2=6, \end{array}\right\}$ or $\left.\begin{array}{l} P=1, \\ Q=2. \end{array}\right\}$

And the transformed result is therefore

$$\frac{\xi_1+2\xi_2}{(3\xi_1^2+4\xi_2^2)\sqrt{5\xi_1^2+6\xi_2^2}}.$$

Ex. 2. In the case

$$\frac{x-3}{(3x^2-30x+79)\sqrt{5x^2-50x+131}},$$

we have

$$\frac{a_1}{a_2}=\frac{b_1}{b_2},$$

$$3x^2-30x+79=3(x-5)^2+4,$$

$$5x^2-50x+131=5(x-5)^2+6.$$

Putting $x-5=\xi$, the transformed result is

$$\frac{\xi+2}{(3\xi^2+4)\sqrt{5\xi^2+6}}.$$

301. Taking the general case then, we suppose for the present $\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \neq 0,$

$$X \equiv a_1x^2+2b_1x+c_1 \equiv p_1\xi_1^2+q_1\xi_2^2,$$

$$Y \equiv a_2x^2+2b_2x+c_2 \equiv p_2\xi_1^2+q_2\xi_2^2,$$

where $\xi_1=x-x_1, \quad \xi_2=x-x_2,$

so that $\xi_2-\xi_1=x_1-x_2$ and $d\xi_1=d\xi_2=dx.$

Also, we are to use the transformation

$$y=\frac{Y}{X}, \quad i.e. \quad =\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1} \quad \text{or} \quad \frac{p_2\xi_1^2+q_2\xi_2^2}{p_1\xi_1^2+q_1\xi_2^2},$$

and

$$\frac{1}{2y}\frac{dy}{dx}=\frac{p_2\xi_1+q_2\xi_2}{p_2\xi_1^2+q_2\xi_2^2}-\frac{p_1\xi_1+q_1\xi_2}{p_1\xi_1^2+q_1\xi_2^2}$$

$$=\begin{vmatrix} p_1, & p_2 \\ q_1, & q_2 \end{vmatrix}(\xi_1-\xi_2)\frac{\xi_1\xi_2}{XY};$$

$$\therefore \frac{dy}{dx}=2\begin{vmatrix} p_1, & p_2 \\ q_1, & q_2 \end{vmatrix}(x_2-x_1)\frac{\xi_1\xi_2}{X^2}=-2K\frac{\xi_1\xi_2}{X^2}, \text{ say.}$$

Now $\left.\begin{array}{l} p_1+q_1=a_1, \\ p_1x_1+q_1x_2=-b_1, \end{array}\right\}$ and $\left.\begin{array}{l} p_2+q_2=a_2, \\ p_2x_1+q_2x_2=-b_2; \end{array}\right\}$

$$\therefore\ C \equiv a_1b_2-a_2b_1=(p_2+q_2)(p_1x_1+q_1x_2)-(p_1+q_1)(p_2x_1+q_2x_2)$$
$$=(p_1q_2-p_2q_1)(x_1-x_2)=+K,$$

i.e. $$K=+\begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \equiv C$$

when expressed in terms of the original coefficients.

The points on the graph of

$$y=\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1};$$

where the ordinate has a maximum or a minimum value, *i.e.* the "turning-points," are given by

$$\frac{dy}{dx}=0, \quad i.e. \text{ by } \xi_1\xi_2=0,$$

and are therefore at $\xi_1=0$ and $\xi_2=0$, *i.e.* at $x=x_1$ and $x=x_2$; and the values of the corresponding ordinates, viz. y_1 and y_2, are plainly

$$y_1=\frac{q_2}{q_1} \quad \text{and} \quad y_2=\frac{p_2}{p_1}.$$

We shall suppose the graph such that $x=x_1$ gives the minimum ordinate and $x=x_2$ the maximum, and that $x_2>x_1$.

Again, clearly $y=\dfrac{a_2}{a_1}$ is an asymptote, and the curve cuts the y-axis where $y=\dfrac{c_2}{c_1}$. It cuts the x-axis where

$$a_2x^2+2b_2x+c_2=0,$$

i.e. in real points P, Q if $b_2^2>a_2c_2$,
in unreal points if $b_2^2<a_2c_2$.

It cuts the asymptote where

$$\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}=\frac{a_2}{a_1},$$

i.e. where $$x=-\frac{1}{2}\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}=\frac{1}{2}\frac{B}{C},$$

i.e. at a point R at a finite distance from the y-axis, unless $a_1b_2-a_2b_1=0$, a case for the present excluded.

There are three cases with which we are concerned, *i.e.* in which some portion of the graph lies on the upper side of the x-axis, otherwise $\sqrt{Y}$ would be unreal.

(1)

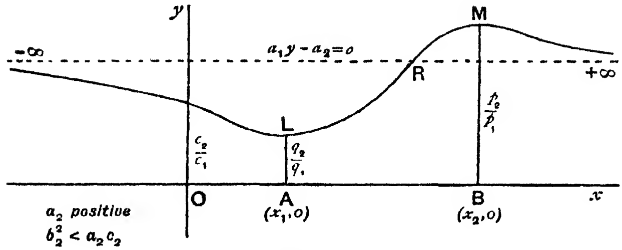


Fig. 17.

(2)

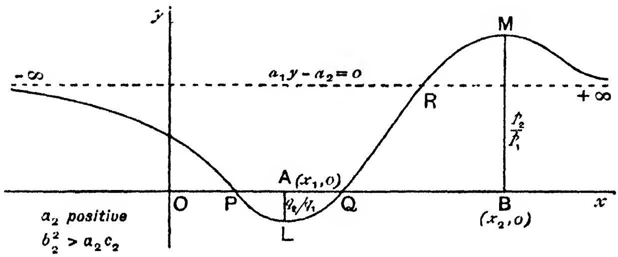


Fig. 18.

(3)

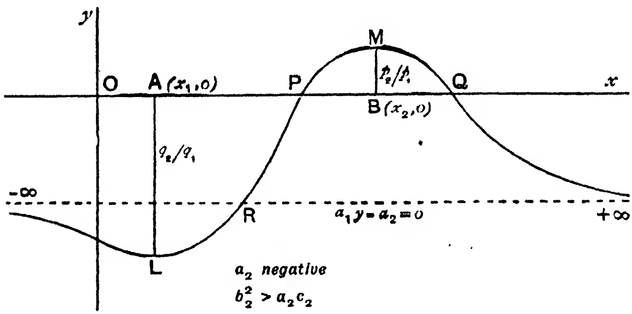


Fig. 19.

302. These are *typical* cases. It will be seen that we have taken $x_2 > x_1$ and the turning-points both on the right-hand side of the y-axis, *i.e.* x_1 and x_2 both positive. The student will have no difficulty in making the necessary modifications for any particular case in which the numerical values of the several constants are given. It is to be observed that p_1 and q_1 are necessarily both positive, for a_1 has been taken positive, and the roots of

$$a_1x^2+2b_1x+c_1=0, \quad \textit{i.e.}\ p_1\xi_1^2+q_1\xi_2^2=0,$$

are imaginary; also that p_2 and q_2 cannot both be negative, for $\sqrt{p_2\xi_1^2+q_2\xi_2^2}$ is to be real. Moreover, p_2 is the positive one, for p_2/p_1 is being regarded as the *maximum* ordinate.

As unreal values of $X\sqrt{Y}$ (*i.e.* $X^{\frac{3}{2}}\sqrt{y}$) are to be excluded, and X is to be regarded as positive, it will be clear that we shall only be concerned with those portions of these graphs in which y is positive, and the limits of integration of

$$\int\frac{Mx+N}{X\sqrt{Y}}\,dx$$

must be such as to lie within the boundaries of such regions as make this true.

In Fig. 17, y is positive from $x=-\infty$ to $x=+\infty$, and the limits may therefore be any real quantities whatever.

In Fig. 18, y is negative between $x=OP$ $(=\lambda_1)$ and $x=OQ$ $(=\lambda_2)$; therefore the limits may be anything between $-\infty$ and λ_1, or between λ_2 and $+\infty$, both limits to lie in the same region.

In Fig. 19, y is only positive between $x=\lambda_1$ and $x=\lambda_2$, and the limits must both lie between these values of x.

303. The Integration after Preparation.

We are now in a position to proceed with the integration of

$$I\equiv\int\frac{Mx+N}{X\sqrt{Y}}\,dx,$$

which we shall, to begin with, suppose to have been transformed as explained to the form

$$I\equiv\int\frac{P\xi_1+Q\xi_2}{(p_1\xi_1^2+q_1\xi_2^2)\sqrt{p_2\xi_1^2+q_2\xi_2^2}}\,dx.$$

Putting $\dfrac{Y}{X}=y$, we have $dx=-\dfrac{1}{2K}\dfrac{X^2}{\xi_1\xi_2}\,dy$;

also, $$y_2-y=\frac{p_2}{p_1}-\frac{p_2\xi_1^2+q_2\xi_2^2}{p_1\xi_1^2+q_1\xi_2^2}=-\begin{vmatrix}p_1, & q_1\\ p_2, & q_2\end{vmatrix}\frac{\xi_2^2}{p_1X}=\frac{K}{x_2-x_1}\frac{\xi_2^2}{p_1X},$$

$$y-y_1=\frac{p_2\xi_1^2+q_2\xi_2^2}{p_1\xi_1^2+q_1\xi_2^2}-\frac{q_2}{q_1}=-\begin{vmatrix}p_1, & q_1\\ p_2, & q_2\end{vmatrix}\frac{\xi_1^2}{q_1X}=\frac{K}{x_2-x_1}\frac{\xi_1^2}{q_1X};$$

$$\left.\begin{aligned}\therefore\quad \frac{\sqrt{X}}{\xi_2}&=\pm\frac{1}{\sqrt{p_1}}\sqrt{\frac{K}{x_2-x_1}}\frac{1}{\sqrt{y_2-y}},\\ \frac{\sqrt{X}}{\xi_1}&=\pm\frac{1}{\sqrt{q_1}}\sqrt{\frac{K}{x_2-x_1}}\frac{1}{\sqrt{y-y_1}},\end{aligned}\right\}$$ the signs of the ambiguities being governed by the signs of ξ_2 and ξ_1.

i.e. both $+^{ve}$ if $x_1<x_2<x$,
first $-^{ve}$, second $+^{ve}$ if $x_1<x<x_2$,
both $-^{ve}$ if $x<x_1<x_2$.

As the typical case we take $x_1<x_2<x$ and both signs positive, and note that $x_2-x_1=\dfrac{\sqrt{B^2-4AC}}{C}$ if expressed in terms of the original coefficients.

Substituting in the integral

$$I=-\frac{1}{2K}\int\frac{P\xi_1+Q\xi_2}{X^{\frac{3}{2}}\sqrt{y}}\cdot\frac{X^2}{\xi_1\xi_2}\,dy$$

$$=-\frac{1}{2K}\int\left(\frac{P}{\xi_2}+\frac{Q}{\xi_1}\right)\frac{X^{\frac{1}{2}}}{\sqrt{y}}\,dy$$

$$=-\frac{1}{2K}\sqrt{\frac{K}{x_2-x_1}}\left[\frac{P}{\sqrt{p_1}}\int\frac{dy}{\sqrt{y(y_2-y)}}+\frac{Q}{\sqrt{q_1}}\int\frac{dy}{\sqrt{y(y-y_1)}}\right]$$

$$=\frac{1}{\sqrt{K(x_2-x_1)}}\left[+\frac{P}{\sqrt{p_1}}\cos^{-1}\sqrt{\frac{y}{y_2}}-\frac{Q}{\sqrt{q_1}}\cosh^{-1}\sqrt{\frac{y}{y_1}}\right]$$

if y_1 be $+^{ve}$; or,

$$\frac{1}{\sqrt{K(x_2-x_1)}}\left[-\frac{P}{\sqrt{p_1}}\sin^{-1}\sqrt{\frac{y}{y_2}}-\frac{Q}{\sqrt{q_1}}\sinh^{-1}\sqrt{\frac{y}{-y_1}}\right],$$

if y_1 be negative.

And the suitable modification is to be made in these general results as to signs of radicals and reality of form in each numerical case which may present itself.

304. The Integration without a Preliminary Transformation.

If it be preferred to pass directly to the integration without the preliminary transformation, we proceed as follows:

$$I\equiv\int\frac{Mx+N}{(a_1x^2+2b_1x+c_1)\sqrt{a_2x^2+2b_2x+c_2}}\,dx.$$

Let
$$y=\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}.$$

Then
$$\frac{1}{2y}\frac{dy}{dx}=\frac{a_2x+b_2}{a_2x^2+2b_2x+c_2}-\frac{a_1x+b_1}{a_1x^2+2b_1x+c_1}$$

$$=\frac{(a_2x+b_2)(b_1x+c_1)-(a_1x+b_1)(b_2x+c_2)}{XY}$$

$$=-\frac{1}{4}\frac{J}{XY}=-\frac{1}{4}\frac{J}{X^2y},$$

where J is the Jacobian of the two quadratic expressions

$$a_1x^2+2b_1x+c_1, \quad a_2x^2+2b_2x+c_2,$$

viz.

$$J \equiv \begin{vmatrix} 2a_1x+2b_1, & 2b_1x+2c_1 \\ 2a_2x+2b_2, & 2b_2x+2c_2 \end{vmatrix}.$$

Hence
$$\frac{dy}{dx} = -\frac{J}{2X^2}.$$

Let x_1, x_2 be the roots of the equation $J=0$, and y_1, y_2 the corresponding values of y. Then the points (x_1, y_1), (x_2, y_2) are the "turning-points" of y, *i.e.* the points of maximum or minimum ordinates of the graph. Let y_1 be the minimum ordinate, y_2 the maximum.

The equation giving x_1, x_2, *i.e.* $J=0$, is obviously

$$(a_1b_2-a_2b_1)x^2-(c_1a_2-c_2a_1)x+(b_1c_2-b_2c_1)=0,$$

i.e.
$$Cx^2-Bx+A=0,$$

where A, B, C are the cofactors of a, b, c in the standard determinant

$$\Delta = \begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix},$$

and we may write $J \equiv +4C(x-x_1)(x-x_2)$.

Again, any straight line $y=\mu$ will cut the graph of

$$y=\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}$$

in two points which are coincident in the two cases $\mu=y_1$ and $\mu=y_2$.

Also $y-\mu \equiv \dfrac{Y}{X}-\mu=\dfrac{(a_2-a_1\mu)x^2+2(b_2-b_1\mu)x+c_2-c_1\mu}{X}.$

Hence, when $\mu=y_1$ or y_2 the numerator must contain $(x-x_1)^2$ or $(x-x_2)^2$ as a factor, and the equation

$$(a_2-a_1\mu)x^2+2(b_2-b_1\mu)x+c_2-c_1\mu=0$$

must have in these cases equal roots.

Hence, the necessary values of μ, viz. y_1 and y_2, are the roots of the quadratic

$$(b_2-b_1\mu)^2=(a_2-a_1\mu)(c_2-c_1\mu),$$

i.e.
$$(b_1^2-a_1c_1)\mu^2+(a_1c_2+a_2c_1-2b_1b_2)\mu+(b_2^2-a_2c_2)=0,$$

and
$$y-y_1=(a_2-a_1y_1)\frac{(x-x_1)^2}{X},$$
$$y_2-y=(a_1y_2-a_2)\frac{(x-x_2)^2}{X}$$

[a_2 supposed positive, $y_1<\frac{a_2}{a_1}$, $y_2>\frac{a_2}{a_1}$, y intermediate between y_1 and y_2, Figs. 17 and 18].

Thus
$$x-x_1=\pm\frac{X^{\frac{1}{2}}}{(a_2-a_1y_1)^{\frac{1}{2}}}\sqrt{y-y_1},$$
$$x-x_2=\pm\frac{X^{\frac{1}{2}}}{(a_1y_2-a_2)^{\frac{1}{2}}}\sqrt{y_2-y},$$

the signs of the right-hand sides

being both positive, if $x_1<x_2<x$;

the first positive, the second negative, if $x_1<x<x_2$;

both negative, if $x<x_1<x_2$.

Substituting in the original integral, and taking $x_1<x_2<x$ as the standard case, we have

$$\begin{aligned}
I &= \int\frac{Mx+N}{X\sqrt{Y}}\,dx=-\int\frac{Mx+N}{X\sqrt{Y}}\,\frac{2X^2}{J}\,dy \\
&=-2\int\frac{Mx+N}{\sqrt{y}}\,\frac{X^{\frac{1}{2}}}{J}\,dy \\
&=-\frac{2}{4C}\int\frac{Mx+N}{(x-x_1)(x-x_2)}\,\frac{X^{\frac{1}{2}}}{\sqrt{y}}\,dy \\
&=-\frac{1}{2C}\int\left[\frac{Mx_1+N}{x_1-x_2}\,\frac{1}{x-x_1}\,\frac{X^{\frac{1}{2}}}{\sqrt{y}}+\frac{Mx_2+N}{x_2-x_1}\,\frac{1}{x-x_2}\,\frac{X^{\frac{1}{2}}}{\sqrt{y}}\right]dy \\
&=-\frac{1}{2C}\,\frac{Mx_1+N}{x_1-x_2}\sqrt{a_2-a_1y_1}\int\frac{dy}{\sqrt{y(y-y_1)}} \\
&\qquad -\frac{1}{2C}\,\frac{Mx_2+N}{x_2-x_1}\sqrt{a_1y_2-a_2}\int\frac{dy}{\sqrt{y(y_2-y)}} \\
&=+F\cosh^{-1}\sqrt{\frac{y}{y_1}}+G\cos^{-1}\sqrt{\frac{y}{y_2}},\ \text{if } y_1 \text{ be positive,}
\end{aligned}$$

or $$=+F\sinh^{-1}\sqrt{\frac{y}{-y_1}}-G\sin^{-1}\sqrt{\frac{y}{y_2}},\ \text{if } y_1 \text{ be negative,}$$

where F and G are constants, viz.

$$F=\frac{Mx_1+N}{\sqrt{B^2-4AC}}\sqrt{a_2-a_1y_1},\quad G=\frac{Mx_2+N}{\sqrt{B^2-4AC}}\sqrt{a_1y_2-a_2},$$

for it has been seen above that $C(x_2-x_1)=\sqrt{B^2-4AC}$.

The suitable modification is to be made in these general results as to sign of radicals and reality of form in each numerical case which may present itself.

305. Comparison of the Processes. Construction of Examples.

Considerable arithmetical simplification accrues from the treatment shown in Art. 303, but of course at the cost of the initial reduction to the canonical form.

The method there shown indicates a method of construction of such examples, for the values of p_1, q_1, p_2, q_2, P, Q, x_1, x_2 are there all at choice, due care being taken that p_1, q_1 are both taken positive, and that p_2, q_2 are not both negative, as explained in Art. 302.

[See a paper by Russell, cited by Greenhill, *Chapter on the Integral Calculus.*]

306. Various Forms of the Coefficients.

The two coefficients may be thrown into various forms:

for since $y-\mu\equiv\dfrac{(a_2-a_1\mu)x^2+2(b_2-b_1\mu)x+c_2-c_1\mu}{X}$ (Art. 304)

is a fraction with $(x-x_1)^2$ as a factor in the numerator when $\mu=y_1$, or with $(x-x_2)^2$ in the numerator when $\mu=y_2$, we have by comparison of coefficients

$$(a_2-a_1y_1)x_1+(b_2-b_1y_1)=0$$

and

$$(a_2-a_1y_2)x_2+(b_2-b_1y_2)=0,$$

so

$$y_1=\frac{a_2x_1+b_2}{a_1x_1+b_1}\quad\text{and}\quad y_2=\frac{a_2x_2+b_2}{a_1x_2+b_1}$$

$$a_2-a_1y_1=-\frac{a_1b_2-a_2b_1}{a_1x_1+b_1}\quad\text{and}\quad a_1y_2-a_2=\frac{a_1b_2-a_2b_1}{a_1x_2+b_1},$$

and

$$a_1b_2-a_2b_1=K=C\quad\text{(Art. 301)}.$$

Also

$$\left.\begin{aligned}p_1+q_1&=a_1,\\ p_1x_1+q_1x_2&=-b_1\end{aligned}\right\}\quad\therefore\ p_1=\frac{a_1x_2+b_1}{x_2-x_1},\quad q_1=-\frac{a_1x_1+b_1}{x_2-x_1};$$

$$\left.\begin{aligned}p_2+q_2&=a_2,\\ p_2x_1+q_2x_2&=-b_2;\end{aligned}\right\}\quad\therefore\ p_2=\frac{a_2x_2+b_2}{x_2-x_1},\quad q_2=-\frac{a_2x_1+b_2}{x_2-x_1};$$

whence we have the following modifications of the coefficients in Art. 303, viz.:

$$\frac{P}{\sqrt{Kp_1(x_2-x_1)}}=\frac{P}{\sqrt{K(a_1x_2+b_1)}}=\frac{P}{\sqrt{p_1}\sqrt[4]{B^2-4AC}}=\frac{P}{K}\sqrt{a_1y_2-a_2}$$
$$=\frac{Mx_2+N}{(x_2-x_1)\sqrt{K(a_1x_2+b_1)}}=\frac{Mx_2+N}{\sqrt{a_1x_2+b_1}}\sqrt{\frac{C}{B^2-4AC}}$$
$$=\frac{Mx_2+N}{\sqrt{B^2-4AC}}\sqrt{a_1y_2-a_2},\text{ etc.}$$

And similarly for the coefficient involving Q.

307. **Convenient General Form of the Result.**

It appears then that if y_1 and y_2 be respectively the minimum and maximum ordinates of

$$y=\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}\left(\equiv\frac{Y}{X}\right),$$

and x_1, x_2 the corresponding abscissae, and if $Mx+N$ be written in the form $P(x-x_1)+Q(x-x_2)$, then the integral

$$I\equiv\int\frac{Mx+N}{X\sqrt{Y}}\,dx$$

can be written, amongst many other ways, in the convenient form

$$PP_1\cos^{-1}\sqrt{\frac{y}{y_2}}-QQ_1\cosh^{-1}\sqrt{\frac{y}{y_1}}$$

or

$$-PP_1\sin^{-1}\sqrt{\frac{y}{y_2}}-QQ_1\sinh^{-1}\sqrt{\frac{y}{-y_1}},$$

according as y_1 is $+^{\text{ve}}$ or $-^{\text{ve}}$,

where $$P_1=\frac{\sqrt{a_1y_2-a_2}}{a_1b_2-a_2b_1}\quad\text{and}\quad Q_1=\frac{\sqrt{a_2-a_1y_1}}{a_1b_2-a_2b_1},$$

provided $a_1b_2-a_2b_1\neq 0$.

308. **Remark.**

It is further to be noticed that the two quadratics involved in this discussion, viz.

$$(b_1^2-a_1c_1)y^2+(a_1c_2+a_2c_1-2b_1b_2)y+(b_2^2-a_2c_2)=0,$$
$$(a_1b_2-a_2b_1)x^2-(c_1a_2-c_2a_1)x+(b_1c_2-b_2c_1)=0,$$

are transformable the one into the other by the homographic transformation

$$y=\frac{a_2x+b_2}{a_1x+b_1}.$$

The one gives the ordinates (y_1, y_2), the other the abscissae (x_1, x_2) of the turning points. [See Salmon, *Higher Algebra*, p. 173.]

309. A Special Case.

It remains to discuss the case we have so far excluded,

$$\text{viz. when } \frac{a_1}{a_2}=\frac{b_1}{b_2}.$$

In this case the asymptote of the graph of

$$y=\frac{a_2x^2+2b_2x+c_2}{a_1x^2+2b_1x+c_1}\left(\equiv\frac{Y}{X}\right), \text{ viz. } y=\frac{a_2}{a_1},$$

does not meet the graph at a finite distance from the y-axis, and one of the two turning points has disappeared. It has been seen that the expression can, however, be written

$$y=\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}, \quad \text{where} \quad \begin{aligned} p_1&=a_1 && =+^{ve} \text{ by Art. 294},\\ p_2&=a_2, &&\\ \xi&=x-x_1 && x_1=-\frac{b_2}{a_2}=-\frac{b_1}{a_1},\\ q_1&=c_1-\frac{b_1^2}{a_1} && =+^{ve} \text{ by Art. 294},\\ q_2&=c_2-\frac{b_2^2}{a_2}. && \end{aligned}$$

$$\text{Now } \frac{1}{2y}\frac{dy}{dx}=\frac{p_2\xi}{p_2\xi^2+q_2}-\frac{p_1\xi}{p_1\xi^2+q_1}=-(p_1q_2-p_2q_1)\frac{\xi}{X^2y}$$

and

$$\frac{dy}{dx}=-2(p_1q_2-p_2q_1)\frac{\xi}{X^2}.$$

Also $\xi=0$ gives the turning point, viz. $x=x_1$, $y=y_1$; and y_1 obviously is $=\frac{q_2}{q_1}$. The only forms of the graph with which we are concerned are the four following. Cases, in which the graph lies entirely below the x-axis, give rise to entirely unreal values of $\sqrt{Y}$. Note the symmetry in all cases of the graph about an ordinate through the turning point.

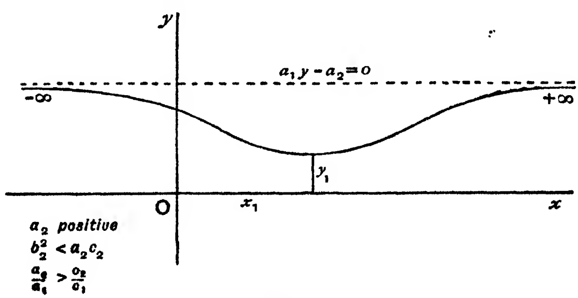


Fig. 20.

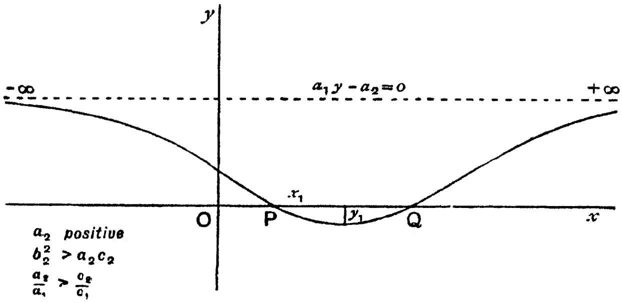


Fig. 21.

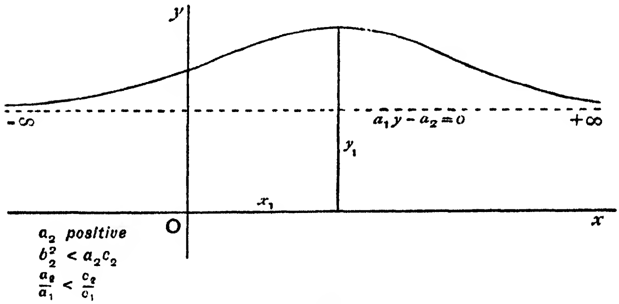


Fig. 22.

with corresponding forms if a_2 be negative, viz.

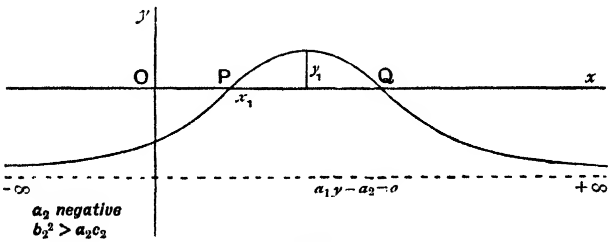


Fig. 23.

In the case a_2 negative, $b_2^2<a_2c_2$, the graph is entirely below the x-axis.

310. When the graph cuts the x-axis, as in Figs. 21 and 23, at points P, Q, $\sqrt{Y}$ is unreal for the value of x intermediate between P and Q, *i.e.* intermediate between the roots of

$$a_2x^2+2b_2x+c_2=0,$$

if a_2 be positive, but real for such values of x, if a_2 be negative. Hence, in the first and third cases (Figs. 20 and 22) the limits of integration may be any whatever; in the second case (Fig. 21) both limits must be in the region from $-\infty$ to the smaller root of the quadratic, or in the region from the larger root to $+\infty$; or in the fourth case (Fig. 23), a_2 being negative, $\sqrt{Y}$ is unreal for all values of x which are not intermediate between these roots. Thus in the fourth case the integration is only to be considered when both limits lie intermediate between the roots of

$$a_2x^2+2b_2x+c_2=0.$$

And in the fifth case, viz. a_2 $-^{\text{ve}}$, $b_2{}^2<a_2c_2$, $\sqrt{Y}$ is unreal for the whole range $x=-\infty$ to $x=+\infty$.

We have also

$$y-y_1=\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}-\frac{q_2}{q_1}=-(p_1q_2-p_2q_1)\frac{\xi^2}{q_1X}.$$

$$I\equiv\int\frac{P\xi+Q}{(p_1\xi^2+q_1)\sqrt{p_2\xi^2+q_2}}d\xi$$ splits up into two integrals, viz.

$$=\frac{P}{2}\int\frac{d\xi^2}{(p_1\xi^2+q_1)\sqrt{p_2\xi^2+q_2}}+Q\int\frac{d\xi}{(p_1\xi^2+q_1)\sqrt{p_2\xi^2+q_2}}.$$

The first falls under the class discussed in Art. 277, and

$$=\frac{P}{2}\frac{2}{\sqrt{p_1(p_2q_1-p_1q_2)}}\sin^{-1}\sqrt{\frac{p_1}{p_2}}\sqrt{\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}}$$

or
$$\frac{P}{2}\frac{2}{\sqrt{p_1(p_1q_2-p_2q_1)}}\cosh^{-1}\sqrt{\frac{p_1}{p_2}}\sqrt{\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}}$$

or
$$\frac{P}{2}\frac{2}{\sqrt{p_1(p_1q_2-p_2q_1)}}\sinh^{-1}\sqrt{\frac{p_1}{-p_2}}\sqrt{\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}},$$

the real form to be chosen.

For the second integral,

$$Q\int\frac{dx}{X\sqrt{Y}}=\frac{Q}{2(p_1q_2-p_2q_1)}\int\frac{1}{X^{\frac{3}{2}}\sqrt{y}}\frac{X^2}{\xi}dy$$

$$=-\frac{Q}{2(p_1q_2-p_2q_1)}\int\frac{1}{\sqrt{y}}\frac{\sqrt{X}}{\xi}dy$$

$$\left.\begin{aligned}&=\frac{Q}{2}\frac{1}{\sqrt{q_1(p_2q_1-p_1q_2)}}\int\frac{dy}{\sqrt{y(y-y_1)}}\\ \text{or}\quad&=-\frac{Q}{2}\frac{1}{\sqrt{q_1(p_1q_2-p_2q_1)}}\int\frac{dy}{\sqrt{y(y_1-y)}},\end{aligned}\right\}\text{according as } y> \text{ or } <y_1,$$

$$\left.\begin{aligned} i.e. \quad &= Q\frac{1}{\sqrt{q_1(p_2q_1-p_1q_2)}}\cosh^{-1}\sqrt{\frac{y}{y_1}} \\ \text{or} \quad &= \frac{Q}{\sqrt{q_1(p_2q_1-p_1q_2)}}\sinh^{-1}\sqrt{\frac{y}{-y_1}} \\ \text{or} \quad &= Q\frac{1}{\sqrt{q_1(p_1q_2-p_2q_1)}}\cos^{-1}\sqrt{\frac{y}{y_1}}, \end{aligned}\right\} \text{the real form to be chosen.}$$

Hence we have

(1) $(p_1q_2-p_2q_1)$ $+^{\text{ve}}$, y_1 and $\therefore$ q_2 $+^{\text{ve}}$, p_2 $+^{\text{ve}}$,

$$I=\frac{1}{\sqrt{p_1q_2-p_2q_1}}\left[-\frac{P}{\sqrt{p_1}}\cosh^{-1}\sqrt{\frac{p_1}{p_2}y}+\frac{Q}{\sqrt{q_1}}\cos^{-1}\sqrt{\frac{q_1}{q_2}y}\right];$$

(2) $(p_1q_2-p_2q_1)$ $-^{\text{ve}}$, y_1 and $\therefore$ q_2 $+^{\text{ve}}$, p_2 $+^{\text{ve}}$,

$$I=\frac{1}{\sqrt{p_2q_1-p_1q_2}}\left[-\frac{P}{\sqrt{p_1}}\cos^{-1}\sqrt{\frac{p_1}{p_2}y}+\frac{Q}{\sqrt{q_1}}\cosh^{-1}\sqrt{\frac{q_1}{q_2}y}\right];$$

(3) $(p_1q_2-p_2q_1)$ $+^{\text{ve}}$, y_1 and $\therefore$ q_2 $+^{\text{ve}}$, p_2 $-^{\text{ve}}$,

$$I=\frac{1}{\sqrt{p_1q_2-p_2q_1}}\left[-\frac{P}{\sqrt{p_1}}\sinh^{-1}\sqrt{\frac{p}{-p_2}y}-\frac{Q}{\sqrt{q_1}}\sin^{-1}\sqrt{\frac{q_1}{q_2}y}\right];$$

(4) $(p_1q_2-p_2q_1)$ $-^{\text{ve}}$, y_1 and $\therefore$ q_2 $-^{\text{ve}}$, p_2 $+^{\text{ve}}$,

$$I=\frac{1}{\sqrt{p_2q_1-p_1q_2}}\left[\frac{P}{\sqrt{p_1}}\sin^{-1}\sqrt{\frac{p_1}{p_2}y}+\frac{Q}{\sqrt{q_1}}\sinh^{-1}\sqrt{\frac{q_1}{-q_2}y}\right];$$

results of similar forms to those obtained when $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$, and again the coefficients may be expressed in various forms.

311. We note that the first of the two integrals, which has been referred for its integration to Art. 277, for which the substitution would be $p_2\xi^2+q_2=y^2$, might equally well be obtained by the substitution

$$\frac{p_2\xi^2+q_2}{p_1\xi^2+q_1}=y,$$

i.e. the same as used in the second integral. Upon this substitution being made in the integral I, we get a result of form

$$A\int\frac{\sqrt{X}}{\sqrt{y}}\,dy+B\int\frac{\sqrt{X}}{\xi}\frac{dy}{\sqrt{y}}$$

In the first of these we substitute for $\sqrt{X}$ its value in terms of y, viz.
$$\sqrt{\frac{p_1q_2-p_2q_1}{p_1y-p_2}}.$$

In the second we form $\quad y_1-y=+\frac{(p_1q_2-p_2q_1)}{q_1}\frac{\xi^2}{X}$,

and substitute for $\frac{\sqrt{X}}{\xi}$ its value, viz. $\sqrt{\frac{p_1q_2-p_2q_1}{q_1}}\frac{1}{\sqrt{y_1-y}}$,
as shown.

312. Illustrative Examples.

Ex. 1. Consider the integral
$$I=\int\frac{3x-1}{(3x^2-2x+1)\sqrt{2x^2-2x+1}}dx,$$

(*a*) without reduction to the canonical form, (*b*) first reducing it as in Art. 298.

(*a*) Putting
$$y=\frac{2x^2-2x+1}{3x^2-2x+1}\left(=\frac{Y}{X}\right),$$
$$\frac{1}{2y}\frac{dy}{dx}=\frac{2x-1}{2x^2-2x+1}-\frac{3x-1}{3x^2-2x+1}$$
$$=\frac{x(x-1)}{XY}=\frac{x(x-1)}{X^2y}.$$

The turning points are given by $x=0$ and $x=1$. If $x=0$, $y=1$; if $x=1$, $y=\frac{1}{2}$.
$$1-y=1-\frac{2x^2-2x+1}{3x^2-2x+1}=\frac{x^2}{X},$$
$$y-\frac{1}{2}=\frac{2x^2-2x+1}{3x^2-2x+1}-\frac{1}{2}=\frac{(x-1)^2}{2X};$$
$$\therefore\ \frac{\sqrt{X}}{x}=\frac{1}{\sqrt{1-y}},\quad \frac{\sqrt{X}}{x-1}=\frac{1}{\sqrt{2}\sqrt{y-\frac{1}{2}}},\quad \text{assuming } x>1;$$
$$\therefore\ I=\int\frac{3x-1}{X\sqrt{Xy}}\cdot\frac{X^2dy}{2x(x-1)}$$
$$=\frac{1}{2}\int\left(\frac{1}{x}+\frac{2}{x-1}\right)\frac{\sqrt{X}\,dy}{\sqrt{y}}$$
$$=\frac{1}{2}\int\left(\frac{1}{\sqrt{y(1-y)}}+\frac{\sqrt{2}}{\sqrt{y(y-\frac{1}{2})}}\right)dy$$
$$=-\cos^{-1}\sqrt{y}+\sqrt{2}\cosh^{-1}\sqrt{2y}$$
$$=-\cos^{-1}\sqrt{\frac{2x^2-2x+1}{3x^2-2x+1}}+\sqrt{2}\cosh^{-1}\sqrt{2\frac{2x^2-2x+1}{3x^2-2x+1}}.$$

The graph of the transformation formula in this case is shown in Fig. 24.

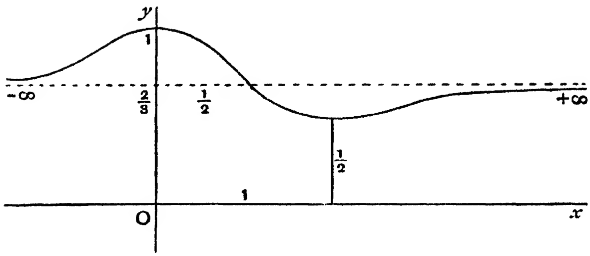


Fig. 24.

And the signs selected refer to values of $x > 1$, and we have

$$I_1 = \Big[-\cos^{-1}\sqrt{y} + \sqrt{2}\cosh^{-1}\sqrt{2y} \Big]_{\lambda_1}^{\lambda_2}$$

if λ_1, λ_2 be the limits and both > 1.

If x lie between 0 and 1, we have

$$\frac{\sqrt{X}}{x} = \frac{1}{\sqrt{1-y}} \quad \text{and} \quad \frac{\sqrt{X}}{x-1} = -\frac{1}{\sqrt{2}\sqrt{y-\frac{1}{2}}},$$

and we shall have

$$I_2 = \Big[-\cos^{-1}\sqrt{y} - \sqrt{2}\cosh^{-1}\sqrt{2y} \Big]_{\lambda_1}^{\lambda_2}$$

if λ_1, λ_2 both lie between 0 and 1.

If x lie between $-\infty$ and 0,

$$\frac{\sqrt{X}}{x} = -\frac{1}{\sqrt{1-y}} \quad \text{and} \quad \frac{\sqrt{X}}{x-1} = -\frac{1}{\sqrt{2}\sqrt{y-\frac{1}{2}}},$$

and we shall have

$$I_3 = \Big[+\cos^{-1}\sqrt{y} - \sqrt{2}\cosh^{-1}\sqrt{2y} \Big]_{\lambda_1}^{\lambda_2}$$

if λ_1, λ_2 be both negative.

If one limit, λ_1, falls on one side of a turning point, say $x=1$, and $0 < \lambda_1 < 1$, and the other, λ_2, on the opposite side, *i.e.* $\lambda_2 > 1$, the integration should be conducted from the lower limit to the turning point with the corresponding result, say I_2, and from the turning point to the upper limit with the result I_1.

(*b*) Next let us transform to the canonical form

$$\int \frac{P\xi_1 + Q\xi_2}{(p_1\xi_1^2 + q_1\xi_2^2)\sqrt{p_2\xi_1^2 + q_2\xi_2^2)}}\,dx$$

before integration.

Here, by the rule of Art. 297,

$$\left.\begin{array}{l} 3x_1x_2 - (x_1+x_2) + 1 = 0, \\ 2x_1x_2 - (x_1+x_2) + 1 = 0; \end{array}\right\} ; \quad \therefore \quad \left.\begin{array}{ll} x_1x_2 = 0, & x_1 + x_2 = 1, \\ x_1 = 0, & x_2 = 1; \end{array}\right\}$$

and
$$p_1+q_1=3, \quad p_2+q_2=2, \quad P+Q=3,$$
$$q_1=1, \quad q_2=1, \quad Q=1;$$
$$\therefore \left.\begin{matrix} p_1=2, \\ q_1=1, \end{matrix}\right\} \quad \left.\begin{matrix} p_2=1, \\ q_2=1, \end{matrix}\right\} \quad \left.\begin{matrix} P=2, \\ Q=1; \end{matrix}\right\}$$
$$\therefore \; I=\int \frac{2\xi_1+\xi_2}{(2\xi_1^2+\xi_2^2)\sqrt{\xi_1^2+\xi_2^2}}\,dx.$$

Let $y=\dfrac{\xi_1^2+\xi_2^2}{2\xi_1^2+\xi_2^2}$, $\dfrac{dy}{2y\,dx}=\dfrac{\xi_1+\xi_2}{\xi_1^2+\xi_2^2}-\dfrac{2\xi_1+\xi_2}{2\xi_1^2+\xi_2^2}=\dfrac{\xi_1\xi_2}{X^2y}$; for $\xi_1-\xi_2=1$;
$$\therefore \; dx=\frac{X^2}{2\xi_1\xi_2}\,dy.$$
$$\xi_1=0 \quad \text{gives} \quad y=1,$$
$$\xi_2=0 \quad \text{gives} \quad y=\tfrac{1}{2}.$$
$$1-y=\frac{\xi_1^2}{X},$$
$$y-\tfrac{1}{2}=\frac{\xi_2^2}{2X};$$
$$\therefore \; I=\frac{1}{2}\int \frac{2\xi_1+\xi_2}{\xi_1\xi_2}\frac{\sqrt{X}}{\sqrt{y}}\,dy$$
$$=\frac{1}{2}\int \left(\frac{2}{\xi_2}+\frac{1}{\xi_1}\right)\frac{\sqrt{X}}{\sqrt{y}}\,dy$$
$$=\frac{1}{2}\int \frac{dy}{\sqrt{y(1-y)}}+\frac{1}{\sqrt{2}}\int \frac{dy}{\sqrt{y(y-\frac{1}{2})}}$$
$$=-\cos^{-1}\sqrt{y}+\sqrt{2}\cosh^{-1}\sqrt{2y},$$
as before.

Ex. 2. As a case where $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$, consider the integration of
$$I=\int \frac{5x-9}{(x^2-6x+10)\sqrt{6x-x^2}}\,dx.$$

Writing $x-3=\xi$, $\qquad dx=d\xi$,
$$I=\int \frac{5\xi+6}{(\xi^2+1)\sqrt{9-\xi^2}}\,d\xi=\int \frac{5\xi+6}{X\sqrt{Y}}\,d\xi, \text{ say.}$$

Let
$$y=\frac{9-\xi^2}{\xi^2+1};$$
$$\therefore \; \frac{dy}{2y\,d\xi}=\frac{-\xi}{9-\xi^2}-\frac{\xi}{\xi^2+1}=\frac{-10\xi}{X^2y} \quad \text{and} \quad d\xi=-\frac{1}{20}\frac{X^2}{\xi}\,dy;$$
$$\therefore \; I=-\frac{1}{20}\int \frac{5\xi+6}{\sqrt{y}}\frac{\sqrt{X}}{\xi}\,dy$$
$$=-\frac{1}{4}\int \frac{\sqrt{X}}{\sqrt{y}}\,dy-\frac{3}{10}\int \frac{\sqrt{X}}{\xi}\frac{dy}{\sqrt{y}}.$$

Now $(\xi^2+1)y=9-\xi^2$; $\therefore\ \xi^2=\dfrac{9-y}{1+y}$ and $X=\xi^2+1=\dfrac{10}{1+y}$.

Therefore $$\int\frac{\sqrt{X}}{\sqrt{y}}dy=\sqrt{10}\int\frac{dy}{\sqrt{y(y+1)}}=2\sqrt{10}\sinh^{-1}\sqrt{y}$$
$$=2\sqrt{10}\sinh^{-1}\sqrt{\frac{9-\xi^2}{1+\xi^2}}.$$

Also, at the maximum ordinate, $\xi=0$ and $y=9$.

And $$9-y=9-\frac{9-\xi^2}{1+\xi^2}=\frac{10\xi^2}{X};$$
$$\therefore\ \frac{\sqrt{X}}{\xi}=\frac{\sqrt{10}}{\sqrt{9-y}},$$
taking $x>3$, *i.e.* ξ as $+^{ve}$.

Therefore, in the second integral,
$$\int\frac{\sqrt{X}}{\xi}\frac{dy}{\sqrt{y}}=\sqrt{10}\int\frac{dy}{\sqrt{y(9-y)}}=2\sqrt{10}\sin^{-1}\frac{\sqrt{y}}{3};$$
$$\therefore\ I=-\frac{1}{2}\sqrt{10}\sinh^{-1}\sqrt{\frac{9-\xi^2}{1+\xi^2}}-\frac{6}{\sqrt{10}}\sin^{-1}\frac{1}{3}\sqrt{\frac{9-\xi^2}{1+\xi^2}},$$
i.e. $$=-\frac{\sqrt{10}}{2}\left(\sinh^{-1}\sqrt{\frac{6x-x^2}{x^2-6x+10}}+\frac{6}{5}\sin^{-1}\frac{1}{3}\sqrt{\frac{6x-x^2}{x^2-6x+10}}\right).$$

The graph of the substitution formula,
$$y=\frac{6x-x^2}{x^2-6x+10},$$
is shown in Fig. 25,

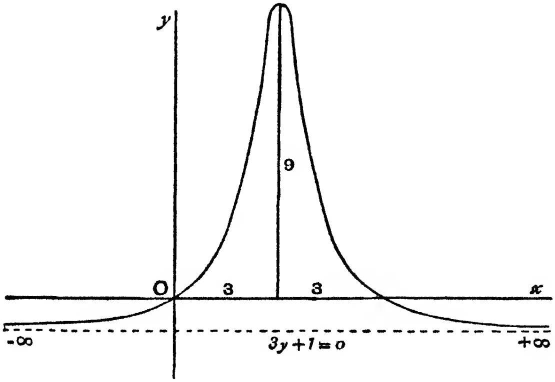


Fig. 25.

y attaining its maximum value 9 when $x=3$, and being negative for all values of x except such as lie between 0 and 6, and as we confine the

integration to real values of $\sqrt{Y}$ the limits of integration are to be such that both lie within the region from 0 to 6. Also the sign of $\frac{\sqrt{X}}{\xi}$ changes as x increases through the value 3. Hence the signs adopted above when we take $\frac{\sqrt{X}}{\xi}=+\frac{\sqrt{10}}{\sqrt{9-y}}$ apply to values of x between 3 and 6. For values between 0 and 3 we must use $\frac{\sqrt{X}}{\xi}=-\frac{\sqrt{10}}{\sqrt{9-y}}$ and make the corresponding change in the sign of the part of the result dependent thereon.

313. **Forms reducible to Case IV.**

As in previous cases, Arts. 281, 286, 288, attention is called to the varieties of form of integrals deducible from the case just considered, viz. X quadratic, Y quadratic.

(1) Thus $\int\frac{L\sin\theta+M}{a\sin^2\theta+2b\sin\theta+c}\,d\theta$ reduces to

$$\int\frac{Lx+M}{ax^2+2bx+c}\,\frac{dx}{\sqrt{1-x^2}},\quad \text{if } \sin\theta=x.$$

(2) $\int\frac{L\sin\theta+M\cos\theta+N}{a\sin^2\theta+2b\sin\theta+c}\,d\theta$ reduces to

$$\int\frac{Lx+N}{ax^2+2bx+c}\,\frac{dx}{\sqrt{1-x^2}}+\int\frac{M\,dx}{ax^2+2bx+c}.$$

(3) $\int\frac{L\sin\theta+M\cos\theta+N}{a\cos^2\theta+2b\cos\theta+c}\,d\theta,$ similarly.

(4) $\int\frac{L\sin\theta+M\cos\theta}{a\sin^2\theta+2b\sin\theta\cos\theta+c\cos^2\theta}\,d\theta,$ by putting $\tan\theta=x$.

(5) $\int\frac{L\sinh u+M}{a\sinh^2 u+2b\sinh u+c}\,du,$ similarly to (1).

(6) $\int\frac{L\sinh u+M\cosh u+N}{a\sinh^2 u+2b\sinh u+c}\,du,$ similarly to (2).

(7) $\int\frac{L\sinh u+M\cosh u}{a\sinh^2 u+2b\sinh u\cosh u+c\cosh^2 u}\,du,$ by $\tanh u=x$.

(8) If in $$I\equiv\int\frac{Mx+N}{(a_1x^2+2b_1x+c_1)}\,\frac{dx}{\sqrt{a_2x^2+2b_2x+c_2}}$$

we put $x=z+\frac{1}{z},\quad dx=\left(1-\frac{1}{z^2}\right)dz,$

$$I\equiv\int\frac{\left[M\left(z+\frac{1}{z}\right)+N\right]\left(1-\frac{1}{z^2}\right)dz}{\left[a_1\left(z^2+\frac{1}{z^2}\right)+2b_1\left(z+\frac{1}{z}\right)+d_1\right]\sqrt{a_2\left(z^2+\frac{1}{z^2}\right)+2b_2\left(z+\frac{1}{z}\right)+d_2}},$$

where d_1, d_2 are written for c_1+2a_1, c_2+2a_2 respectively ; so that

$$I=\int\frac{(Mz^2+Nz+M)(z^2-1)\,dz}{(a_1z^4+2b_1z^3+d_1z^2+2b_1z+a_1)\sqrt{a_2z^4+2b_2z^3+d_2z^2+2b_2z+a_2}}.$$

Hence, if F be a "reciprocal" quadratic function of z, and X, Y reciprocal quartic expressions in z, we can integrate

$$I=\int\frac{F}{X\sqrt{Y}}(z^2-1)\,dz \quad \text{by the substitution } z+\frac{1}{z}=x.$$

(9) Similarly

$$\int\frac{(Mz^2+Nz-M)(z^2+1)\,dz}{(a_1z^4+2b_1z^3+d_1z^2-2b_1z+a_1)\sqrt{a_2z^4+2b_2z^3+d_2z^2-2b_2z+a_2}}$$

integrates by the substitution $z-\frac{1}{z}=x.$

314. The Case of $Y\equiv$ a Reciprocal Quartic.

Let Y be any reciprocal binary quartic expression

$$\equiv ax^4+4bx^3+6cx^2+4bx+a.$$

Then $I=\int\frac{x^2-1}{x}\frac{dx}{\sqrt{Y}}$ reduces at once to the form

$$\int\frac{dz}{\sqrt{\text{Quadratic}}},$$

by the substitution $x+\frac{1}{x}=z$, whence $\left(1-\frac{1}{x^2}\right)dx=dz.$

For
$$\begin{aligned}Y&=x^2\left[a\left(x^2+\frac{1}{x^2}\right)+4b\left(x+\frac{1}{x}\right)+6c\right]\\&=x^2[az^2+4bz+6c-2a]\\&=ax^2\left[\left(z+\frac{2b}{a}\right)^2-\frac{4b^2-6ac+2a^2}{a^2}\right]\\&=ax^2\left[\left(z+\frac{2b}{a}\right)^2-\frac{K}{a^2}\right],\quad\text{where } K=2(2b^2-3ac+a^2);\end{aligned}$$

$$\therefore\ I=\frac{1}{\sqrt{a}}\int\frac{dz}{\sqrt{\left(z+\frac{2b}{a}\right)^2-\frac{K}{a^2}}}\quad\text{or}\quad\frac{1}{\sqrt{-a}}\int\frac{dz}{\sqrt{\frac{K}{a^2}-\left(z+\frac{2b}{a}\right)^2}},$$

which, by Arts. 80, 81,

$$\left.\begin{aligned}&=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{\sqrt{aY}}{x\sqrt{K}},\quad\text{if } K \text{ be } +^{\text{ve}},\\ \text{or}\quad&\frac{1}{\sqrt{a}}\cosh^{-1}\frac{\sqrt{aY}}{x\sqrt{-K}},\quad\text{if } K \text{ be } -^{\text{ve}},\end{aligned}\right\}\text{ and } a+^{\text{ve}},$$

$$\text{or}\quad\frac{1}{\sqrt{-a}}\cos^{-1}\frac{\sqrt{-aY}}{x\sqrt{K}},\quad\text{if } a \text{ be } -^{\text{ve}}.$$

Note that if K be positive, the factors of Y as expressed in terms of z are real;
if K be negative, unreal;
and that $aY+Kx^2$ is a perfect square.

315. **A Similar Case.**

In the same way, if

$$Y_1 \equiv ax^4+4bx^3+6cx^2-4bx+a,$$

the integration of

$I_1=\int\frac{x^2+1}{x}\frac{dx}{\sqrt{Y_1}}$ can be effected by the substitution $x-\frac{1}{x}=z$.

$$\text{For}\quad Y_1=x^2\left[a\left(x^2+\frac{1}{x^2}\right)+4b\left(x-\frac{1}{x}\right)+6c\right]$$

$$=x^2[az^2+4bz+6c+2a]$$

$$=ax^2\left[\left(z+\frac{2b}{a}\right)^2-\frac{4b^2-6ac-2a^2}{a^2}\right]$$

$$=ax^2\left[\left(z+\frac{2b}{a}\right)^2-\frac{K_1}{a^2}\right],\text{ where } K_1=2(2b^2-3ac-a^2);$$

$$\therefore\ I_1=\frac{1}{\sqrt{a}}\int\frac{dz}{\sqrt{\left(z+\frac{2b}{a}\right)^2-\frac{K_1}{a^2}}}\quad\text{or}\quad\frac{1}{\sqrt{-a}}\int\frac{dz}{\sqrt{\frac{K_1}{a^2}-\left(z+\frac{2b}{a}\right)^2}}$$

$$\left.\begin{array}{ll}=\frac{1}{\sqrt{a}}\sinh^{-1}\frac{\sqrt{aY_1}}{x\sqrt{K_1}}, & \text{if } K_1 \text{ be } +^{\text{ve}},\\ \text{or}\quad\frac{1}{\sqrt{a}}\cosh^{-1}\frac{\sqrt{aY_1}}{x\sqrt{-K_1}}, & \text{if } K_1 \text{ be } -^{\text{ve}},\end{array}\right\}\text{ and } a\ +^{\text{ve}},$$

$$\text{or}\quad\frac{1}{\sqrt{-a}}\cos^{-1}\frac{\sqrt{-aY_1}}{x\sqrt{K_1}},\text{ if } a \text{ be } -^{\text{ve}};$$

also, Y_1 expressed in terms of z has real or unreal factors, as K_1 is $+^{\text{ve}}$ or $-^{\text{ve}}$, and $aY_1+K_1x^2$ is a perfect square.

In the integrations of these two articles, since the final form exhibited is arrived at by the conversion of a function of z into a function of Y, or of Y_1, in which process a square root is extracted

$$\left(e.g.\ \sin^{-1}\frac{az+2b}{\sqrt{K}}=\cos^{-1}\sqrt{1-\frac{(az+2b)^2}{K}}=\text{etc.}\right),$$

it is desirable to check by direct differentiation the sign of all numerical results obtained.

316. Other Forms.

The substitutions

$$x+\frac{1}{x}=\frac{1}{z}, \quad x-\frac{1}{x}=\frac{1}{z}$$

respectively reduce

$$\int\frac{x^2-1}{x^2+1}\frac{dx}{\sqrt{Y}} \quad \text{and} \quad \int\frac{x^2+1}{x^2-1}\frac{dx}{\sqrt{Y_1}},$$

Y and Y_1 denoting the same quartic functions as before. [See Greenhill's *Chapter on the Integral Calculus*, p. 41.]

For taking $x+\frac{1}{x}=\frac{1}{z}$ we have, differentiating logarithmically,

$$\frac{1-\frac{1}{x^2}}{x+\frac{1}{x}}dx=-\frac{dz}{z}, \quad i.e. \ \frac{x^2-1}{x^2+1}dx=-\frac{x\,dz}{z},$$

and $$\sqrt{Y}=x\,[az^{-2}+4bz^{-1}+6c-2a]^{\frac{1}{2}};$$

$$\therefore \quad \int\frac{x^2-1}{x^2+1}\frac{dx}{\sqrt{Y}}=-\int\frac{dz}{\sqrt{a+4bz+(6c-2a)z^2}},$$

whose integral can be written down by Art. 80.

And similarly, if $x-\frac{1}{x}=\frac{1}{z}$,

$$\frac{x^2+1}{x^2-1}dx=-\frac{x\,dz}{z}$$

and $$\int\frac{x^2+1}{x^2-1}\frac{dx}{\sqrt{Y_1}}=-\int\frac{dz}{\sqrt{a+4bz+(6c+2a)z^2}},$$

whose integral can be written down as before.

The integrals

$$\int\frac{(x^2-1)}{a_1x^2+b_1x+a_1}\frac{dx}{\sqrt{Y}}, \qquad \int\frac{x^2+1}{a_1x^2+b_1x-a_1}\frac{dx}{\sqrt{Y_1}},$$

$$\int\frac{x^2-1}{x^2+1}\frac{x}{a_1x^2+b_1x+a_1}\frac{dx}{\sqrt{Y}}, \quad \int\frac{x^2+1}{x^2-1}\frac{x}{a_1x^2+b_1x-a_1}\frac{dx}{\sqrt{Y_1}},$$

are reduced to forms already considered by the same substitutions, and are therefore integrable.

Similarly, if $Y \equiv ap^2x^4+4bpx^3+6cx^2+4bqx+aq^2$,

$Y_1 \equiv ap^2x^4+4bpx^3+6cx^2-4bqx+aq^2$,

the integrations of

$$\left.\begin{array}{l} \displaystyle\int\frac{px^2-q}{x}\cdot\frac{dx}{\sqrt{Y}}, \\ \displaystyle\int\frac{px^2-q}{px^2+q}\cdot\frac{dx}{\sqrt{Y}}, \\ \displaystyle\int\frac{px^2+q}{x}\cdot\frac{dx}{\sqrt{Y_1}}, \\ \displaystyle\int\frac{px^2+q}{px^2-q}\cdot\frac{dx}{\sqrt{Y_1}}, \end{array}\right\} \text{can be effected by the respective substitutions} \left\{\begin{array}{l} px+\dfrac{q}{x}=z, \\ px+\dfrac{q}{x}=\dfrac{1}{z}, \\ px-\dfrac{q}{x}=z, \\ px-\dfrac{q}{x}=\dfrac{1}{z}. \end{array}\right.$$

Ex. Consider the integral

$$I\equiv\int\frac{(x^4-1)\,dx}{(x^4+6x^2+1)\sqrt{x^4+x^2+1}}.$$

Here
$$I=\int\frac{x^3\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)dx}{x^3\left\{\left(x+\frac{1}{x}\right)^2+4\right\}\sqrt{\left(x+\frac{1}{x}\right)^2-1}};$$

and putting $x+\dfrac{1}{x}=z$, $\quad I=\displaystyle\int\frac{z\,dz}{(z^2+4)\sqrt{z^2-1}}.$

Put $z^2-1=w^2$, $z\,dz=w\,dw$;

$$\therefore\ I=\int\frac{dw}{5+w^2}=\frac{1}{\sqrt{5}}\tan^{-1}\frac{w}{\sqrt{5}}$$
$$=\frac{1}{\sqrt{5}}\sin^{-1}\frac{w}{\sqrt{w^2+5}}=\frac{1}{\sqrt{5}}\sin^{-1}\sqrt{\frac{z^2-1}{z^2+4}}$$
$$=\frac{1}{\sqrt{5}}\sin^{-1}\sqrt{\frac{x^4+x^2+1}{x^4+6x^2+1}}.$$

317. **Summing up.**

It will now be clear that any integration of the form

$$\int\frac{\phi(x)}{\psi(x)}\,\frac{dx}{\sqrt{ax^2+bx+c}}$$

can be effected, where ϕ and ψ are rational integral algebraic functions of x.

For if $\dfrac{\phi(x)}{\psi(x)}$ be put into the form

$$\sum A_nx^n+\sum\frac{\lambda}{x-a}+\sum\frac{\mu}{(x-\beta)^r}+\sum\frac{\lambda'x+\mu'}{Ax^2+Bx+C}$$
$$+\sum\frac{\lambda''x+\mu''}{(A'x^2+B'x+C')^s},$$

as explained in the chapter on Partial Fractions, then of the resulting integrals

(1) $\int \frac{x^n\,dx}{\sqrt{ax^2+bx+c}}$ is reducible to a lower order by Art. 240, and integrable.

(2) $\int \frac{dx}{(x-\alpha)\sqrt{ax^2+bx+c}}$ has been considered in Art. 287.

(3) $\int \frac{dx}{(x-\beta)^r\sqrt{ax^2+bx+c}}$ reduces by the method of Art. 290.

(4) $\int \frac{(\lambda' x+\mu')\,dx}{(Ax^2+Bx+C)\sqrt{ax^2+bx+c}}$ has been considered in Art. 291.

and (5) $\int \frac{(\lambda'' x+\mu'')\,dx}{(Ax^2+Bx+C)^s\sqrt{ax^2+bx+c}}$

is best got by differentiation with regard to C of the result for the case where $s=1$, as will be explained later. This method may also be adopted in (3).

318. General Consideration of the Position.

We have therefore now completed the integration of the most general function of x of form

$$\frac{A+B\sqrt{R}}{C+D\sqrt{R}},$$

where $A, B\ C, D$ are rational integral algebraic functions of x of any degree, and R is a rational integral algebraic function of x of degree 1 or 2.

For rationalizing the denominator,

$$\frac{A+B\sqrt{R}}{C+D\sqrt{R}}=\frac{(A+B\sqrt{R})(C-D\sqrt{R})}{C^2-D^2R}$$

$$=\frac{AC-BDR}{C^2-D^2R}+\frac{(BC-AD)R}{C^2-D^2R}\cdot\frac{1}{\sqrt{R}}$$

$$=\frac{P}{Q}+\frac{M}{N}\cdot\frac{1}{\sqrt{R}},\text{ say,}$$

where P, Q, M, N are rational integral algebraic functions of x.

Now $\int \frac{P}{Q}\,dx$ is integrable by the methods of partial fractions; and if $\frac{M}{N}$ be put into partial fractions, $\int \frac{M}{N}\frac{1}{\sqrt{R}}\,dx$ can, as has

been explained, be expressed as the sum of a finite number of such terms as have been discussed in the present chapter, and each term may then be integrated.

Hence the theory of the integration of

$$\int \frac{A+B\sqrt{R}}{C+D\sqrt{R}}\,dx$$

is now complete, where R is linear or quadratic. And it will be noted that the integration has been in all cases effected in terms of *the known* algebraic, logarithmic, inverse circular or inverse hyperbolic functions.

When R is of higher degree than the second, it has been seen that in *some special cases* the integration can still be effected in terms of the elementary functions, but for the *general* discussion of the cases where R is cubic or quartic, we shall require the elliptic functions, and in general for forms of R of higher degree than the fourth, we should require the functions known as hyperelliptic.

GENERAL EXAMPLES.

1. Obtain the following integrals:

(i) $\int (1+x)^{-1}x^{-\frac{1}{2}}\,dx$. (ii) $\int (1+x)^{-1}(1+2x)^{-\frac{1}{2}}\,dx$.

(iii) $\int x^{-1}(2-3x+x^2)^{-\frac{1}{2}}\,dx$. (iv) $\int (1+x)^{-1}(1+x+x^2)^{-\frac{1}{2}}\,dx$.

(v) $\int \frac{\sqrt{1+x+x^2}}{1+x}\,dx$. (vi) $\int \frac{x^2+x-1}{(x+1)\sqrt{x^2-1}}\,dx$.

(vii) $\int \frac{dx}{x\sqrt{a^n+x^n}}$. (viii) $\int \frac{1-x}{1+x}\,\frac{1}{\sqrt{x+x^2+x^3}}\,dx$.

2. Integrate (i) $\int \frac{dx}{(x+2)\sqrt{x^2-1}}$. (ii) $\int \frac{\sqrt{x^2-1}}{x+2}\,dx$.

[BARNES SCHOL., 1887.]

3. Show that

$$\int \frac{dx}{(x-p)\sqrt{a+2bx+cx^2}} = \frac{1}{\{-(a+2bp+cp^2)\}^{\frac{1}{2}}}\sin^{-1}\left\{\frac{(a+bx)+p(b+cx)}{(x-p)\sqrt{b^2-ac}}\right\}$$

where p lies between the roots of $a+2bx+cx^2=0$, supposed real.

[TRINITY, 1886 and 1891.]

4. Show that

$$\int \frac{dx}{(x^2+a^2)\sqrt{x^2+b^2}} = \frac{1}{a\sqrt{b^2-a^2}} \cos^{-1}\frac{a}{b}\sqrt{\frac{x^2+b^2}{x^2+a^2}}, \quad \text{if } a<b,$$

$$\text{and } = \frac{1}{a\sqrt{a^2-b^2}} \cosh^{-1}\frac{a}{b}\sqrt{\frac{x^2+b^2}{x^2+a^2}}, \quad \text{if } a>b.$$

5. Prove that

$$\int \frac{(x+1)\,dx}{(2x^2-2x+1)\sqrt{3x^2-2x+1}} = \cosh^{-1}\sqrt{\frac{3x^2-2x+1}{2x^2-2x+1}}$$
$$+ 2\cos^{-1}\frac{1}{\sqrt{2}}\sqrt{\frac{3x^2-2x+1}{2x^2-2x+1}}.$$

6. Integrate

(i) $\displaystyle\int \frac{(3x+4)\,dx}{(5x^2+8x)\sqrt{4x^2-2x+1}}$, (ii) $\displaystyle\int \frac{dx}{(x^2+2ax+b^2)\sqrt{x^2+2ax+c^2}}$,

where $a<b<c$.

7. Integrate (i) $\displaystyle\int \frac{x\,dx}{(a^2+b^2-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}}$. [ST. JOHN'S, 1888.]

(ii) $\displaystyle\int \frac{(x+b)\,dx}{(x^2+a^2)\sqrt{x^2+c^2}}$ $(a>c)$. [ST. JOHN'S, 1889.]

(iii) $\displaystyle\int \frac{d\theta}{\sin\theta\sqrt{a\cos^2\theta+b\sin^2\theta+c}}$. [TRINITY, 1888.]

8. Find the values of

(i) $\displaystyle\int \frac{\sin x\,dx}{(\cos x+\cos\alpha)\sqrt{(\cos x+\cos\beta)(\cos x+\cos\gamma)}}$. [$\gamma$, 1890.]

(ii) $\displaystyle\int \frac{dx}{\cos(x+\alpha)\sqrt{\cos(x+\beta)\cos(x+\gamma)}}$. [$\gamma$, 1890.]

9. Integrate $\displaystyle\int \frac{a^2-x^2}{(a^2-ax+x^2)(a^4+a^2x^2+x^4)^{\frac{1}{2}}}\,dx$,

transforming by the substitution

$$x^2+ax+a^2=y^2(x^2-ax+a^2). \quad [\alpha, 1884.]$$

10. Integrate (i) $\displaystyle\int \frac{(8x-13)\,dx}{(3x^2-10x+9)\sqrt{-x^2+10x-13}}$.

(ii) $\displaystyle\int \frac{dx}{(x-1)(x-2)\sqrt{(x-3)(x-4)}}$. [COLL., 1892.]

(iii) $\displaystyle\int \frac{(x+9)\,dx}{(x^2-5x+4)\sqrt{x^2-2x+2}}$.

(iv) $\displaystyle\int \frac{(x-a)\,dx}{(x-b)(x-c)(x-d)\sqrt{x-e}}$.

(v) $\displaystyle\int \frac{(x+3)\,dx}{(x^2+x+1)\sqrt{x^2+x+2}}$.

11. Integrate (i) $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$; [Coll., 1901.]

(ii) $\int \frac{x^{-1}(x^2-1)}{\sqrt{x^4+x^2+1}}\,dx.$

12. Show that

$$\int\sqrt{\frac{\sin(x-a)}{\sin(x+a)}}\,dx=\cos a\cos^{-1}\left(\frac{\cos x}{\cos a}\right)-\sin a\cosh^{-1}\left(\frac{\sin x}{\sin a}\right).$$

[Coll., 1901.]

13. Integrate (i) $\int \frac{1-2x^2}{1+2x^2}\sqrt{\frac{1+x^2}{1-x^2}}\,x\,dx.$ [R. P.]

(ii) $\int \frac{dx}{(1+x^4)\sqrt{(\sqrt{1+x^4}-x^2)}}.$ [R. P.; Euler, *C.I.*, vol. iv.]

(iii) $\int \frac{dx}{(a^2-ax-x^2)\sqrt{ax+x^2}}.$ [Oxf. I. P., 1900.]

14. Evaluate the integral

$$\int \frac{dx}{(a^2-\tan^2 x)(b^2-\tan^2 x)^{\frac{1}{2}}}.$$ [Math. Trip., 1886.]

15. Prove that

$$\int \frac{dx}{\sin x\sin^{\frac{1}{2}}(2x+a)}=-\frac{1}{\sqrt{\sin a}}\cosh^{-1}\frac{\sin(x+a)}{\sin x}.$$ [Coll., 1892.]

16. Show how to integrate

$$\int (fx+g)(ax^2+bx+c)^{\frac{n}{2}}\,dx,$$

where n is any positive or negative integer. [a, 1890.]

17. Show that $\int \frac{(a+bx)\,dx}{(a+2bx+cx^2)^{\frac{3}{2}}}=\frac{x}{(a+2bx+cx^2)^{\frac{1}{2}}}.$ [Trinity, 1889.]

18. Prove by the substitution

$$y^2=(ax^2+2bx+c)/(Ax^2+2Bx+C),$$

where A and $AC-B^2$ are positive, that the integral

$$\int \frac{(Mx+N)\,dx}{(Ax^2+2Bx+C)\sqrt{ax^2+2bx+c}}$$

becomes of the form

$$P_1\int\frac{dy}{\sqrt{\lambda_1-y^2}}+P_2\int\frac{dy}{\sqrt{y^2-\lambda_2}},$$

where λ_1 and λ_2 are the roots of the quadratic

$$(a-\lambda A)(c-\lambda C)-(b-\lambda B)^2=0,$$

and P_1, P_2 are definite constants.

Integrate completely the function

$$\frac{x+3}{(2x^2-10x+17)\sqrt{4x^2-26x+49}}.$$

[MATH. TRIP., 1891.]

19. Prove $\int_0^a \sqrt{\tanh^2 a - \tanh^2 x}\, dx = \frac{\pi}{2}(1 - \operatorname{sech} a)$. [$\beta$, 1889.]

20. Integrate $\int (x^2+b^2)^{-1}(x^2+a^2+b^2)^{-\frac{1}{2}}\, dx$. [OX. II. P., 1902.]

21. Integrate $\int \frac{\sin^3 x\, dx}{(1+\cos^2 x)\sqrt{1+\cos^2 x+\cos^4 x}}$, [ST. JOHN'S, 1882.]

and evaluate $\int_{-1}^{1} \frac{dx}{(a^2+c^2x^2)\sqrt{1-x^2}}$.

22. Show that $\int \frac{dx}{(x-a)\sqrt{Ax^2+2Bx+C}}$

is transcendental unless $Aa^2+2Ba+C=0$. [J. M. SCH. OX., 1904.]

Establish the results

$$\text{(i)} \int \frac{dx}{(x-1)\sqrt{x^2-4x+5}} = \frac{1}{\sqrt{2}} \cosh^{-1} \frac{\sqrt{2(x^2-4x+5)}}{x-1}$$

and

$$\text{(ii)} \int \frac{dx}{(x-2)\sqrt{(3+2x-2x^2)}} = \sin^{-1} \frac{\sqrt{3+2x-2x^2}}{\sqrt{7}(x-2)}.$$

[COLL. α, 1890.]

23. Show that

$$\int_{-1}^{+1} \frac{(1-ax)(1-bx)}{(1-2ax+a^2)(1-2bx+b^2)} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \frac{2-ab}{1-ab}.$$ [δ, 1884.]

24. Describe the steps whereby the integral of a rational function of a single variable, x, can be obtained.

Prove that if the sign of summation refer to the suffixes 1, 2, 3 in cyclical order, the integral

$$\int dx \sqrt{(x-a)(x-b)} \Sigma (c_2 - c_3) \left[\frac{2(c_1-a)(c_1-b)}{(x-c_1)^2} - \frac{2c_1-a-b}{x-c_1} \right]$$

is a certain constant multiple of

$$(x-a)^{\frac{3}{2}}(x-b)^{\frac{3}{2}}(x-c_1)^{-1}(x-c_2)^{-1}(x-c_3)^{-1}.$$

[MATH. TRIP., 1896.]

25. Determine the degenerate form of the elliptic integral

$$\int \frac{ds}{\sqrt{4(s-s_1)(s-s_2)(s-s_3)}}, \quad s_1 > s_2 > s_3,$$

when s_2 is made to coincide with s_1 or with s_3. [INT. ARTS, LONDON.]

26. Prove, by means of the substitution $\dfrac{x-\beta}{x-\alpha}=y^2$, that

$$\int\frac{dx}{(x-\gamma)\sqrt{(x-\alpha)(x-\beta)}}=\frac{-2}{\sqrt{(\alpha-\gamma)(\gamma-\beta)}}\tan^{-1}\sqrt{\frac{\alpha-\gamma}{\gamma-\beta}\cdot\frac{x-\beta}{x-\alpha}}$$

$$\text{or }=\frac{2}{\sqrt{(\alpha-\gamma)(\beta-\gamma)}}\,\text{th}^{-1}\sqrt{\frac{\alpha-\gamma}{\beta-\gamma}\cdot\frac{x-\beta}{x-\alpha}}.$$

[INT. ARTS, LONDON.]

27. Prove that

$$\int_0^1\frac{dx}{(1+x)(2+x)\sqrt{x(1-x)}}=\pi\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}}\right).$$

[MATH. TRIP. I., 1912.]

28. Show that the integral

$$\int\frac{dx}{x\sqrt{3x^2+2x-1}}$$

is rationalized by the assumption $x=(1+y^2)/(3-y^2)$, and hence, or otherwise, find its value.

Prove that if m be a positive proper fraction, the value of the above integral when taken between limits $\dfrac{1}{3}$ and $\dfrac{1}{2+m}$ is the same as when taken between limits $\dfrac{1}{2+m}$ and $\dfrac{1}{m(2+m)}$.

[MATH. TRIP. I., 1910.]

29. Prove by means of the substitution

$$\frac{a-x}{x-b}=\frac{a-d}{b-c}\,\frac{c-y}{y-d},$$

that, if m be any positive quantity, and $a>b>c>d$,

$$\int_b^a\frac{\{(a-x)(x-c)\}^{m-1}}{\left\{\dfrac{(a-x)(x-d)}{a-d}+\dfrac{(x-b)(x-c)}{b-c}\right\}^m}dx$$

$$=\int_d^c\frac{\{(a-x)(c-x)\}^{m-1}}{\left\{\dfrac{(a-x)(x-d)}{a-d}+\dfrac{(b-x)(c-x)}{b-c}\right\}^m}dx.$$

[MATH. TRIP., 1878.]

[See Wolstenholme's *Mathematical Problems*, Numbers 1900-1903 for a group of similar examples.]

30. By the transformation $px+\dfrac{q}{x}=\sqrt{2pq}/z$, integrate

$$\int\frac{px^2-q}{px^2+q}\,\frac{dx}{\sqrt{p^2x^4+q^2}}.$$ [Cf. EULER, *C.I.*, iv., p. 22.]

31. Apply the transformation $x^2 + \dfrac{1}{x^2} = 2/z^2$ to integrate

$$\text{(i)} \int \frac{\sqrt{1+x^4}}{1-x^4}\,dx, \quad \text{(ii)} \int \frac{x^2\,dx}{(1-x^4)\sqrt{1+x^4}}.$$

[EULER, *C.I.*, iv.]

32. Show that

$$\int \frac{2\,d\theta}{\sin\theta\sqrt[4]{\cos 2\theta}} = \tanh^{-1}\frac{\cos\theta}{\sqrt[4]{\cos 2\theta}} + \tan^{-1}\frac{\cos\theta}{\sqrt[4]{\cos 2\theta}}.$$

33. Show that the transformation

$$\{(a+bx^n)^\lambda - b^\lambda x^{\lambda n}\} = \left(\frac{x}{u}\right)^{\lambda n}$$

will reduce the integration

$$\int \frac{x^{m-1}\,dx}{(a+bx^n)\{(a+bx^n)^\lambda - b^\lambda x^{\lambda n}\}^{\frac{m}{\lambda n}}} \quad \text{to the form} \quad \frac{1}{a}\int \frac{u^{m-1}du}{1+b^\lambda u^{\lambda n}}.$$

[EULER, *C.I.*, iv., 53 and 56; PEACOCK, p. 305.]

34. (i) Show that

$$\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}\,dx = e^x\sqrt{\frac{1+x}{1-x}}, \qquad \text{[PEACOCK, p. 309.]}$$

and (ii) integrate
$$\int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\,dx.$$

35. Integrate

$$\text{(i)} \int \frac{2-3x}{2+3x}\sqrt{\frac{1+x}{1-x}}\,dx. \qquad \text{(ii)} \int \frac{\sin^5\theta + 2\cos^5\theta}{\cos\theta\sin 4\theta}\,d\theta.$$

[ST. JOHN'S, 1881.]

36. Show that

$$\text{(i)} \int \sqrt{\frac{a^2-c^2x^2}{a^2-x^2}}\frac{dx}{x} = \frac{1}{2}\log\left\{\frac{y-1}{y+1}\frac{(y+c)^c}{(y-c)^c}\right\},$$

where
$$y = \sqrt{\frac{a^2-c^2x^2}{a^2-x^2}}.$$

$$\text{(ii)} \int \frac{x^2+1}{x^2-1}\frac{dx}{\sqrt{1-ax^2+x^4}} = \frac{1}{\sqrt{a-2}}\cos^{-1}\frac{x\sqrt{a-2}}{x^2-1}.$$

[HALL, *I.C.*, p. 325.]

37. If $F(x, y)$ be a rational algebraic function of x and y, show that
$$\int F(x, \sqrt{1+x^2})(x+\sqrt{1+x^2})^{\frac{p}{q}}\,dx$$
may be integrated by the transformation $x = \sinh(q\log z)$.

38. Show that

(i) $\int (\cos 2\theta)^{\frac{3}{2}} \cos\theta \, d\theta$

$$= \frac{1}{8}\sin\theta(3 + 2\cos 2\theta)\sqrt{\cos 2\theta} + \frac{3}{8\sqrt{2}}\sin^{-1}(\sqrt{2}\sin\theta).$$

(ii) $\int_a^\theta \frac{\sin\theta}{\sqrt{\sin^2 a - \sin^2\theta}} d\theta = \log\left(\frac{\cos a}{\cos\theta + \sqrt{\sin^2 a - \sin^2\theta}}\right).$

(iii) $\int_0^\pi e^{2x\cos\theta} d\theta = \pi\left[1 + \frac{x^2}{(1\,!)^2} + \frac{x^4}{(2\,!)^2} + \frac{x^6}{(3\,!)^2} + \dots\right].$

39. Show that

(i) $\int_0^1 x^{a \pm cx} dx = \frac{1}{a+1} \mp \frac{c}{(a+2)^2} + \frac{c^2}{(a+3)^3} \mp \frac{c^3}{(a+4)^4} + \dots .$

(ii) $\int_0^1 x^{\pm x^2} dx = 1 \pm \frac{1}{3^2} + \frac{1}{5^3} \mp \frac{1}{7^4} + \dots .$ [ANGLIN.]

40. If $\phi(x) = a_0 + \frac{1}{2}a_2x^2 + \frac{1 \cdot 3}{2 \cdot 4}a_4x^4 + \dots$, show that

(i) $\frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos^2\theta\,\phi(\sin\theta)\,d\theta = 1^2 . a_0 + \frac{1}{2}\left(\frac{1}{2}\right)^2 a_2 + \frac{1}{3}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 a_4 + \dots .$

(ii) $\frac{20}{9\pi} = \frac{1}{2}.1^2 + \frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{1.3}{2.4}\right)^2 + \frac{1}{5}\left(\frac{1.3.5}{2.4.6}\right)^2 + \dots .$

(iii) $\frac{4}{\pi} - 1 = \frac{1}{1^2}\left(\frac{1}{2}\right)^2 + \frac{1}{3^2}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \frac{1}{5^2}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + \dots$

[ANGLIN.]

41. Integrate

(i) $\int \frac{\sin 2\theta \, d\theta}{\sqrt{\sin^4\theta + 4\sin^2\theta\cos^2\theta + 2\cos^4\theta}}.$

(ii) $\int \frac{(z^2 - 1)\,dz}{(z^2 + z + 1)\sqrt{z^4 + 4z^3 + 4z^2 + 4z + 1}}.$

42. If J be the Jacobian of two quadratic functions of x, viz.

$$u_1 \equiv a_1x^2 + 2b_1x + c_1, \quad u_2 \equiv a_2x^2 + 2b_2x + c_2,$$

$$J \equiv \begin{vmatrix} 2(a_1x + b_1), & 2(b_1x + c_1) \\ 2(a_2x + b_2), & 2(b_2x + c_2) \end{vmatrix},$$

show that if $u_1 = 0$, $u_2 = 0$ have no positive roots, then

$$\int_0^\infty \frac{J}{u_1u_2} dx = 2\log\frac{a_1c_2}{a_2c_1}.$$

43. By means of the identity

$$\int_0^{\frac{\pi}{2}} (a+\sin^2 x)^n \cos x\,dx = \int_0^{\frac{\pi}{2}} (1+a-\sin^2 x)^n \sin x\,dx,$$

prove that

$$a^n + {}^nC_1\frac{a^{n-1}}{3} + {}^nC_2\frac{a^{n-2}}{5} + {}^nC_3\frac{a^{n-3}}{7} + \dots$$

$$= (1+a)^n - 2\frac{n}{3}(1+a)^{n-1} + 2^2\frac{n(n-1)}{3.5}(1+a)^{n-2}$$

$$- 2^3\frac{n(n-1)(n-2)}{3.5.7}(1+a)^{n-3} + \dots.$$

[WOLSTENHOLME, *Problems*, No. 1929; WIGGINS, *E. Times*, No. 13323.]

44. Show that

(i) $$a^n + {}^nC_1\frac{1}{p+2}a^{n-1} + {}^nC_2\frac{1.3}{(p+2)(p+4)}a^{n-2}$$

$$+ {}^nC_3\frac{1.3.5}{(p+2)(p+4)(p+6)}a^{n-3} + \dots$$

$$= (1+a)^n - {}^nC_1\frac{p+1}{p+2}(1+a)^{n-1} + {}^nC_2\frac{(p+1)(p+3)}{(p+2)(p+4)}(1+a)^{n-2}$$

$$- {}^nC_3\frac{(p+1)(p+3)(p+5)}{(p+2)(p+4)(p+6)}(1+a)^{n-3} + \dots.$$

(ii) $$a^n + {}^nC_1\frac{1}{p+2}a^{n-1}b + {}^nC_2\frac{1.3}{(p+2)(p+4)}a^{n-2}b^2$$

$$+ {}^nC_3\frac{1.3.5}{(p+2)(p+4)(p+6)}a^{n-3}b^3 + \dots$$

$$= (a+b)^n - {}^nC_1\frac{p+1}{p+2}(a+b)^{n-1}b + {}^nC_2\frac{(p+1)(p+3)}{(p+2)(p+4)}(a+b)^{n-2}b^2$$

$$- {}^nC_3\frac{(p+1)(p+3)(p+5)}{(p+2)(p+4)(p+6)}(a+b)^{n-3}b^3 + \dots.$$

45. (i) Integrate

$$\int \frac{2x^3-1}{x^6+2x^3-x^2+1}\,dx.$$

[Ox. I. P., 1903.]

(ii) Integrate

$$\int \frac{(3x^4-1)\,dx}{x^8+2x^4-16x^2+1}.$$

(iii) Prove that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1-\sin\theta\cos\theta+\sin^2\theta)\cos^2\theta\,d\theta}{(1+\sin\theta\cos\theta)^3} = \frac{10\pi}{9}\sqrt{3}.$$

46. (i) Show that

$$\int_0^a x^{n-1}(1-a+x)^{-n-1}\,dx=\frac{1}{n}\frac{a^n}{1-a}.$$

(ii) Evaluate

$$\int_a^{a+h}\frac{(a+h-x)^{n-1}}{(n-1)!}\frac{d^n}{dx^n}f(x)\,dx.$$

How could your result be applied to the summation of series?

[a, 1886.]

47. Discuss the integration of

$$\int f\left[x,\ \left(\frac{a+bx}{a'+b'x}\right)^{\frac{m}{n}},\ \left(\frac{a+bx}{a'+b'x}\right)^{\frac{p}{q}},\ \left(\frac{a+bx}{a'+b'x}\right)^{\frac{r}{s}}\right]dx,$$

where f denotes a rational integral algebraic function of the quantities indicated. [LACROIX, *C.I.*, ii., p. 35.]

48. If $F(x)$ be a rational integral algebraic function of x, show that $\int_{-1}^{+1}\frac{F(x)}{\sqrt{1-x^2}}=\pi k$, where k is the coefficient of $\frac{1}{a}$ in the product

$$F(a)\left[\frac{1}{a}+\frac{1}{2}\cdot\frac{1}{a^3}+\frac{1.3}{2.4}\cdot\frac{1}{a^5}+\dots\right];$$ [ST. JOHN'S, 1891.]

or where k is the constant term in the expansion of $\frac{x\,F(x)}{(x^2-1)^{\frac{1}{2}}}$.

[COLL., 1892.]

49. If $f(x)$ be an arbitrary algebraic polynomial of degree $n-1$, and

$$P_n(x)\equiv A\frac{d^n}{dx^n}(x-a)^n(x-b)^n,$$

where A is a constant, then

$$\int_a^b f(x)\,P_n(x)\,dx=0.$$ [LOND. UNIV.]

50. Prove that

$$\int_0^a\frac{x\,dx}{\cos x\cos(a-x)}=\frac{a}{\sin a}\log\sec a.$$ [COLL., 1896.]

51. Show that if a be less than unity,

$$\int_0^\pi\frac{x\sin x\,dx}{1+a^2\cos^2 x}=\pi\frac{\tan^{-1}a}{a}.$$ [a, 1891.]

52. Integrate (i) $\displaystyle\int \frac{b^4x+x^5}{(b^4-x^4)^2}\sin^{-1}\frac{x^2}{b^2}\,dx.$ [δ, 1881.]

(ii) $\displaystyle\int \frac{d\phi}{\cos\phi}\sqrt{1-\tfrac{1}{2}\sin^2\phi}.$ [ST. JOHN'S, 1885.]

53. From the definition of a Bessel's function, viz.

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left[1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2.4(2n+2)(2n+4)}-\dots\right],$$

derive the results

$$\frac{\sin x}{x}=\int_0^{\frac{\pi}{2}} J_0(x\sin\theta)\sin\theta\,d\theta,$$

$$\frac{1-\cos x}{x}=\int_0^{\frac{\pi}{2}} J_1(x\sin\theta)\,d\theta.$$ [COLL., 1896.]

54. Integrate (i) $\dfrac{x^3}{(x+1)^3(x^3-1)}.$

(ii) $\dfrac{1}{\sqrt{1+3\sin x\cos x+2\sin^2 x\cos^2 x}}.$

(iii) $\dfrac{1}{\sqrt{(1+\sin x)(2+\sin x)}}.$ [MATH. TRIP., 1897.]

55. Show that

$$\int \sin^m x\sin nx\,dx=\sin^2 nx\,\frac{d}{dx}\left\{\frac{\phi(\sin x)}{\sin nx}\right\},$$

where the form of the function ϕ is defined by the relation

$$\phi(z)=\frac{z^m}{n^2-m^2}-\frac{m(m-1)}{(n^2-m^2)\{n^2-(m-2)^2\}}z^{m-2}$$
$$+\frac{m(m-1)(m-2)(m-3)}{(n^2-m^2)\{n^2-(m-2)^2\}\{n^2-(m-4)^2\}}z^{m-4}-\dots;$$

m being a positive integer and n not being of the form $\pm(m-2r)$, where r is a positive integer not greater than $\dfrac{m}{2}$.

[MATH. TRIP., 1897.]

56. Draw graphs of the transformation formula

$$(a_2x^2+2b_2x+c_2)y^2=a_1x^2+2b_1x+c_1$$

corresponding to those of Arts. 301 and 309 for

$$(a_2x^2+2b_2x+c_2)y=a_1x^2+2b_1x+c_1.$$

CHAPTER IX.

GENERAL THEOREMS.

319. Various Limiting Forms expressed as Definite Integrals. The definition of an integral, viz.

$$\int_a^b \phi(x)dx = Lt_{h=0} h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b)]$$

where $b=a+nh$ may be expressed as

$$Lt_{n=\infty} \sum_{\frac{r}{n}=0}^{\frac{r}{n}=1} \frac{b-a}{n}\phi\left(a+r\frac{b-a}{n}\right),$$

and can be used for the evaluation of a certain class of limiting forms.

Ex. Find the value of

$$Lt_{n=\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\frac{3^2}{n^3+3^3}+\ldots+\frac{n^2}{n^3+n^3}\right].$$

This may be written as

$$Lt_{n=\infty} \sum_{\frac{r}{n}=\frac{1}{n}}^{\frac{r}{n}=\frac{n}{n}=1} \frac{1}{n}\cdot\frac{\frac{r^2}{n^2}}{1+\frac{r^3}{n^3}},$$

and taking $\frac{r}{n}$ as x and $\frac{1}{n}$ as dx

$$=\int_0^1 \frac{x^2}{1+x^3}dx = \tfrac{1}{3}\Big[\log(1+x^3)\Big]_0^1 = \tfrac{1}{3}\log_e 2.$$

320. In the same way

$$Lt_{n=\infty}\{\phi(a)\phi(a+h)\phi(a+2h)\ldots\phi(a+nh)\}^{\frac{1}{n}},$$

where $h=\dfrac{b-a}{n}$ may be evaluated.

Let $\qquad u=\{\phi(a)\phi(a+h)\phi(a+2h)\ldots\phi(a+nh)\}^{\frac{1}{n}};$

then $\log u=\frac{1}{n}\{\log\phi(a)+\log\phi(a+h)+\ldots+\log\phi(a+rh)$

$$+\ldots+\log\phi(b)\}$$

$$=\sum_{\frac{r}{n}=\frac{0}{n}}^{\frac{r}{n}=\frac{n}{n}=1}\frac{1}{n}\log\phi\left\{a+(b-a)\frac{r}{n}\right\};$$

and therefore if we write

$$a+(b-a)\frac{r}{n}=x$$

and $\qquad (b-a)\frac{1}{n}=dx,$

the limit of $\log u$ is $\qquad \int_a^b\frac{\log\phi(x)}{b-a}\,dx.$

Hence $\quad Lt_{n=\infty}\{\phi(a)\phi(a+h)\phi(a+2h)\ldots\phi(a+nh)\}^{\frac{1}{n}},$

where $h=\frac{b-a}{n}$,

$$=e^{\frac{1}{b-a}\int_a^b\log\phi(x)\,dx}$$

[see *Diff. Calc.*, p. 6, Ex. 3].

Ex. Find the limit when $n=\infty$ of

$$\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{\frac{1}{n}}.$$

Calling this expression u,

$$\log u=\frac{1}{n}\left\{\log\left(1+\frac{1^2}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\ldots+\log\left(1+\frac{n^2}{n^2}\right)\right\}$$

$$=\sum_{\frac{r}{n}=\frac{1}{n}}^{\frac{r}{n}=\frac{n}{n}=1}\frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right),$$

and $\qquad Lt\log u=\int_0^1\log(1+x^2)\,dx$

$$=\Big[x\log(1+x^2)\Big]_0^1-2\int_0^1\frac{x^2}{1+x^2}\,dx$$

$$=\log 2-2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\log 2-2+2\,.\,\frac{\pi}{4}=\frac{\pi}{2}+\log 2-2;$$

$$\therefore\ Lt\,u=e^{\log 2+\frac{\pi-4}{2}}=2e^{\frac{\pi-4}{2}}$$

EXAMPLES.

1. Determine by integration the limiting values of the sums of the following series when n is infinitely great:

(i) $$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{n+n}.$$ [a, 1884.]

(ii) $$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\dots+\frac{n}{n^2+n^2}.$$ [OXFORD, 1888.]

(iii) $$\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\frac{1}{\sqrt{6n-3^2}}+\dots+\frac{1}{\sqrt{2n^2-n^2}}.$$ [CLARE, ETC., 1882.]

(iv) $$\frac{1}{n}\left\{\sin^{2k}\frac{\pi}{2n}+\sin^{2k}\frac{2\pi}{2n}+\sin^{2k}\frac{3\pi}{2n}+\dots+\sin^{2k}\frac{\pi}{2}\right\},$$

k being a positive integer. [ST. JOHN'S, 1886.]

2. Show that the limit when n is increased indefinitely of

$$\frac{(n-m)^{\frac{1}{3}}}{n}+\frac{(2^2n-m)^{\frac{1}{3}}}{2n}+\frac{(3^2n-m)^{\frac{1}{3}}}{3n}+\dots+\frac{(n^3-m)^{\frac{1}{3}}}{n^2} \quad\text{is}\quad \frac{3}{2}.$$

[COLLEGES, 1892.]

3. Find the limit when n is indefinitely great of the series

$$\frac{\sqrt{n-1}}{n}+\frac{\sqrt{2n-1}}{2n}+\frac{\sqrt{3n-1}}{3n}+\dots+\frac{\sqrt{n^2-1}}{n^2}.$$

[COLLEGES, 1890.]

4. Evaluate

$$Lt_{n=\infty}\left[\frac{1}{\sqrt{2a^2n-1}}+\frac{1}{\sqrt{4a^2n-1}}+\frac{1}{\sqrt{6a^2n-1}}+\dots+\frac{1}{\sqrt{2a^2n^2-1}}\right].$$

5. Evaluate

$$Lt_{n=\infty}\left[\frac{n^2}{(n^2+1^2)^{\frac{3}{2}}}+\frac{n^2}{(n^2+2^2)^{\frac{3}{2}}}+\dots+\frac{n^2}{\{n^2+(n-1)^2\}^{\frac{3}{2}}}\right].$$

[C. S., 1901.]

GENERAL THEOREMS ON INTEGRATION.

321. Various Propositions.

There are certain general propositions on integration, many of which are almost self-evident from the definition of integration or from geometrical considerations, the truth of some of which the student will have noticed for himself, but which require to be definitely stated. It will be assumed that all functions occurring in the following theorems are finite and continuous between the limits ascribed, unless the contrary be specified:

322. I. $$\int_a^b \phi(x)\,dx = \int_a^b \phi(z)\,dz,$$

for if $\psi(x)$ be such that

$$\phi(x) = \frac{d}{dx}\psi(x),$$

and therefore such that

$$\phi(z) = \frac{d}{dz}\psi(z),$$

each integral is equal to $\psi(b) - \psi(a)$.

In other words, the result being necessarily eventually independent of x or z, it is plainly immaterial whether the letter x or the letter z is used in the process of obtaining the indefinite integral previous to the substitution of the limits.

323. II. $$\int_a^b \phi(x)\,dx = \int_a^c \phi(x)\,dx + \int_c^b \phi(x)\,dx.$$

For if $\psi(x)$ be the indefinite integral of $\phi(x)$,

the left side is $\psi(b) - \psi(a)$

and the right side is

$$\{\psi(c) - \psi(a)\} + \{\psi(b) - \psi(c)\},$$

which is the same thing.

Further, it is equally clear that

$$\int_a^b \phi(x)\,dx = \int_a^c \phi(x)\,dx + \int_c^d \phi(x)\,dx + \int_d^e \phi(x)\,dx + \dots + \int_k^b \phi(x)\,dx,$$

where $c, d, e, f, \dots k$ are any real quantities which lie in the region from a to b for which $\phi(x)$ has been assumed to be finite and continuous.

Let us illustrate the fact geometrically.

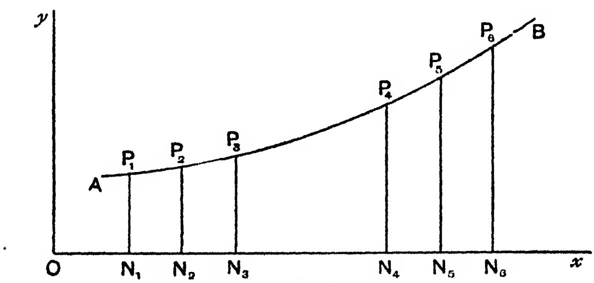


Fig. 26.

Let the curve drawn be the graph of $y=\phi(x)$, and let the equations of the ordinates

$$N_1P_1,\ N_2P_2,\ N_3P_3,\ \dots\ N_5P_5,\ N_6P_6$$

be $$x=a,\ x=c,\ x=d,\ \dots\ x=k,\ x=b$$

respectively.

Then the above theorem in integration expresses the obvious fact that

$$\text{Area } N_1N_6P_6P_1 = \text{Area } N_1N_2P_2P_1 + \text{Area } N_2N_3P_3P_2 + \dots + \text{Area } N_5N_6P_6P_5.$$

324. III. $$\int_a^b \phi(x)\,dx = -\int_b^a \phi(x)\,dx.$$

For, with the same notation as before,

the left side is $\psi(b)-\psi(a)$

and the right side is $-\{\psi(a)-\psi(b)\}$.

An interchange of the limits, therefore, changes the sign of the integral.

325. IV. $$\int_0^a \phi(x)\,dx = \int_0^a \phi(a-x)\,dx.$$

For if we put $x=a-X$, we have $dx=-dX$; and

if $x=a$, $X=0$;

if $x=0$, $X=a$.

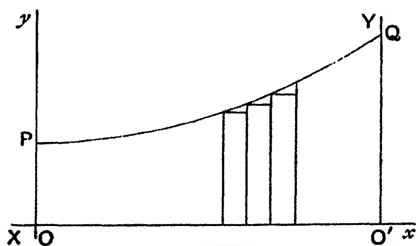


Fig. 27.

Hence
$$\int_0^a \phi(x)\,dx = -\int_a^0 \phi(a-X)\,dX$$
$$= \int_0^a \phi(a-X)\,dX, \text{ (by III.)},$$
$$= \int_0^a \phi(a-x)\,dx, \quad \text{(by I.)}.$$

Geometrically this expresses the obvious fact that, in estimating the area $OO'QP$ (Fig. 27) between the y and x-axes, an ordinate $O'Q$, and the curve PQ, which is the graph of $y=\phi(x)$, we *may if we like take our origin at* O', $O'Q$ *as our* Y-*axis and* $O'X$ *as our* X-*axis*, as it cannot affect the result, whether the elements of area are added up from left to right, or from right to left.

326. V. $\displaystyle\int_0^{2a} \phi(x)\,dx = \int_0^a \phi(x)\,dx + \int_0^a \phi(2a-x)\,dx.$

For, by II.,

$$\int_0^{2a} \phi(x)\,dx = \int_0^a \phi(x)\,dx + \int_a^{2a} \phi(x)\,dx,$$

and if in the second term we put $x=2a-X$, we have $dx=-dX$, and

$$\text{when } x=a, \quad X=a;$$
$$\text{when } x=2a, \quad X=0.$$

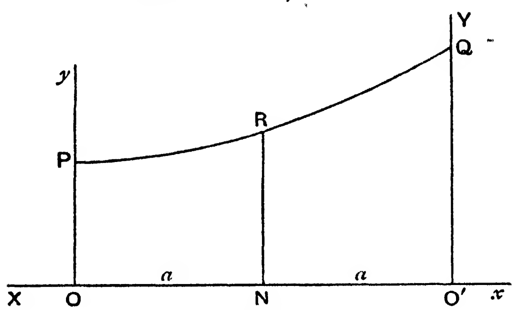


Fig. 28.

Thus the second integral on the right side, viz.

$$\begin{aligned}\int_a^{2a} \phi(x)\,dx &= -\int_a^0 \phi(2a-X)\,dX \\ &= \int_0^a \phi(2a-X)\,dX \quad \text{(by III.)} \\ &= \int_0^a \phi(2a-x)\,dx \quad \text{(by I.)};\end{aligned}$$

$$\therefore \int_0^{2a} \phi(x)\,dx = \int_0^a \phi(x)\,dx + \int_0^a \phi(2a-x)\,dx.$$

The geometrical interpretation is, that if we are estimating the area $OO'QP$ (Fig. 28) between the y and x axes, an ordinate $O'Q$, viz. $x=2a$, and the graph of $y=\phi(x)$, viz. the curve QP, we

may if we like take Ox and Oy for our axes for the portion $ONRP$, NR being the mid-ordinate, and $O'X$, $O'Y$ for axes in the second portion, thus finding each part separately, and then adding together, a fact obviously true.

327. VI. Plainly, if $\phi(x)$ be such that

$$\phi(2a-x)=\phi(x),$$

this proposition takes the form

$$\int_0^{2a}\phi(x)\,dx=2\int_0^a\phi(x)\,dx\,;$$

and if $\phi(x)$ be such that

$$\phi(2a-x)=-\phi(x),$$

$$\int_0^{2a}\phi(x)\,dx=0.$$

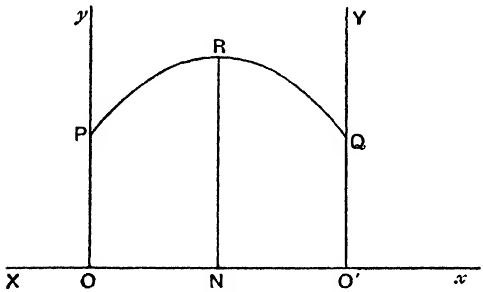


Fig. 29.

In the first case there is symmetry about the mid-ordinate NR (Fig. 29), and the whole area $OO'QRP$ in such a case is double that of $ONRP$.

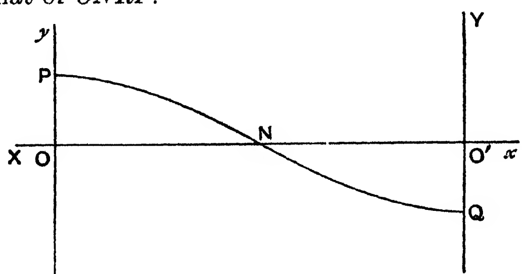


Fig. 30.

In the second case $\phi(a)=-\phi(a)$, *i.e.* $\phi(a)=0$, and the curve cuts the x-axis at N (Fig. 30), viz. where $x=a$, and though

the regions ONP, $O'NQ$ are equal in absolute area, the second integral of Art. 326, viz. $\int_0^a \phi(2a-x)dx$, which is referred to $O'X$ and $O'Y$ as axes, represents (—the area $O'NQ$), for all the ordinates are affected by a negative sign.

Hence, the algebraic sum of the two is zero, the one cancelling the other.

There is now symmetry about the point N.

328. This principle is very useful in the integrals of the trigonometric or of any periodic functions.

Thus, since $\sin^n x = \sin^n(\pi - x)$,

$$\int_0^\pi \sin^n x\, dx = 2\int_0^{\frac{\pi}{2}} \sin^n x\, dx.$$

And since $\cos^{2n+1} x = -\cos^{2n+1}(\pi - x)$,

$$\int_0^\pi \cos^{2n+1} x\, dx = 0;$$

so also since $\cos^{2n} x = \cos^{2n}(\pi - x)$,

$$\int_0^\pi \cos^{2n} x\, dx = 2\int_0^{\frac{\pi}{2}} \cos^{2n} x\, dx.$$

We may express these propositions in words, thus:

To add up all terms of the form $\sin^n x\, dx$ at equal indefinitely small intervals from 0 to π is to add up all such terms from 0 to $\frac{\pi}{2}$ and double the result. For the second quadrant sines are merely repetitions of the first quadrant sines in the reverse order.

Or geometrically, the curve $y = \sin^n x$ being symmetrical about the ordinate $x = \frac{\pi}{2}$, the whole area between the ordinates 0 and π is double that between 0 and $\frac{\pi}{2}$.

Similarly, the second quadrant cosines are repetitions of the first quadrant cosines with opposite signs, and therefore a term of form $\cos^{2n+1} x\, dx$ in the first quadrant is cancelled by the corresponding term in the second quadrant, but a term $\cos^{2n} x\, dx$, the index being now even, is duplicated by the corresponding term in the second quadrant.

Similar remarks and geometrical illustrations apply to other cases and for wider limits of integration.

Thus
$$\int_0^{2\pi} \sin^{2n+1} x\,dx = 0,$$
for the third and fourth quadrant elements cancel those from the first and second.
$$\int_0^{2\pi} \sin^{2n} x\,dx = 4\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx,$$
$$\int_0^{2\pi} \cos^{2n+1} x\,dx = 0,$$
$$\int_0^{2\pi} \cos^{2n} x\,dx = 4\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx,$$
and so on.

329. VII. **A Periodic Function.**

If
$$\phi(x) = \phi(a+x),$$
$$\int_0^{na} \phi(x)\,dx = n\int_0^a \phi(x)\,dx.$$

For, drawing the graph of $y=\phi(x)$, it is clear that it consists of an infinite series of repetitions of the part lying between the ordinates OP_0, $(x=0)$, and N_1P_1, $(x=a)$, (Fig. 31), for
$$\phi(x) = \phi(x+a),$$
and therefore writing $x+a$ for x,
$$\phi(x+a) = \phi(x+2a) = \phi(x+3a) = \text{etc.}$$
Also the areas bounded by the successive portions of the curve, the corresponding ordinates and the x-axis are all equal.

Thus $$\int_0^a \phi(x)dx = \int_a^{2a} \phi(x)dx = \int_{2a}^{3a} \phi(x)dx = \text{etc.}$$
and $$\int_0^{na} \phi(x)dx = \int_0^a \phi(x)dx + \int_a^{2a} \phi(x)dx + \dots + \int_{(n-1)a}^{na} \phi(x)dx$$
$$= n\int_0^a \phi(x)dx.$$

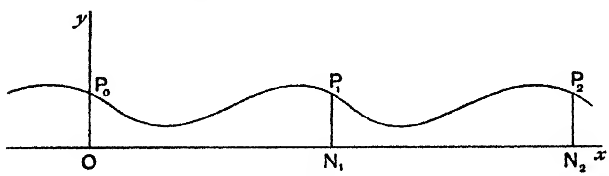


Fig. 31.

Thus, for instance, since $\sin^{2n} x = \sin^{2n}(\pi + x)$,
$$\int_0^{4\pi} \sin^{2n} x\,dx = 4\int_0^{\pi} \sin^{2n} x\,dx = 8\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = 8\frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}.$$

330. VIII. **Arbitrary Change of the Limits.**

In estimating $\int_a^b \phi(x)dx$, the limits may be altered arbitrarily to p, q, provided x be transformed linearly in a suitable manner.

Take $x=A+B\xi$. Let A and B be chosen so that

$$\left.\begin{aligned} a&=A+Bp,\\ b&=A+Bq,\end{aligned}\right\} \text{ whence } A=\frac{aq-bp}{q-p}, \quad B=\frac{b-a}{q-p},$$

i.e.
$$x=\frac{aq-bp}{q-p}+\frac{b-a}{q-p}\xi$$

and
$$dx=\frac{b-a}{q-p}d\xi.$$

Then
$$\int_a^b \phi(x)dx=\frac{b-a}{q-p}\int_p^q \phi\left(\frac{aq-bp}{q-p}+\frac{b-a}{q-p}\xi\right)d\xi$$
$$=\frac{b-a}{q-p}\int_p^q \phi\left(\frac{aq-bp}{q-p}+\frac{b-a}{q-p}x\right)dx \quad \text{(by I.).}$$

The geometrical significance of this is that instead of finding the area of $y=\phi(x)$ from $x=a$ to $x=b$, we may find the area of
$$\eta=\frac{b-a}{q-p}\phi\left(\frac{aq-bp}{q-p}+\frac{b-a}{q-p}\xi\right)$$
from $\xi=p$ to $\xi=q$.

Let the two graphs be drawn (Fig. 32), and let AA', BB', two ordinates, viz. $x=a$, $x=b$ in the one, correspond to PP',

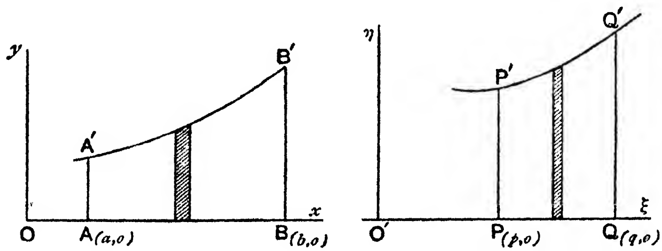


Fig. 32.

QQ', two ordinates, viz. $\xi=p$, $\xi=q$ in the other; then each element of the distance AB is reduced to a corresponding element of PQ in the ratio $\frac{q-p}{b-a}$, whilst there is a transference

of the origin a distance $\dfrac{aq-bp}{q-p}$ in the positive direction of the x-axis if this quantity be positive, or in the opposite direction if negative. This alteration in the graph leaves the number of units of area in the portion of the graph considered unaltered, the effect being merely that of drawing the graph on a different scale, the ordinates being altered in the ratio $\dfrac{b-a}{q-p}$, whilst the breadths of the elementary strips are altered in the inverse ratio, leaving the areas unchanged.

331. IX. If $\phi(x)$, $\psi(x)$ be single-valued continuous and finite functions of x, of which the latter retains the same sign between a and b, then

$$\int_a^b \phi(x)\,\psi(x)\,dx=\phi(\xi)\int_a^b \psi(x)\,dx,$$

where
$$a<\xi<b.$$

For $\int_a^b \phi(x)\psi(x)\,dx$, by the definition of an integral (Art. 11),

$$=Lt_{h=0}h\,[\phi(a)\,\psi(a)+\phi(a+h)\,\psi(a+h)+\phi(a+2h)\,\psi(a+2h)+\dots$$
$$+\phi(b-h)\,\psi(b-h)].$$

Now, of all the expressions

$$\phi(a),\quad \phi(a+h),\quad \phi(a+2h),\ \dots\ \phi(b-h),$$

let $\phi(\xi_1)$ be the greatest and $\phi(\xi_2)$ the least.

Then $\phi(a)\psi(a)+\phi(a+h)\psi(a+h)+\dots+\phi(b-h)\psi(b-h)$

$$<\phi(\xi_1)[\psi(a)+\psi(a+h)+\psi(a+2h)+\dots+\psi(b-h)]$$

and $>\phi(\xi_2)[\psi(a)+\psi(a+h)+\psi(a+2h)+\dots+\psi(b-h)]$.

Hence
$$\int_a^b \phi(x)\,\psi(x)\,dx<\phi(\xi_1)\int_a^b \psi(x)\,dx$$

and
$$>\phi(\xi_2)\int_a^b \psi(x)\,dx,$$

and therefore must
$$=\phi(\xi)\int_a^b \psi(x)\,dx,$$

where $\phi(\xi)$ is intermediate between $\phi(\xi_1)$ and $\phi(\xi_2)$. And ξ is a value of x somewhere between a and b.

It has been assumed that $\psi(x)$ is positive for the range from a to b. If $\psi(x)$ be negative throughout, the order of the inequalities is reversed, but the final result remains the same.

332. Cor. I. As a case of this theorem write $\phi'(x)$ for $\phi(x)$, and 1 for $\psi(x)$.

Then $$\int_a^b \phi'(x)\,dx = \phi'(\xi)\int_a^b 1\,dx = (b-a)\,\phi'(\xi),$$

i.e. $$\phi(b) - \phi(a) = (b-a)\,\phi'(\xi);$$

or putting $b = a + h$ and $\xi = a + \theta h$, where θ is a positive proper fraction,

$$\phi(a+h) = \phi(a) + h\phi'(a+\theta h),$$

subject to the condition that $\phi(x)$ and $\phi'(x)$ are finite and continuous functions of x for the whole range of values of x from a to $a+h$. [See *Diff. Calc.*, Art. 139.]

333. Cor. II. If $\phi(x)$ has a finite value for all values of x, $a < x < b$, it follows that $I \equiv \int_a^b \phi(x)\,dx$ is finite if a and b are finite, for if $\phi(\xi_1)$ be the greatest and $\phi(\xi_2)$ the least of the values of $\phi(x)$, I lies between $\phi(\xi_1)(b-a)$ and $\phi(\xi_2)(b-a)$, and is therefore finite.

334. Cor. III. If $u_1, u_2, u_3, \ldots$ be all single-valued functions of x, finite and continuous for all values of x between a and b, and if the series $u_1 + u_2 + u_3 + u_4 + \ldots$ to an infinite number of terms be uniformly and unconditionally convergent for all values of x between these limits, and $f(x)$ the limit towards which it converges, then the series

$$\int_a^x u_1\,dx + \int_a^x u_2\,dx + \int_a^x u_3\,dx + \ldots$$

is also convergent for values of x between a and b, and converges to the limit $\int_a^x f(x)\,dx$. [This theorem has already been proved in Art. 34 from a slightly different point of view.]

Let R_n be the remainder after n terms of the given series, so that

$$u_1 + u_2 + u_3 + \ldots + u_n + R_n = f(x).$$

Then

$$\int_a^x u_1\,dx + \int_a^x u_2\,dx + \int_a^x u_3\,dx + \ldots + \int_a^x R_n\,dx = \int_a^x f(x)\,dx.$$

Now, by supposition, R_n is finite. Let R_n' and R_n'' be the greatest and least values of R_n as x changes continuously from a to b.

Then $\int_a^x R_n\,dx$ lies between $R_n'(x-a)$ and $R_n''(x-a)$.

Moreover, R_n vanishes by hypothesis when n is indefinitely increased, whence R_n' and R_n'' also vanish in the limit;

$\therefore \int_a^x R_n\,dx$ vanishes in the limit.

Hence $$\int_a^x u_1\,dx+\int_a^x u_2\,dx+\int_a^x u_3\,dx+\dots$$

converges to the limit $\int_a^x f(x)\,dx$.

[SERRET, *Calcul Intég.*, p. 108.]

335. Cor. IV. If a continuous function $f(x)$ can be expanded in a series of powers of x convergent for values of x between 0 and a,

say, $$A_0+A_1x+A_2x^2+\dots,$$

then $$A_0x+A_1\frac{x^2}{2}+A_2\frac{x^3}{3}+\dots$$

is also a continuous and convergent series tending to the limit

$$\int_0^x f(x)\,dx. \qquad \text{[Cf. Art. 34.]}$$

336. Cor. V.

$$\int_0^x f(x)\,dx=\int_0^x \left[f(0)+xf'(0)+\frac{x^2}{\underline{|2}}f''(0)+\dots\right]dx$$

$$=xf(0)+\frac{x^2}{\underline{|2}}f'(0)+\frac{x^3}{\underline{|3}}f''(0)+\dots,$$

convergent between the same limits for which Maclaurin's series, which has been used, is convergent.

This gives a means of expressing an integration by means of a series.

337. LEMMA. A THEOREM DUE TO ABEL. If S_r be the sum of the first r terms, and S_r' the sum of the last r terms of the series

$$u_1+u_2+u_3+\dots+u_r+\dots+u_n,$$

each term being real and finite, but not necessarily all of the same sign, and if

Σ and σ be the greatest and least values of S_r,

and Σ' and σ' be the greatest and least values of S_r',

and if $a_1, a_2, a_3, \ldots a_n$ be n positive finite quantities arranged in descending order of magnitude, and if

$$S = a_1u_1 + a_2u_2 + a_3u_3 + \ldots + a_nu_n,$$

then we shall have $\quad a_1\Sigma > S > a_1\sigma;$

and if $a_1, a_2, a_3, \ldots a_n$ be arranged in ascending order of magnitude, then

$$a_n\Sigma' > S > a_n\sigma'.$$

For

$$\begin{aligned} S &= a_1u_1 + a_2u_2 + a_3u_3 + \ldots + a_nu_n \\ &= a_1(S_1) + a_2(S_2 - S_1) + a_3(S_3 - S_2) \\ &\qquad + \ldots + a_{n-1}(S_{n-1} - S_{n-2}) + a_n(S_n - S_{n-1}) \\ &= S_1(a_1 - a_2) + S_2(a_2 - a_3) + S_3(a_3 - a_4) \\ &\qquad + \ldots + S_{n-1}(a_{n-1} - a_n) + S_na_n, \end{aligned}$$

and $a_1 - a_2,\ a_2 - a_3, \ldots a_{n-1} - a_n,\ a_n$ are all positive quantities;

$$\therefore\ S < \Sigma[(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \ldots + (a_{n-1} - a_n) + a_n]$$

and $\quad > \sigma[(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \ldots + (a_{n-1} - a_n) + a_n],$

i.e. $\quad S < a_1\Sigma$ and $S > a_1\sigma$, $\quad$ *i.e.* $a_1\Sigma > S > a_1\sigma$.

In the same way, writing the series from the other end, and if $a_n, a_{n-1}, a_{n-2}, \ldots a_1$ be in descending order of magnitude,

$$a_n\Sigma' > S > a_n\sigma'.$$

This theorem in inequalities is due to ABEL.

We note also that if $a_1, a_2, a_3, \ldots a_n$ were all negative, the same theorems would still hold, except that the inequalities would have been reversed, viz.

$$a_1\Sigma < S < a_1\sigma \quad \text{and} \quad a_n\Sigma' < S < a_n\sigma'.$$

338. X. Applying Abel's inequality theorem to the case of the integral

$$\int_a^b \phi(x)\,\psi(x)\,dx,$$

where $\phi(x)$ and $\psi(x)$ are finite and continuous functions of x for all values of x between the limits a and b, and $\phi(x)$ positive and continually decreasing throughout that range, and writing

$$\phi(a),\ \phi(a+h),\ \phi(a+2h),\ \ldots\ \phi(b-h)$$

respectively for $\quad a_1, \quad a_2, \quad a_3, \quad \ldots \quad a_n,$

and $\quad h\psi(a),\ h\psi(a+h),\ h\psi(a+2h),\ \ldots\ h\psi(b-h)$

for $\quad u_1, \quad u_2, \quad u_3, \quad \ldots \quad u_n,$

and taking the limit when h is indefinitely small, we have

$$S = \int_a^b \phi(x)\psi(x)\,dx,$$

$$a_1\Sigma = \phi(a)\int_a^{\xi_1} \psi(x)\,dx,$$

$$a_1\sigma = \phi(a)\int_a^{\xi_2} \psi(x)\,dx,$$

where ξ_1, ξ_2 are the limits corresponding to the greatest and least values of $\int_a^{\xi} \psi(x)\,dx$ for different values of ξ between a and b;

$$\therefore\ \phi(a)\int_a^{\xi_1} \psi(x)\,dx > \int_a^b \phi(x)\psi(x)\,dx > \phi(a)\int_a^{\xi_2} \psi(x)\,dx,$$

and therefore $\int_a^b \phi(x)\psi(x)\,dx = \phi(a)\int_a^{\xi} \psi(x)\,dx$

for some value of ξ intermediate between a and b.

Similarly, if $\phi(x)$ be a continually increasing function,

$$\phi(b)\int_{\xi_1'}^b \psi(x)\,dx > \int_a^b \phi(x)\psi(x)\,dx > \phi(b)\int_{\xi_2'}^b \psi(x)\,dx,$$

where ξ_1', ξ_2' are the values of ξ which make $\int_{\xi}^b \psi(x)\,dx$ greatest or least, and therefore

$$\int_a^b \phi(x)\psi(x)\,dx = \phi(b)\int_{\xi}^b \phi(x)\,dx,$$

where ξ is intermediate between a and b.

339. From the last remark of Art. 337 it appears that the same theorem will be true when $\phi(x)$ is negative throughout. That is, that provided $\phi(x)$ be continually positive or continually negative from $x=a$ to $x=b$, and $\phi'(x)$ retains the same sign throughout this range,

$$\int_a^b \phi(x)\psi(x)\,dx = \phi(a)\int_a^{\xi} \psi(x)\,dx$$

or

$$\int_a^b \phi(x)\psi(x)\,dx = \phi(b)\int_{\xi}^b \psi(x)\,dx,$$

according as $\phi'(x)$ is negative or positive, where ξ is some value of x between a and b, *i.e.* $\xi = a + \theta(b-a)$, where θ is some positive proper fraction.

340. **A Theorem due to Ossian Bonnet.**

If $\phi'(x)$ be negative, *i.e.* $\phi(x)$ decreasing, but $\phi(x)$ changing sign in the interval from $x=a$ to $x=b$, and therefore $\phi(b)$ negative and $\phi(a)$ positive, write

$$\phi(x)-\phi(b)=\chi(x);$$

then $\chi'(x)$ is negative and $\chi(x)$ is positive from a to b.

$$\begin{aligned}\therefore\quad \int_a^b \phi(x)\psi(x)\,dx &= \int_a^b [\phi(b)+\chi(x)]\psi(x)dx \\ &= \phi(b)\int_a^b \psi(x)\,dx + \chi(a)\int_a^\xi \psi(x)\,dx \\ &= \left[\phi(b)\int_a^\xi \psi(x)\,dx + \phi(b)\int_\xi^b \psi(x)\,dx\right] + \chi(a)\int_a^\xi \psi(x)\,dx \\ &= [\phi(b)+\chi(a)]\int_a^\xi \psi(x)\,dx + \phi(b)\int_\xi^b \psi(x)\,dx \\ &= \phi(a)\int_a^\xi \psi(x)\,dx + \phi(b)\int_\xi^b \psi(x)\,dx.\end{aligned}$$

341. Finally, if $\phi'(x)$ be positive, *i.e.* $\phi(x)$ increasing, but changing sign in the interval between a and b, and therefore $\phi(a)$ negative and $\phi(b)$ positive, write

$$\phi(x)-\phi(a)=\chi(x);$$

then $\chi'(x)$ is positive and $\chi(x)$ is positive from a to b.

$$\begin{aligned}\therefore\quad &\int_a^b \phi(x)\psi(x)dx \\ &= \int_a^b [\phi(a)+\chi(x)]\psi(x)dx \\ &= \phi(a)\int_a^b \psi(x)dx + \chi(b)\int_\xi^b \psi(x)dx \\ &= \phi(a)\left[\int_a^\xi \psi(x)dx + \int_\xi^b \psi(x)dx\right] + \chi(b)\int_\xi^b \psi(x)dx \\ &= \phi(a)\int_a^\xi \psi(x)dx + [\phi(a)+\chi(b)]\int_\xi^b \psi(x)dx \\ &= \phi(a)\int_a^\xi \psi(x)dx + \phi(b)\int_\xi^b \psi(x)dx.\end{aligned}$$

Hence, in all cases where the differential coefficient of $\phi(x)$ is a continuous function, retaining one sign between the limits, though $\phi(x)$ itself may change sign,

$$\int_a^b \phi(x)\psi(x)dx = \phi(a)\int_a^\xi \psi(x)dx + \phi(b)\int_\xi^b \psi(x)dx$$

for some value of ξ intermediate between a and b, ϕ and ψ being finite and continuous throughout.

This theorem is due to Ossian Bonnet.

342. XI. (i) Since

$$(a_1^2+a_2^2+a_3^2+\dots+a_n^2)(b_1^2+b_2^2+\dots+b_n^2)h^2$$
$$\nless (a_1b_1+a_2b_2+\dots+a_nb_n)^2h^2,$$

we have upon putting

$$a_1=\phi(a),\quad a_2=\phi(a+h)\dots a_n=\phi(b-h),$$
$$b_1=\psi(a),\quad b_2=\psi(a+h)\dots b_n=\psi(b-h),$$

and taking the limit when h is indefinitely small,

$$\int_a^b [\phi(x)]^2dx\int_a^b[\psi(x)]^2dx \nless \left[\int_a^b \phi(x)\psi(x)dx\right]^2.$$

(ii) If $\qquad a_1,\ a_2,\ a_3,\ \dots\ a_n,$

and $\qquad b_1,\ b_2,\ b_3,\ \dots\ b_n,$

be two sets of positive quantities, both in descending or both in ascending order of magnitude,

$$\Sigma a_r \Sigma a_r^2 b_r - \Sigma a_r^2 \Sigma a_r b_r \nless 0$$

[for $\Sigma a_r a_s(a_r-a_s)(b_r-b_s)$ is positive].

And it follows as in (i) that if $\phi(x)$ and $\psi(x)$ be finite, continuous, and positive, and $\phi'(x)$ and $\psi'(x)$ be both positive or both negative from $x=a$ to $x=b$, then

$$\int_a^b \phi(x)dx\int_a^b[\phi(x)]^2\psi(x)dx \nless \int_a^b[\phi(x)]^2dx\int_a^b[\phi(x)][\psi(x)]dx.$$

If ϕ' and ψ' are of opposite signs the order of the inequality is reversed.

General and Principal Values of an Integral. Cauchy.

343. XII. **The Definition of Integration. Modifications.**

In our summation definition of integration, as

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\dots+\phi(b-h)],$$

which has been denoted by

$$\int_a^b \phi(x)dx,$$

we have assumed

(1) $\phi(x)$ finite and continuous and single-valued for the whole range from $x=a$ to $x=b$.

(2) a and b to be both finite quantities.

This definition will fail when these conditions are not satisfied, and will require modification.

We have also (Art. 18) extended our notation so as to let $\int_a^\infty \phi(x)dx$ stand for the limit when b is indefinitely increased of $\psi(b)-\psi(a)$ where $\frac{d\psi(x)}{dx}=\phi(x)$, with a similar extension when the lower limit becomes infinitely large. The subject of integration itself, viz. $\phi(x)$, has been so far, however, in all cases, understood to be *finite*, single-valued, and continuous for the whole range of integration from a to b, whether that range be finite or infinite.

344. **Infinities of the Integrand.** General and Principal Values. Cauchy.

When $\phi(x)$ becomes infinite between the limits of integration, say at the point $x=c$, where $a<c<b$, and nowhere else between a and b, our definition holds

from $x=a$ to $x=c-\epsilon$

and from $x=c+\eta$ to $x=b$,

where ϵ and η are two positive quantities which may be taken as small as we please.

The integral $\int_a^b \phi(x)dx$ is now to be understood as meaning

$$\underset{\substack{\epsilon=0\\ \eta=0}}{Lt}\left[\int_a^{c-\epsilon}\phi(x)dx+\int_{c+\eta}^b \phi(x)dx\right].$$

This limit may be finite, infinite, or of undetermined value.

It is called the General Value of the Integral.

When $\eta=\epsilon$, Cauchy has named the limiting form derived, the Principal Value of the Integral, viz.

$$Lt_{\epsilon=0}\left[\int_a^{c-\epsilon}\phi(x)dx+\int_{c+\epsilon}^b \phi(x)dx\right],$$

which may be finite or infinite.

A similar modification of the original definition will obviously be necessary when the subject of integration, viz. $\phi(x)$, attains an infinite value more than once between the extreme limits of the integration, viz. between a and b.

If the infinity of $\phi(x)$ occurs *at* one of the limits, say at the upper one, then the integral $\int_a^b \phi(x)\,dx$ is to be understood to mean

$$Lt_{\epsilon=0}\int_a^{b-\epsilon} \phi(x)\,dx.$$

Again when the upper *limit* is infinite we shall understand $\int_a^\infty \phi(x)\,dx$ to mean

$$Lt_{\epsilon=0}\int_a^{\frac{1}{\epsilon}} \phi(x)\,dx$$

and when the lower *limit* is infinite we shall understand $\int_\infty^b \phi(x)\,dx$ to mean

$$Lt_{\epsilon=0}\int_{\frac{1}{\epsilon}}^{b} \phi(x)\,dx.$$

When the integration is from $-\infty$ to $+\infty$ we shall consider the integration $\int_{-\infty}^{\infty} \phi(x)\,dx$ to mean

$$Lt_{\substack{\epsilon=0\\ \eta=0}}\int_{-\frac{1}{\eta}}^{+\frac{1}{\epsilon}} \phi(x)\,dx,$$

which we shall refer to as its **General** value; *i.e.*

$$Lt_{\substack{\epsilon=0\\ \eta=0}}\left[\psi\left(\frac{1}{\epsilon}\right)-\psi\left(-\frac{1}{\eta}\right)\right], \quad \text{where } \frac{d\psi}{dx}=\phi(x),$$

ϵ and η being small positive quantities independent of each other; and when $\eta=\epsilon$ we shall refer to

$$Lt_{\epsilon=0}\int_{-\frac{1}{\epsilon}}^{\frac{1}{\epsilon}} \phi(x)\,dx$$

as its **Principal** value; *i.e.*

$$Lt_{\epsilon=0}\left[\psi\left(\frac{1}{\epsilon}\right)-\psi\left(-\frac{1}{\epsilon}\right)\right].$$

345. Geometrical Illustrations.

Let a graph be drawn of $y=\phi(x)$, and let $OA=a$, $OC=c$, $OB=b$. Then at C $(x=c)$ there is an asymptote parallel to the y-axis. The graph may be such as to approach the asymptote from opposite sides at the same extremity (Fig. 33), or from opposite sides at opposite extremities (Fig. 34). In the first case there is no change of sign of $\phi(x)$ as x passes

through the value c. In the second, $\phi(x)$ does change sign. Let the inscribed rectangles be drawn as in Art. 11. Let P_rN_r and P_sN_s be the ordinates at distances $p\epsilon$ and $q\epsilon$ on opposite sides of the asymptote; then it is clear that Cauchy's "General Value" of $\int_a^b \phi(x)\,dx$ is the limit of

$$\text{area } AN_rP_rP_1 \pm \text{area } N_sBP_nP_s,$$

where ϵ is indefinitely decreased, *i.e.* where N_rC, CN_s are indefinitely decreased in such a manner as to retain a definite, but arbitrary ratio to each other, viz. $p:q$, whilst

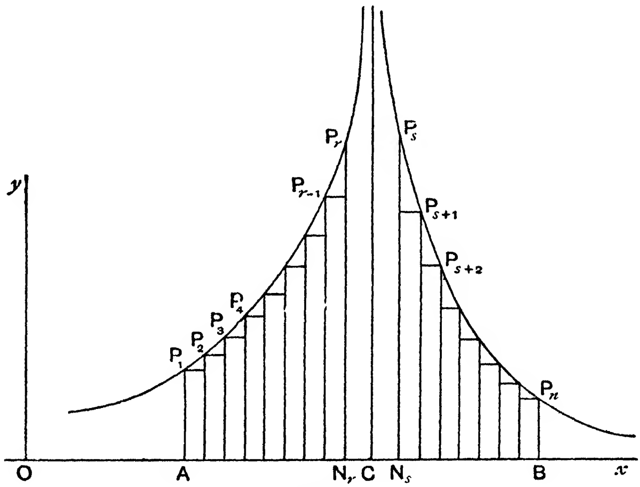


Fig. 33.

the "Principal Value" is what this becomes when N_rC, CN_s ultimately vanish in a ratio of equality.

This treatment in either case excludes the area bounded by $P_rN_rN_sP_s\infty P_r$ in Fig. 33, where $\phi(x)$ retains the same sign or the difference of the areas $N_rC\infty P_rN_r$, $N_sC(-\infty)P_sN_s$, where $\phi(x)$ changes sign as x passes through C, as in Fig. 34, when both ordinates N_rP_r and N_sP_s are made to approach indefinitely closely to the asymptote.

There is no advantage in prescribing beforehand the relative speeds at which the ordinates N_rP_r, N_sP_s are made to approach the asymptote, viz. by making the approach in the ratio of some definite but arbitrarily chosen quantities p, q. We

leave the choice of these relative speeds till after integration, and thereby retain command of the mode in which the ordinates are made to close up.

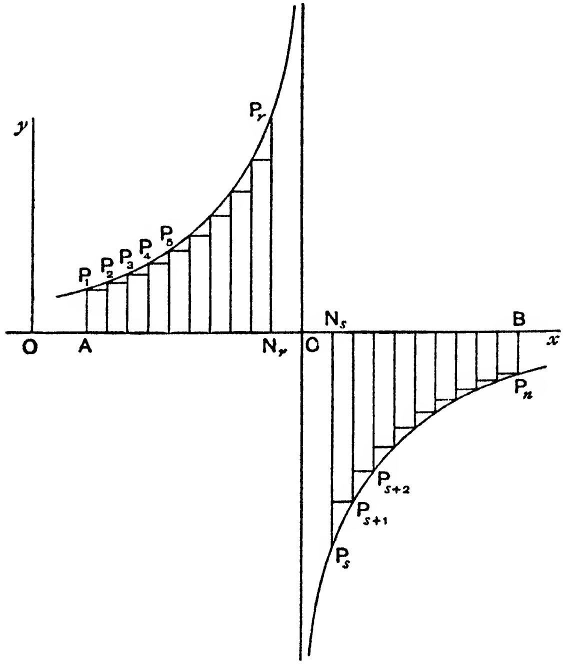


Fig. 34.

In understanding $\int_a^b \phi(x)\,dx$ to mean

$$\underset{\substack{\epsilon=0\\ \eta=0}}{Lt}\left[\int_a^{c-\epsilon}\phi(x)\,dx+\int_{c+\eta}^{b}\phi(x)\,dx\right],$$

where ϵ, η are two positive quantities, we can ultimately make $\frac{\epsilon}{p}=\frac{\eta}{q}$ in our investigations of the "General Value," and if we take $p=q$, that is $\epsilon=\eta$, we shall have Cauchy's "Principal Value."

346. When the inscribed and circumscribed rectangles are drawn in the Newtonian manner (Art. 11), the pairs in immediate contiguity with the asymptote are in area [Fig. 35]

$$\epsilon\phi(c-\epsilon),\ \epsilon\phi(c) \quad \text{and} \quad \eta\phi(c+\eta),\ \eta\phi(c).$$

The circumscribed rectangles are numerically greater than the inscribed ones. They are of infinite length $\phi(c)$, and of infinitesimal breadths ϵ and η respectively (Fig. 35).

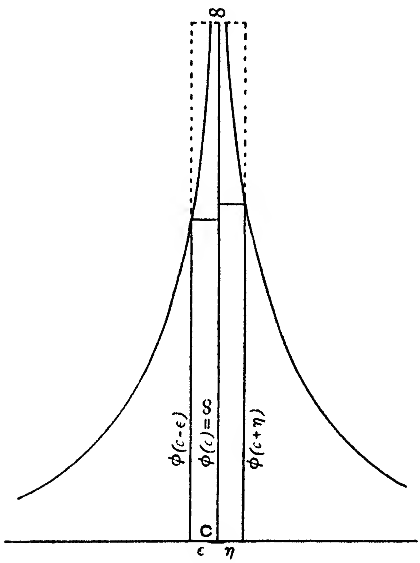


Fig. 35.

These areas then are "undetermined" quantities until we know the nature of $\phi(c)$. If the orders of the infinitesimals ϵ, η be higher than the order of the infinity $\phi(c)$ their limits are zero. If of lower order their limits are infinite. But, in the latter case, if $\phi(x)$ change sign as x passes through the value c, we may be only concerned with the difference of these infinities, which may be finite.

347. If $\phi(x)$ becomes infinite at a point $x=c$, the general way in which it does so is by the vanishing of a factor in its denominator.

Let $\phi(x)=\dfrac{F(x)}{(x-c)^n}$, where $F(x)$ contains no factor $x-c$, and therefore retains the same sign as x increases through the value c, and n is positive.

We are only concerned to discuss the behaviour of this function in the immediate vicinity of the asymptote.

Therefore we may take our limits a, b so near to $x=c$ that $F'(x)$ retains the same sign throughout, and if A and B are the greatest and least values of $F'(x)$ in this interval,

$\int_a^b \phi(x)\,dx$ is intermediate between $A\int_a^b \frac{dx}{(x-c)^n}$ and $B\int_a^b \frac{dx}{(x-c)^n}$.

Hence we may confine our discussion to $\int_a^b \frac{dx}{(x-c)^n}$. And it will be convenient to push forward our origin to the point $(c, 0)$, so that the y-axis coincides with the asymptote, and we then have to discuss the limit of

$$\int_{-\alpha}^{-\epsilon} \frac{dx}{x^n} + \int_{\eta}^{\beta} \frac{dx}{x^n}, \qquad \text{where } \alpha = c-a, \quad \beta = b-c.$$

This expression has the value

$$-\frac{1}{n-1}\left\{\left[\frac{1}{x^{n-1}}\right]_{-\alpha}^{-\epsilon} + \left[\frac{1}{x^{n-1}}\right]_{\eta}^{\beta}\right\}$$

$$= -\frac{1}{n-1}\left\{\frac{1}{(-\epsilon)^{n-1}} - \frac{1}{(-\alpha)^{n-1}} + \frac{1}{(\beta)^{n-1}} - \frac{1}{(\eta)^{n-1}}\right\}.$$

(a) When n is <1, *i.e.* $0<n<1$, the limit is finite, viz.

$$-\frac{1}{n-1}\left[-(-\alpha)^{1-n} + \beta^{1-n}\right],$$

and is independent of the limiting value of $\frac{\epsilon}{\eta}$. This is then both the "General Value" and the "Principal Value."

The first and last elements in the summations, viz. $\epsilon\cdot\frac{1}{\epsilon^n}$ and $\eta\cdot\frac{1}{\eta^n}$, being respectively ϵ^{1-n} and η^{1-n} $(n<1)$ vanish *independently of each other.*

(b) If $n>1$, the limit to be discussed is that of

$$-\frac{1}{n-1}\left\{\frac{1}{(-\epsilon)^{n-1}} - \frac{1}{(-\alpha)^{n-1}} + \frac{1}{\beta^{n-1}} - \frac{1}{\eta^{n-1}}\right\},$$

which is infinite in general, when ϵ and η diminish independently and ultimately vanish in any arbitrary ratio of inequality. Hence the "General Value" is infinite.

But when n is odd or of the form $\frac{2\lambda+1}{2\mu+1}$, ($\lambda$ and μ being

integers and $\lambda > \mu$), the infinities will cancel each other when ϵ, η ultimately vanish in a ratio of equality.

The Principal Value is therefore finite, and

$$=-\frac{1}{n-1}\left[\frac{1}{\beta^{n-1}}-\frac{1}{\alpha^{n-1}}\right],$$

when n is odd or of the form $\frac{2\lambda+1}{2\mu+1}$, $(\lambda > \mu)$, and infinite if n is an even integer or of the form $\frac{2\lambda}{2\mu+1}$, $(\lambda > \mu)$.

(c) When $n=1$ we have to discuss the limit of

$$\int_{-\alpha}^{-\epsilon}\frac{dx}{x}+\int_{\eta}^{\beta}\frac{dx}{x},$$

or putting $x=-\xi$ in the first integral,

$$Lt\left\{\int_{\alpha}^{\epsilon}\frac{d\xi}{\xi}+\int_{\eta}^{\beta}\frac{dx}{x}\right\},\quad i.e.\ Lt\left\{\Big[\log\xi\Big]_{\alpha}^{\epsilon}+\Big[\log x\Big]_{\eta}^{\beta}\right\};$$

i.e. $$\log\frac{\beta}{\alpha}+Lt\log\frac{\epsilon}{\eta}.$$

This limit depends entirely upon the mode of approach of the ordinates N_rP_r, N_sP_s (Fig. 34) to the asymptote, and is undetermined till that is settled.

When $\frac{\epsilon}{p}=\frac{\eta}{q}$, where p, q are any finite quantities to be chosen, the limit is $\log\frac{\beta}{\alpha}+\log\frac{p}{q}$, and is arbitrary, depending upon the choice of p and q.

When p and q have been chosen equal, that is when ϵ, η vanish in a ratio of equality, the limit becomes $\log\frac{\beta}{\alpha}$.

Hence the General Value is an arbitrary quantity; the Principal Value is $\log\frac{\beta}{\alpha}$.

If n be of the form $\frac{2\lambda+1}{2\mu}$, $\frac{1}{x^n}$ becomes unreal when x is negative and the first integral is unreal, from $-\alpha$ to $-\epsilon$. Excluding this we are then only concerned with

$$Lt_{\eta=0}\int_{\eta}^{\beta}\frac{dx}{x^n},\quad i.e.\ -\frac{1}{n-1}Lt\left[\frac{1}{x^{n-1}}\right]_{\eta}^{\beta},$$

or $$-\frac{1}{n-1}Lt\left[\frac{1}{\beta^{n-1}}-\frac{1}{\eta^{n-1}}\right],$$

which is real and $= -\frac{1}{n-1}\frac{1}{\beta^{n-1}}$ if $n < 1$, and infinite if $n > 1$, and may be referred to as the Principal Value of the real part.

348. We next consider the case when the infinite value of $\phi(x)$ occurs *at* one of the limits, say b.

$\int_a^b \phi(x)\,dx$ is then to be interpreted as $Lt_{\epsilon=0}\int_a^{b-\epsilon}\phi(x)\,dx$, which is called the "Principal Value."

Let $\phi(x) = \frac{f(x)}{(x-b)^n}$, where $f(x)$ does not contain the factor $x-b$, and therefore does not vanish when $x=b$; and let n be positive. Then,

(a) if n be < 1 and if we can find some quantity γ between a and b such that throughout the range of values of x from γ to b the numerical value of $f(x)$ does not exceed some finite quantity A, the Principal Value will be finite.

For
$$\int_a^{b-\epsilon}\phi(x)\,dx = \int_a^{\gamma}\phi(x)\,dx + \int_{\gamma}^{b-\epsilon}\phi(x)\,dx.$$

The first of these two integrals is finite, and in the limit the numerical value of the second is not greater than

$$Lt_{\epsilon=0}A\int_{\gamma}^{b-\epsilon}\frac{dx}{(x-b)^n};$$

moreover
$$\int_{\gamma}^{b-\epsilon}\frac{dx}{(x-b)^n} = \frac{1}{1-n}\Big[(x-b)^{1-n}\Big]_{\gamma}^{b-\epsilon}$$
$$= \frac{1}{1-n}[(-\epsilon)^{1-n} - (\gamma-b)^{1-n}],$$

the limit of which, when $\epsilon=0$, is $-\frac{1}{1-n}(\gamma-b)^{1-n}$ and therefore finite.

(b) If, however, $n > 1$, and if we can find some quantity γ between a and b, such that throughout the range of values of x from γ to b the numerical value of $f(x)$ is greater than some finite quantity B throughout this range of values of x, and if $f(x)$ preserves the same sign throughout that range, the Principal Value of the integral will be infinite.

For, as before,

$$\int_a^{b-\epsilon} \phi(x)\,dx = \int_a^{\gamma} \phi(x)\,dx + \int_{\gamma}^{b-\epsilon} \phi(x)\,dx,$$

the first of the two integrals being finite.

But the numerical value of $Lt_{\epsilon=0}\int_{\gamma}^{b-\epsilon} \phi(x)\,dx$ is greater than the numerical value of

$$Lt_{\epsilon=0} B\int_{\gamma}^{b-\epsilon} \frac{dx}{(x-b)^n}$$

and
$$\int_{\gamma}^{b-\epsilon} \frac{dx}{(x-b)^n} = -\frac{1}{n-1}\left[\frac{1}{(-\epsilon)^{n-1}} - \frac{1}{(\gamma-b)^{n-1}}\right],$$

which becomes infinite when ϵ vanishes.

(*c*) Lastly, if $n=1$, and if, as in the last case (*b*), such a quantity γ can be found as there described, the numerical value of $Lt_{\epsilon=0}\int_{\gamma}^{b-\epsilon} \phi(x)\,dx$ is greater than the numerical value of

$$Lt_{\epsilon=0} B\int_{\gamma}^{b-\epsilon} \frac{dx}{x-b}, \quad \text{and} \quad \int_{\gamma}^{b-\epsilon} \frac{dx}{x-b} = \log_b \frac{\epsilon}{b-\gamma};$$

the numerical value of which is infinite, and therefore the Principal Value of $\int_a^b \phi(x)\,dx$ is in this case, also, infinite.

349. To sum up these Statements.*

If it be possible to find a quantity γ between a and b such that the numerical value of $\phi(x)(x-b)^n$, that is $f(x)$, does not exceed some finite quantity A throughout the range from γ to b, and if $n<1$, then the Principal Value of $\int_a^b \phi(x)\,dx$ is finite. If it be possible to find a quantity γ between a and b such that the numerical value of $\phi(x)(x-b)^n$ does exceed some finite quantity B throughout that range, and if $\phi(x)(x-b)^n$ does not change sign throughout that range, then if $n \nless 1$ the Principal Value of $\int_a^b \phi(x)\,dx$ will be infinite.

Obviously a similar rule holds for the lower limit by reversing the order of integration, *i.e.* interchanging the limits.

* Serret, *Calcul Intégral*, p. 100.

350. (α) Consider $$\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

Here the subject of integration, viz. $\frac{1}{\sqrt{1-x^2}}$, is infinite at the upper limit.

We have to consider $$Lt_{\epsilon=0}\int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}}.$$

Let $\phi(x)=\frac{1}{\sqrt{1-x^2}}$. Then $\phi(x)\sqrt{1-x}=\frac{1}{\sqrt{1+x}}$, which is <1 for the whole range $0<x \leqq 1$ or for any part of it, and the index of the factor $1-x$ is $\frac{1}{2}$, which is <1. Hence by Art. 348 (a) the Principal Value is finite.

It is of course obviously equal to $Lt_{\epsilon=0}\Big[\sin^{-1}x\Big]_0^{1-\epsilon}$,

i.e. $$Lt_{\epsilon=0}\{\sin^{-1}(1-\epsilon)-\sin^{-1}0\}=\sin^{-1}1-\sin^{-1}0=\frac{\pi}{2}.$$

(β) Consider $$\int_0^1 \frac{dx}{(1-x^2)^{\frac{3}{2}}}.$$

Here the subject of integration, viz. $\frac{1}{(1-x^2)^{\frac{3}{2}}}$, is infinite at the upper limit. Let $\phi(x)=\frac{1}{(1-x^2)^{\frac{3}{2}}}$. Then $\phi(x)(1-x)^{\frac{3}{2}}=\frac{1}{(1+x)^{\frac{3}{2}}}$, which is $\not< \frac{1}{2\sqrt{2}}$ and does not change sign for all values of x from $x=0$ to $x=1$ or for any part of that range. Also the index of the factor $1-x$ is $\frac{3}{2}$, *i.e.* >1. Hence, by Art. 348 (b), the Principal Value of this integral is ∞.

351. Consider $\int_0^1 \frac{\log x}{x^n}dx$, where $0<n<1$. (Serret, *C.I.*, p. 103.)

$$Lt_{x=0}\frac{\log x}{x^n}=\infty.$$

When x is made to approach zero indefinitely closely, the integrand, viz. $\phi(x)\equiv \log x/x^n$, increases numerically without limit. Take a quantity p lying between zero and $1-n$, so that p is positive and <1. Then $x^{p+n}\phi(x)\equiv x^p\log x$ has a turning point at $x=e^{-\frac{1}{p}}$, vanishes at $x=0$, and whilst numerically decreasing to zero as x diminishes from $e^{-\frac{1}{p}}$ to zero is always numerically less than $\frac{1}{pe}$. Moreover $p+n$ is a positive index less than 1.

Hence, by Art. 349, the Principal Value of this integral is finite.

352. Suppose that $\int_a^b f(x)\,dx$ has a value which is finite and determinate, when $f(x)$ becomes ∞ at $x=c$. ($a<c<b$.) Then this value must be

$$Lt_{\epsilon=0}\left\{\int_a^{c-p\epsilon} f(x)\,dx+\int_{c+q\epsilon}^b f(x)\,dx\right\}, \quad \text{..................(A)}$$

whatever may be the ratio of $p:q$, and if this limit were not independent of $p:q$, this General Value would not be determinate.

The Principal Value is the case when $p=q=1$,

$$Lt_{\epsilon=0}\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx\right] \text{..................... (B)}$$

The difference of these expressions A and B is

$$Lt_{\epsilon=0}\left[\int_{c-\epsilon}^{c-p\epsilon} f(x)\,dx + \int_{c+q\epsilon}^{c+\epsilon} f(x)\,dx\right],$$

and this limit must therefore vanish whatever the ratio $p:q$ may be if $\int_a^b f(x)\,dx$ is to have a finite and determinate value.

Cauchy* calls such integrals "Singular Definite" integrals [Intégrales définies singulières], viz. those in which the subject of integration becomes infinitely great at the same time that the limits differ by an infinitesimal.

In order that p and q shall disappear, the first integral must be independent of p, the second of q, when ϵ is indefinitely diminished.

For example, in the case

$$I=\int_a^b \frac{dx}{(x-c)^{\frac{1}{3}}}, \quad \text{where } a<c<b;$$

here

$$\int_{c-\epsilon}^{c-p\epsilon} \frac{dx}{(x-c)^{\frac{1}{3}}} = \frac{3}{2}\left[(x-c)^{\frac{2}{3}}\right]_{c-\epsilon}^{c-p\epsilon}$$

$$=\frac{3}{2}\left[(-p\epsilon)^{\frac{2}{3}} - (-\epsilon)^{\frac{2}{3}}\right],$$

and the limit when $\epsilon=0$ is zero and independent of p.

Similarly for $\int_{c+\epsilon}^{c+q\epsilon} \frac{dx}{(x-c)^{\frac{1}{3}}}$ the limit is independent of q, and the integral I is determinate.

See Williamson, *Int. Calc.*, pages 128-135; Moigno, *Calc. Intég.*; Serret, *Calc. Int.*, pages 91-107; Bertrand, *C.I.*, p. 117, for further information as to General and Principal Values.

353. Successive Integrations.

Successive integrations of a function may be expressed in terms of single integrals.

Let u be any function of x.

Then will

$$n!\frac{1}{D^{n+1}}u = x^n\frac{1}{D}u - {}^nC_1 x^{n-1}\frac{1}{D}xu + {}^nC_2 x^{n-2}\frac{1}{D}x^2u$$

$$-\ldots+(-1)^n\frac{1}{D}x^n u, \quad \text{where } D\equiv\frac{d}{dx}.$$

* Serret, *Calcul Intégral*, p. 107.

For
$$\frac{1}{D^2}u=\int\left[\int u\,dx\right]dx$$
$$=x\int u\,dx-\int xu\,dx$$
$$=x\frac{1}{D}u-\frac{1}{D}xu,$$
and the theorem is therefore true when $n=1$.

Also, integrating each term of the stated result, assumed for the moment true,
$$n!\frac{1}{D^{n+2}}u=\left[\frac{x^{n+1}}{n+1}\frac{1}{D}u-\frac{1}{D}\frac{x^{n+1}}{n+1}u\right]$$
$$-{}^nC_1\left[\frac{x^n}{n}\frac{1}{D}xu-\frac{1}{D}\frac{x^{n+1}}{n}u\right]$$
$$+{}^nC_2\left[\frac{x^{n-1}}{n-1}\frac{1}{D}x^2u-\frac{1}{D}\frac{x^{n+1}}{n-1}u\right]$$
$$+\dots$$
$$+(-1)^n\left[x\frac{1}{D}x^nu-\frac{1}{D}x^{n+1}u\right],$$
and
$$\frac{1}{n+1}-\frac{{}^nC_1}{n}+\frac{{}^nC_2}{n-1}-\dots+(-1)^n\frac{{}^nC_n}{1}$$
$$=\frac{1}{n+1}[1-{}^{n+1}C_1+{}^{n+1}C_2-\dots+(-1)^n\,{}^{n+1}C_n]$$
$$=\frac{1}{n+1}[(1-1)^{n+1}-(-1)^{n+1}]=\frac{-(-1)^{n+1}}{n+1}.$$

Hence, the right-hand members of the several brackets add up to
$$\frac{(-1)^{n+1}}{n+1}\frac{1}{D}x^{n+1}u.$$

Therefore, multiplying by $n+1$,
$$(n+1)!\frac{1}{D^{n+2}}u=x^{n+1}\frac{1}{D}u-{}^{n+1}C_1x^n\frac{1}{D}xu$$
$$+{}^{n+1}C_2x^{n-1}\frac{1}{D}x^2u-\dots+(-1)^{n+1}\frac{1}{D}x^{n+1}u,$$
i.e. if the theorem be true for the operator $\frac{1}{D^{n+1}}$, *i.e.* for $n+1$ integrations, it is true for $\frac{1}{D^{n+2}}$, *i.e.* for $n+2$ integrations; which establishes the inductive proof, for we have

shown that it is true if $n=1$, whence it is true for $n=2$, etc., and generally.

The theorem shows that a repeated integral such as

$$\iiiint u\,dx\,dx\,dx\,dx$$

can be expressed in terms of single integrations of

$$\int u\,dx,\quad \int xu\,dx,\quad \int x^2u\,dx,\quad \int x^3u\,dx.$$

This theorem is given by Todhunter, *Integral Calculus*, p. 72, *q.v.*

MISCELLANEOUS EXAMPLES.

1. Integrate (i) $\displaystyle\int\frac{\sin^2 x\,dx}{(x\cos x-\sin x)^2}$; [L.]

(ii) $\displaystyle\int\frac{\log x\,dx}{x^2(1-\log x)^2}$. [L.]

2. Prove that $\displaystyle\int_{-c}^{c}\frac{x(c^2-x^2)\,dx}{(b^2+c^2-2bx)^{\frac{3}{2}}}=\frac{4}{5}\frac{c^5}{b^4}\quad(b>c).$ [L.]

3. If $X=a+2bx+cx^2$, show that $\displaystyle\int\frac{dx}{X^n}$ can be made to depend upon $\displaystyle\int\frac{dx}{X^{n-1}}$.

Find a reduction formula for $\displaystyle\int\cos mx\sin^n x\,dx$, and apply it to the case $n=4$. [L.]

4. Evaluate $\displaystyle\int\frac{2x-3}{5x^2-16x+14}\,\frac{dx}{\sqrt{3x^2-10x+9}}$. [L.]

5. Prove that

$$\int_0^b u\frac{d^nv}{dx^n}\,dx$$

can be made to depend upon

$$\int_0^b v\frac{d^nu}{dx^n}\,dx.$$

Hence show that if $f(x)$ be an arbitrary polynomial of degree $n-1$, and

$$P_n(x)=\frac{d^n(Ax^2+Bx+C)^n}{dx^n},$$

then

$$\int_\alpha^\beta f(x)P_n(x)\,dx=0,$$

where α, β are the roots, considered real, of the quadratic

$$Ax^2+Bx+C=0.$$

6. Prove that the effect of the operation $p\frac{d}{dt}+q$ on a periodic function $a\cos(nt+\epsilon)$ is to multiply the amplitude a by $\sqrt{p^2n^2+q^2}$, and to increase the angle $nt+\epsilon$ by $\tan^{-1}\frac{pn}{q}$.

Write down the effect of the operation

$$\left(p\frac{d}{dt}+q\right)\Big/\left(P\frac{d}{dt}+Q\right),$$

and generally, of the operation

$$\left(a+\beta\frac{d}{dt}+\gamma\frac{d^2}{dt^2}+\dots\right)\Big/\left(A+B\frac{d}{dt}+C\frac{d^2}{dt^2}+\dots\right)$$

on the same periodic function. [INT. ARTS, LONDON.]

7. When $y^2=ax^2+2bx+c$, prove that

$$\int\frac{dx}{y}=\frac{1}{\sqrt{a}}\operatorname{ch}^{-1}\frac{y\sqrt{a}}{\sqrt{ac-b^2}},\quad \frac{1}{\sqrt{a}}\operatorname{sh}^{-1}\frac{y\sqrt{a}}{\sqrt{b^2-ac}}\quad\text{or}\quad\frac{1}{\sqrt{-a}}\sin^{-1}\frac{y\sqrt{-a}}{\sqrt{b^2-ac}},$$

the real form to be chosen, and deduce the value of the integral in the degenerate case when $a=0$. [INT. ARTS, LONDON.]

8. Find the limiting value of $(n!)^{\frac{1}{n}}/n$, when n is infinite.

9. Find the limiting value when n is infinite of the n^{th} part of the sum of the n quantities

$$\frac{n+1}{n},\ \frac{n+2}{n},\ \frac{n+3}{n},\dots\frac{n+n}{n},$$

and show that it bears to the limiting value of the n^{th} root of the product of the same quantities the ratio $3e:8$, where e is the base of the Napierian logarithms. [OXFORD 1886, and I. P., 1911.]

10. If na is always equal to unity, and n is indefinitely great, show that the limiting value of the product

$$(1+a^4)\{1+(2a)^4\}^{\frac{1}{2}}\{1+(3a)^4\}^{\frac{1}{3}}\{1+(4a)^4\}^{\frac{1}{4}}\dots\{1+(na)^4\}^{\frac{1}{n}}$$

is $$e^{\frac{\pi^2}{48}}.$$ [OXFORD, 1888.]

11. Show that the limit of the sum of n terms of the series

$$\frac{n^3}{(n^2+1^2)(n^2+2.1^2)}+\frac{n^3}{(n^2+2^2)(n^2+2.2^2)}+\dots+\frac{n^3}{(n^2+n^2)(n^2+2n^2)},$$

when n is infinite, is

$$\sqrt{2}\tan^{-1}\sqrt{2}-\frac{\pi}{4}.$$ [γ, 1901.]

12. Find $Lt_{n=\infty}\left[\frac{\sqrt{n-a}}{n-c}+\frac{\sqrt{2n-a}}{2n-c}+\frac{\sqrt{3n-a}}{3n-c}+\dots+\frac{\sqrt{n^2-a}}{n^2-c}\right]$.

13. Find the limiting value, when n is infinite, of

$$\left\{\tan\frac{\pi}{2n}\cdot\tan\frac{2\pi}{2n}\cdot\tan\frac{3\pi}{2n}\dots\tan\frac{n\pi}{2n}\right\}^{\frac{1}{n}}.$$

[OXFORD I. P., 1903.]

14. Show that the limit of the product

$$\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{\frac{1}{2}}\left(1+\frac{3}{n}\right)^{\frac{1}{3}}\dots\left(1+\frac{n}{n}\right)^{\frac{1}{n}},$$

when n is increased indefinitely, is $e^{\frac{\pi^2}{12}}$. [COLLEGES, 1896.]

15. Find the limit, when n is indefinitely increased, of

$$\frac{1}{n}\left\{\sec\frac{x}{n}+\sec\frac{2x}{n}+\dots+\sec\frac{(n-1)x}{n}\right\},$$

where x is $<\frac{\pi}{2}$.

Examine the case when $x>\frac{\pi}{2}$.

16. Find the limiting value of

$$2\log n-\log\left[(1+n^2)^{\frac{1}{n}}(2^2+n^2)^{\frac{1}{n}}\dots(2n^2)^{\frac{1}{n}}\right],$$

when n is indefinitely increased. [OXFORD I. P., 1900.]

17. Show from elementary considerations that when n increases indefinitely,

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n$$

approaches a finite limit intermediate between $\frac{1}{2}$ and 1.

[ST. JOHN'S, 1884.]

18. If $f(x)=f(a+x)$, show that

$$\int_a^{na} f(x)\,dx=(n-1)\int_0^a f(x)\,dx,$$

and illustrate geometrically. [OXFORD I. P., 1888.]

19. Prove that $\displaystyle\int_0^a \phi(x)\,dx=\int_0^a \phi(a-x)\,dx$,

and show that (1) $\displaystyle\int_0^\pi \frac{x\sin^n x}{1+\cos^2 x}\,dx=\frac{\pi}{2}\int_0^\pi\frac{\sin^n x}{1+\cos^2 x}\,dx$,

and evaluate this integral when $n=1$ and when $n=3$.

$$(2)\quad \int_0^{\frac{\pi}{2}}\log\frac{(1+\sin^2 x)^2}{1+\frac{1}{8}\sin^2 2x}\,dx=\frac{\pi}{2}\log 2.$$

20. If $\phi(x)=-\phi(2a-x)$, show that

$$\int_b^{2a}\phi(x)\,dx=-\int_0^b\phi(x)\,dx.$$

[COLLEGES, 1886.]

21. Prove that $\displaystyle\int_b^c \frac{\phi(x-b)}{\phi(c-x)}\,dx = \int_b^c \frac{\phi(c-x)}{\phi(x-b)}\,dx$,

provided $\phi(x)$ remains finite when x vanishes. [St. John's, 1883.]

22. Prove that if $\phi(x)$, $\psi(x)$, $\phi'(x)$, $\psi'(x)$ be continuous and finite from $x=a$ to $x=b$,

$$\int_a^b \phi'(x)\psi'(x)\,dx = \phi\{a+\theta(b-a)\}[\psi(b)-\psi(a)],$$

where θ is a positive proper fraction.

23. Prove that $\displaystyle\int_a^{\pi-a} x f(\sin x)\,dx = \frac{\pi}{2}\int_a^{\pi-a} f(\sin x)\,dx$.

[St. John's, 1883.]

24. Show that

$$\int_a^b f^n(x)\phi(c-x)\,dx - \int_a^b f(x)\phi^n(c-x)\,dx$$
$$= \sum_{r=1}^{r=n}\Big[f^{r-1}(x)\phi^{n-r}(c-x)\Big]_a^b,$$

where $f^n(x)$ means the n^{th} differential coefficient of $f(x)$. [γ, 1893.]

25. Show that, if $\displaystyle\psi(x) = \int_0^x \phi(x)\phi'(2a-x)\,dx$,

then $\psi(2a) - 2\psi(a) = [\phi(a)]^2 - \phi(0)\phi(2a)$. [Trinity, 1895.]

26. If $f(x, y)$ is symmetrical in x and y, prove that

$$\int_{1-b}^b x f(x, 1-x)\,dx = \frac{1}{2}\int_{1-b}^b f(x, 1-x)\,dx.$$

[Colleges α, 1889.]

27. Examine under what limitations the formula

$$\int_b^a \phi(x)\,dx = \int_b^c \phi(x)\,dx + \int_c^a \phi(x)\,dx$$

holds good.

Show that $\displaystyle\int_{-\infty}^{\infty}\left(x+\frac{1}{x}\right)\phi\left(x-\frac{1}{x}\right)\frac{dx}{x} = 2\int_{-\infty}^{\infty}\phi(x)\,dx$.

[Math. Tripos, 1884.]

28. If

$$A_n = 1+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2n+1},$$
$$B_m = \frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2m},$$

show that when n and m are both infinite and the ratio $n:m$ tends to a limit k^2, $A_n - B_m = \log 2 + \log k$. [Colleges α, 1888.]

29. Show that

$$\frac{1}{D^n} e^{ax} \frac{\sin}{\cos}(bx+a) = e^{ax} \frac{\frac{\sin}{\cos}\left(bx + a - n\tan^{-1}\frac{b}{a}\right)}{(a^2+b^2)^{\frac{n}{2}}}$$
$$+ A_0 + A_1 x + A_2 x^2 + \ldots + A_{n-1}x^{n-1},$$

A_0, A_1, etc., being arbitrary constants, and also that it may be written

$$e^{ax}\frac{1}{(D+a)^n}\frac{\sin}{\cos}(bx+a) + A_0 + A_1 x + \ldots + A_{n-1}x^{n-1},$$

and explain how the latter operation is to be conducted.

30. If $$I_1 = \int_0^{\frac{\pi}{2}} \log(1 + a_1 \sin^2\theta)\, d\theta,$$

show that $$I_1 = \frac{\pi}{4}\log(1+a_1) + \tfrac{1}{2} I_2,$$

where $$I_2 = \int_0^{\frac{\pi}{2}} \log(1 + a_2\sin^2\theta)\, d\theta$$

and $$4(1+a_2)(1+a_1) = (2+a_1)^2.$$

Hence show that

$$I_1 = \frac{\pi}{4}\log\left[(1+a_1)(1+a_2)^{\frac{1}{2}}(1+a_3)^{\frac{1}{4}}\ldots\right],$$

where $$4(1+a_{r+1})(1+a_r) = (2+a_r)^2.$$

31. Show that if $n > 1$,

$$\int_0^1 \tanh\frac{1}{nx}\, dx < \frac{1}{n}(1+\log n).$$ [Oxford I. P., 1911.]

32. How is the equation

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

to be interpreted when $f(x)$ is not a single-valued function?

Illustrate your answer by evaluating

$$\int_0^{\frac{1}{2}n\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta},$$

where a and b are real and n is a positive integer.

[Oxford I. P., 1912.]

33. Remembering that $\int_0^\infty$ means the limit tended to by $\int_\epsilon^\eta$ as the first of the two positive quantities ϵ, η tends to zero, and the second to infinity, prove that if $a > 1$, the value of

$$\int_0^\infty (a^n e^{-ax} - e^{-x}) x^{n-1}\, dx$$

is zero if $n > 0$, but not if $n = 0$. [Oxford I. P., 1917.]

34. If $f(x)$ be any function of x which can be put into partial fractions of the form $\frac{A}{a^2-x^2}$, then will

$$\int_0^\infty \frac{f(x)}{1+x^2}\,dx = \frac{\pi}{2} f(\sqrt{-1}).$$ [R. P.]

35. If $0<b<a,\quad a_1=\frac{1}{2}(a+b),\quad b_1=(ab)^{\frac{1}{2}}$,

prove that $b<b_1<a_1<a,\quad a_1-b_1<\frac{1}{2}(a-b)$.

Show that if $(a+b)\tan\theta\cot\phi = a - b\tan^2\theta$,

then $$\int_0^{\frac{\pi}{2}} (a^2\cos^2\theta + b^2\sin^2\theta)^{-\frac{1}{2}}\,d\theta = \int_0^{\frac{\pi}{2}} (a_1^2\cos^2\phi + b_1^2\sin^2\phi)^{-\frac{1}{2}}\,d\phi.$$

[MATH. TRIP., PART II., 1915.]

36. Show that

$$\int_0^{\frac{\pi}{2}} \sin\theta\tan^{-1}(\sin\theta)\,d\theta = \frac{\pi}{2}(\sqrt{2}-1).$$

[MATH. TRIP., 1882.]

37. Show how to evaluate $\int R(x, y)\,dx$, where $R(x, y)$ denotes any rational algebraic function of the coordinates x, y of a point on a conic. [ST. JOHN'S, 1891.]

38. Show that if a be greater than unity,

$$\int_0^\pi \frac{x\,dx}{a^2-\cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}}.$$ [OXF. I. P., 1890.]

39. Prove that

$$\int_0^\infty \phi(x)\,dx = \int_0^1 \left\{\phi(x) + \frac{1}{x^2}\phi\left(\frac{1}{x}\right)\right\}dx.$$

[ST. JOHN'S COLL., 1882.]

40. Prove that $$\int_0^\pi \frac{x\sin x\,dx}{2+\cos 2x} = \frac{\pi}{\sqrt{2}}\tan^{-1}\sqrt{2}.$$ [OXF. I. P., 1889.]

41. Integrate $\int_a^b \frac{dx}{c^2-x^2}$, when c lies between a and b. [R. P.]

42. Prove that

$$\int_0^1 x^n(2-x)^n\,dx = 2^{2n}\int_0^1 x^n(1-x)^n\,dx.$$

[OXF. II. P., 1886.]

43. Prove that

$$\int_a^{\frac{\pi}{2}-a} 2\cos^2 x\,\phi(\sin 2x)\,dx = \int_{2a}^{\frac{\pi}{2}} \phi(\sin x)\,dx.$$

[ST. JOHN'S.]

44. Show that

$$\theta = \int_{a-c}^{r} \frac{a^2 - c^2 + r^2}{r\sqrt{2(a^2+c^2)r^2 - (a^2-c^2)^2 - r^4}}\,dr$$

is the equation of a circle. [MATH. TRIP., 1882.]

45. Find the integrals

$$(a) \quad \int \left\{ \left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x \right\} \log x \, dx\,;$$

$$(b) \quad \int \frac{dx}{x} \sqrt{x^2(x-3)^2 - 4x}\,;$$

$$(c) \quad \int \frac{dx}{\sin^3 x \tan^n \frac{x}{2}}.$$

[ST. JOHN'S, 1887.]

46. Prove that

$$\int_0^a \log(1 + \tan a \tan x)\,dx = a \log \sec a.$$

[COLLS., 1896.]

47. Evaluate

(i) $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx.$ [TRIN., 1891.]

(ii) $\displaystyle\int e^x \frac{x^2+1}{(x+1)^2}\,dx.$ [HALL, *I.C.*]

(iii) $\displaystyle\int \frac{e^{x\sqrt{2}}}{1 - x\sqrt{2}} \cdot \frac{1 - x^2}{\sqrt{1 - 2x^2}}\,dx.$

(iv) $\displaystyle\int \frac{xe^x\,dx}{(e^x - 1)^3}.$ [HALL, *I.C.*]

(v) $\displaystyle\int \frac{dx}{(1 + x\tan x)^2}.$ [TRIN., 1891.]

(vi) $\displaystyle\int_e^{e^2} \frac{\sec x \operatorname{cosec} x}{\log \tan x}\,dx.$ [TRIN., 1884.]

(vii) $\displaystyle\int \frac{\log(1+x^2)}{\sqrt{1-x}}\,dx.$

48. If

$$I_n \equiv \int_{-1}^{1} (1 - x^2)^n \cos ax\,dx,$$

show that $\quad a^2 I_n = 2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2},$

provided $n > 1$.

Show also that $I_n = \dfrac{n!}{a^{2n+1}}\{f(a)\sin a + g(a)\cos a\},$

where $f(a)$ and $g(a)$ are algebraic functions of a, of degrees $\ngtr n$, with integral coefficients. [TRIN., 1892.]

49. Show that

$$(i) \quad \int x^3 \sqrt{\frac{1+x^2}{1-x^2}}\,dx = \tfrac{1}{2}\tan^{-1}\sqrt{\frac{1+x^2}{1-x^2}} - \tfrac{1}{4}(2+x^2)\sqrt{1-x^4}.$$

$$(ii) \quad \int \frac{x^2+1}{x^2-1} \frac{dx}{\sqrt{1 - ax^2 + x^4}} = \frac{1}{\sqrt{a-2}} \cos^{-1} \frac{x\sqrt{a-2}}{x^2-1} \quad (a > 2).$$

[HALL, *I.C.*, p. 325 and p. 346.]

50. Show that

$$\int_0^1 x^{cx^a}\,dx = 1 - \frac{c}{(a+1)^2} + \frac{c^2}{(2a+1)^3} - \frac{c^3}{(3a+1)^4} + \dots.$$

[ANGLIN.]

51. Prove that

(i) $$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+\sin^2\theta}} = 1^2 - \left(\frac{1}{2}\right)^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 - \dots.$$

(ii) $$1 + \frac{1}{2^2}\left(\frac{1}{2}\right)^2 + \frac{1}{3^2}\left(\frac{1\,.\,3}{2\,.\,4}\right)^2 + \dots = \frac{11}{\pi} - 4.$$ [ANGLIN.]

52. If $$\phi(x) = a_1 x + \frac{2}{3}a_3 x^3 + \frac{2\,.\,4}{3\,.\,5}a_5 x^5 + \dots,$$

prove that

(i) $$\int_0^{\frac{\pi}{2}} \cos^2\theta\,\phi(\sin\theta)\,d\theta = \frac{1}{3}a_1 + \frac{1}{5}\left(\frac{2}{3}\right)^2 a_3 + \frac{1}{7}\left(\frac{2\,.\,4}{3\,.\,5}\right)^2 a_5 + \dots.$$

(ii) $$\frac{\pi}{2} = 1 + \frac{1}{3}\,.\,1^2 + \frac{1}{5}\left(\frac{2}{3}\right)^2 + \frac{1}{7}\left(\frac{2\,.\,4}{3\,.\,5}\right)^2 + \dots.$$

(iii) $$\pi - 3 = \frac{1}{3^2}\,.\,1^2 + \frac{1}{5^2}\left(\frac{2}{3}\right)^2 + \frac{1}{7^2}\left(\frac{2\,.\,4}{3\,.\,5}\right)^2 + \dots.$$ [ANGLIN.]

CHAPTER X.

DIFFERENTIATION, ETC., UNDER AN INTEGRATION SIGN.

354. **Differentiation of a Definite Integral with regard to a Parameter.**

A definite integral is by its nature independent of the value of the particular variable in terms of which the integration is effected, and its value depends upon any other quantities which may occur in the integrand or in the limits.

First, let us consider the differentiation with regard to c of the integral $u = \int_a^b \phi(x, c)\,dx$, where a and b are each finite and independent of c. We shall suppose also that $\phi(x, c)$ is *single-valued, finite* and *continuous*, as also its *differential coefficient with regard to c for the range of values of x from a to b*. When c changes to $c+\delta c$, suppose that the consequent change of u is to $u+\delta u$.

Then $$u+\delta u = \int_a^b \phi(x, c+\delta c)\,dx$$

and $$\delta u = \int_a^b [\phi(x, c+\delta c) - \phi(x, c)]\,dx.$$

Now $$\phi(x, c+\delta c) = \phi(x, c) + \delta c \phi'(x, c+\theta\,\delta c),$$

where the accent represents differentiation of $\phi(x, c)$ with regard to c, and θ is a positive proper fraction, $c+\theta\,\delta c$ being written for c after the differentiation is performed, *i.e.*

$$\frac{\partial u}{\partial c} = Lt_{\delta c=0}\frac{\delta u}{\delta c} = Lt_{\delta c=0}\int_a^b \phi'(x, c+\theta\,\delta c)\,dx = \int_a^b \frac{\partial \phi(x, c)}{\partial c}\,dx.$$

[See Arts. 1898, 1902, Vol. II.]

355. Next, let a and b be also functions of c.

Then $u+\delta u=\int_{a+\delta a}^{b+\delta b}\phi(x, c+\delta c)\,dx$

and
$$\delta u=\int_{a+\delta a}^{b+\delta b}\phi(x, c+\delta c)\,dx-\int_a^b\phi(x, c)\,dx$$
$$=\int_b^{b+\delta b}\phi(x, c+\delta c)\,dx+\int_{a+\delta a}^{a}\phi(x, c+\delta c)\,dx$$
$$+\int_a^b[\phi(x, c+\delta c)-\phi(x, c)]\,dx.$$

Now
$$\int_b^{b+\delta b}\phi(x, c+\delta c)\,dx=\phi(b+\theta_1\delta b,\ c+\delta c)\,\delta b \quad \text{(by Art. 332)}$$
and
$$\int_{a+\delta a}^{a}\phi(x, c+\delta c)\,dx=-\phi(a+\theta_2\delta a,\ c+\delta c)\,\delta a,$$
where θ_1 and θ_2 are positive proper fractions.

Also
$$Lt_{\delta c=0}\int_a^b\frac{[\phi(x, c+\delta c)-\phi(x, c)]\,dx}{\delta c}$$
has been discussed in the last article.

Hence, dividing the expression for δu by δc and taking the limit, when δc is indefinitely diminished,
$$\frac{\partial u}{\partial c}=\int_a^b\frac{\partial\phi(x, c)}{\delta c}\,dx+\phi(b, c)\frac{db}{dc}-\phi(a, c)\frac{da}{dc},$$
and the conditions under which this is true have been stated above, viz. $\phi(x, c)$ and $\dfrac{\partial\phi(x, c)}{\partial c}$ are *single-valued, finite* and *continuous* functions of x *throughout the finite range* $x=a$ *to* $x=b$, inclusive.

This is a case of the theorem on partial differentiation, *Diff. Calc.*, Art. 160, viz.
$$\frac{du}{dc}=\frac{\partial u}{\partial c}+\frac{\partial u}{\partial a}\cdot\frac{da}{dc}+\frac{\partial u}{\partial b}\cdot\frac{db}{dc}.$$

356. **Geometrical Meaning of the Process.**

We next examine the geometrical meaning of this differentiation.

Let $\alpha\beta$, $\alpha'\beta'$ be the respective graphs of
$$y=\phi(x, c), \quad y=\phi(x, c+\delta c).$$

Let the ordinates of both curves be drawn at the points

$$x=a, \quad x=b, \quad x=a+\delta a, \quad x=b+\delta b,$$

viz. $\qquad A\alpha\gamma', \quad B\beta\delta', \quad A'\gamma\alpha', \quad B'\delta\beta',$

respectively. Let NQP be any other ordinate, and draw αS, βR parallel to the x-axis. Then $\int_a^b \phi(x, c)\,dx$ is represented by the area $AB\beta\alpha$. We have to differentiate this area with

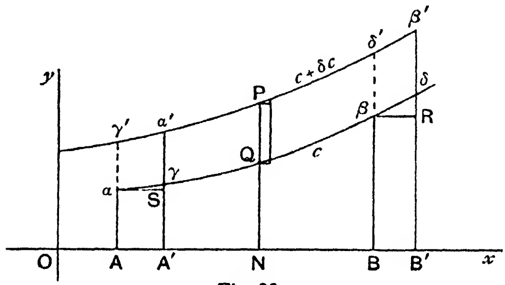


Fig. 36.

regard to c. When c is increased to $c+\delta c$, a and b being both dependent upon c, area $AB\beta\alpha$ is changed to $A'B'\beta'\alpha'$, and

$$\frac{\partial}{\partial c} \text{ area } AB\beta\alpha = Lt_{\delta c=0}\frac{\text{area } A'B'\beta'\alpha' - \text{area } AB\beta\alpha}{\delta c}$$

$$= Lt_{\delta c=0}\frac{\beta\delta'\gamma'\alpha + BB'\beta'\delta' - AA'\alpha'\gamma'}{\delta c}$$

Now

$$Lt\frac{\beta\delta'\gamma'\alpha}{\delta c} = Lt\left[\frac{\int_a^b (NP-NQ)\,dx}{\delta c}\right] = Lt\int_a^b \frac{\phi(x, c+\delta c)-\phi(x, c)}{\delta c}\,dx$$

$$= \int_a^b \frac{\partial\phi(x, c)}{\partial c}\,dx.$$

Also $\qquad Lt\dfrac{BB'\beta'\delta'}{\delta c} = Lt\,\dfrac{BB'R\beta + \beta R\beta'\delta'}{\delta c}$

$$= Lt\,\frac{\phi(b, c)\,\delta b + \beta R\beta'\delta'}{\delta c}$$

$$= \phi(b, c)\frac{db}{dc} + Lt\frac{\beta R\beta'\delta'}{\delta c};$$

and $$Lt\frac{AA'a'\gamma'}{\delta c}=Lt\frac{AA'Sa+aSa'\gamma'}{\delta c}$$

$$=Lt\frac{\phi(a, c)\,\delta a+aSa'\gamma'}{\delta c}$$

$$=\phi(a, c)\frac{da}{dc}+Lt\frac{aSa'\gamma'}{\delta c};$$

$$\therefore\ \frac{d}{dc}\text{area } AB\beta a=\int_a^b\frac{\partial\phi(x, c)}{dc}dx+\phi(b, c)\frac{db}{dc}-\phi(a, c)\frac{da}{dc}$$

$$+Lt\frac{\text{area }\beta R\beta'\delta'-\text{area }aSa'\gamma'}{\delta c}.$$

Now, if the terminal ordinates $A'a'$, Aa and $B'\beta'$, $B\beta$ are finite, as supposed, the portions $\beta R\beta'\delta'$ and $aSa'\gamma'$ are both of the second order of infinitesimals, for their breadths and greatest lengths are both first order infinitesimals; and therefore, when divided by δc, they still remain of the first order of infinitesimals and disappear when the limit is taken.

$$\therefore\ \frac{d}{dc}\int_a^b\phi(x, c)\,dx=\int_a^b\frac{\partial\phi(x, c)}{\partial c}dx+\phi(b, c)\frac{db}{dc}-\phi(a, c)\frac{da}{dc}.$$

The student will see that the truth of this theorem could not be asserted without further examination if any of the ordinates of the figure became infinite, or if either of the graphs were discontinuous, or if either graph were cut by an ordinate in more places than one for any position between the extreme ordinates of the portion considered.

When one of the limits is infinite the theorem may still be true, but special consideration is needed in each case.

357. If the integral to be differentiated with respect to c be "indefinite," *i.e.* the limits not stated, say

$$u=\int\phi(x, c)\,dx+A,$$

where A is an arbitrary constant, then

$$\frac{du}{dc}=\int\frac{\partial\phi(x, c)}{\partial c}dx+\frac{\partial A}{\partial c},$$

and A being an arbitrary constant as regards x, $\frac{\partial A}{\partial c}$ is also

an arbitrary constant as regards x; and we may write the result as

$$\frac{du}{dc}=\int\frac{\partial\phi(x, c)}{\partial c}\,dx+A',$$

where A' is an arbitrary constant.

358. **Integration of a Definite Integral with regard to a Parameter.**

Take the integral $$u=\int_a^b\phi(x, c)\,dx,$$

where a and b are not functions of c.

Then, by the previous articles,

$$\frac{\partial}{\partial c}\int_a^b\left[\int\phi(x, c)\,dc\right]dx=\int_a^b\frac{\partial}{\partial c}\left[\int\phi(x, c)\,dc\right]dx$$

$$=\int_a^b\phi(x, c)\,dx=u:$$

$$\therefore\quad\int u\,dc=\int_a^b\left[\int\phi(x, c)\,dc\right]dx,$$

i.e. $$\int\left[\int_a^b\phi(x, c)\,dx\right]dc=\int_a^b\left[\int\phi(x, c)\,dc\right]dx.$$

359. Supposing that instead of an *indefinite* integration of u we require a *definite* integration between c_0 and c, say, regarded as independent of a and b, then we shall have in general

$$\int_{c_0}^c\left[\int_a^b\phi(x, c)\,dx\right]dc=\int_a^b\left[\int_{c_0}^c\phi(x, c)\,dc\right]dx$$

that is the order of integration is immaterial.

For putting $$\int_{c_0}^c\phi(x, c)\,dc=f(x, c),\text{ say,}$$

then $$\int_a^b\left[\int_{c_0}^c\phi(x, c)\,dc\right]dx=\int_a^b f(x, c)\,dx,$$

and $$\frac{\partial}{\partial c}\int_a^b\left[\int_{c_0}^c\phi(x, c)\,dc\right]dx=\frac{\partial}{\partial c}\int_a^b f(x, c)\,dx$$

$$=\int_a^b\frac{\partial f(x, c)}{\partial c}\,dx$$

$$=\int_a^b\phi(x, c)\,dx,$$

also $$\frac{\partial}{\partial c}\int_{c_0}^c\left[\int_a^b\phi(x, c)\,dx\right]dc=\int_a^b\phi(x, c)\,dx.$$

Hence both $$\int_{c_0}^{c}\left[\int_a^b \phi(x, c)\,dx\right]dc$$

and $$\int_a^b\left[\int_{c_0}^{c} \phi(x, c)\,dc\right]dx$$

have the same differential coefficient with regard to c, and both vanish when $c=c_0$. Hence they are equal.

This theorem may be written

$$\int_{c_0}^{c}\int_a^b \phi(x, c)\,dc\,dx=\int_a^b\int_{c_0}^{c} \phi(x, c)\,dx\,dc,$$

and expresses that the order of the integrations may be changed. The theorem presupposes that the limits of integration c_0 and c are independent of the limits a and b, and also that $\phi(x, c)$ remains single-valued, finite and continuous for all values of the quantities x and c between or at their limits.

360. Notation.

The notation of this "double integration" calls for explanation. It will be noticed that we have written

$$\int_{c_0}^{c}\left[\int_a^b \phi(x, c)\,dx\right]dc \quad \text{as} \quad \int_{c_0}^{c_1}\int_a^b \phi(x, c)\,dc\,dx,$$

inverting the order of the dx and dc. The order of writing these symbols does not appear to be universally agreed upon, some authors adopting the opposite order. For the sake of clearness we may state that throughout this book the *right-hand element and the right-hand integration sign refer to the first operation*, the *left-hand element and the left-hand integration sign refer to the second.*

Thus $\int_{x_0}^{x_1}\int_{y_0}^{y_1} \phi(x, y)\,dx\,dy$ will mean that

(1) $\phi(x, y)$ is to be integrated with regard to y, keeping x constant, between limits $y=y_0$, $y=y_1$.

(2) That the result obtained is then to be integrated with regard to x between limits $x=x_0$ and $x=x_1$.

A notation which carries its own explanation, and used when there is any fear of confusion, is

$$\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\, \phi(x, y).$$

361. **Geometrical Interpretation.**

Writing y where we had c in $\phi(x, c)$ and dy for dc, we have to establish the theorem

$$\int_{c_0}^{c}\left[\int_a^b \phi(x, y)\, dx\right]dy = \int_a^b \left[\int_{c_0}^{c} \phi(x, y)\, dy\right]dx.$$

Imagine the rectangular space bounded by

$$x=a, \quad x=b, \quad y=c_0, \quad y=c$$

to be divided up into infinitesimal rectangles by two families of straight lines, the first set being equidistant from each other and parallel to the x-axis, and the second set being equidistant from each other and parallel to the y-axis, the distance between consecutive lines of each family being infinitesimal.

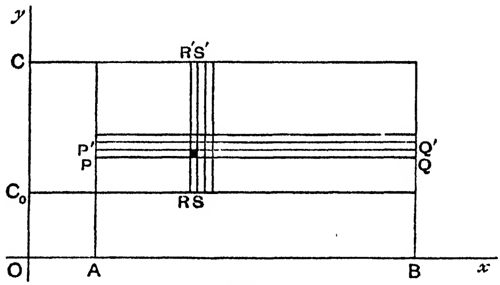


Fig. 37.

Imagine that we have to find the mass of this rectangle, regarded as of variable density, such that $\phi(x, y)$ is the density at any point (x, y), and that the elementary rectangle whose corners are (x, y), $(x+\delta x, y)$, $(x+\delta x, y+\delta y)$ and $(x, y+\delta y)$, and whose area is $\delta x\, \delta y$, is so small that the density may be taken uniform all over it, or, which will amount to the same thing, that the density at any point of the small area $\delta x\, \delta y$ differs from that at x, y by an infinitesimal.

The mass of this small rectangle will be, to the second order of infinitesimals, $\phi(x, y)\, \delta x\, \delta y$. Let $PQQ'P'$ and $RSS'R'$ be the two elementary strips whose common element is $\delta x\, \delta y$. Then, in adding up all the elements of mass along the strip $PQQ'P'$, we have

$$\left[\int_a^b \phi(x, y)\, dx\right]\delta y$$

in the limit when δx is indefinitely small.

Then, if we sum the strips from $y=c_0$ to $y=c$, we have in the limit, when δy is indefinitely small,

$$\int_{c_0}^{c}\left[\int_a^b \phi(x, y)\,dx\right]dy.$$

But if we first sum the elements $\phi(x, y)\delta x\,\delta y$ along the strip $RSS'R'$, we have in the limit, when δy is indefinitely small,

$$\left[\int_{c_0}^{c} \phi(x, y)\,dy\right]\delta x.$$

And if we sum these strips from $x=a$ to $x=b$ we have in the limit, when δx is indefinitely small,

$$\int_a^b\left[\int_{c_0}^{c_1} \phi(x, y)\,dy\right]dx.$$

And as the order of addition of these elements is obviously immaterial we perceive that these two results must be equal. Hence the truth of the theorem, provided $\phi(x, y)$ be finite for all points of the rectangle. [See Art. 1899, Vol. II.]

362. **Successive Differentiation.**

Having established the equation

$$\frac{d}{dc}\int_a^b \phi(x, c)\,dx = \int_a^b \frac{\partial\phi(x, c)}{\partial c}\,dx + \phi(b, c)\frac{db}{dc} - \phi(a, c)\frac{da}{dc}$$

we can differentiate again and again and successively obtain the second, third, etc., differential coefficients with regard to c. The successive results however, in general form, rapidly get complicated. Thus, for instance, we have

$$\begin{aligned}
&\frac{d^2}{dc^2}\int_a^b \phi(x, c)\,dx \\
&\quad= \frac{d}{dc}\left[\int_a^b \frac{\partial\phi(x, c)}{\partial c}\,dx\right] + \frac{d}{dc}\left[\phi(b, c)\frac{db}{dc}\right] - \frac{d}{dc}\left[\phi(a, c)\frac{da}{dc}\right] \\
&\quad= \int_a^b \frac{\partial^2\phi(x, c)}{\partial c^2}\,dx + \frac{\partial\phi(b, c)}{\partial c}\frac{db}{dc} - \frac{\partial\phi(a, c)}{\partial c}\frac{da}{dc} \\
&\quad\quad+ \frac{\partial\phi(b, c)}{\partial b}\left(\frac{db}{dc}\right)^2 + \frac{\partial\phi(b, c)}{\partial c}\frac{db}{dc} + \phi(b, c)\frac{d^2b}{dc^2} \\
&\quad\quad- \frac{\partial\phi(a, c)}{\partial a}\left(\frac{da}{dc}\right)^2 - \frac{\partial\phi(a, c)}{\partial c}\frac{da}{dc} - \phi(a, c)\frac{d^2a}{dc^2},
\end{aligned}$$

which reduces to an expression with seven terms.

Similarly, the third and other differential coefficients may be found when necessary.

In particular cases there may be considerable simplification.

363. Many important results can be derived from these theorems, and new forms deduced, by differentiation or integration with regard to letters which have been regarded as constants in a previous integration.

Ex. 1. For example, taking the case

$$\int_0^\pi \frac{dx}{a+b\cos x}\ (a>b) = \frac{2}{\sqrt{a^2-b^2}}\left[\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right]_0^\pi \text{(Art. 171)} = \frac{\pi}{\sqrt{a^2-b^2}},$$

we have, upon differentiation with regard to a,

$$\int_0^\pi \frac{dx}{(a+b\cos x)^2} = -\frac{\partial}{\partial a}\left(\frac{\pi}{\sqrt{a^2-b^2}}\right) = \frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}.$$

Differentiating again with regard to a,

$$\int_0^\pi \frac{dx}{(a+b\cos x)^3} = -\frac{1}{2}\frac{\partial}{\partial a}\frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}} = \frac{\pi}{2}\frac{2a^2+b^2}{(a^2-b^2)^{\frac{5}{2}}},$$

or, with regard to b,

$$\int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^3} = -\frac{1}{2}\frac{\partial}{\partial b}\frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}} = -\frac{3\pi}{2}\frac{ab}{(a^2-b^2)^{\frac{5}{2}}}.$$

Hence, $$\int_0^\pi \frac{a'+b'\cos x}{(a+b\cos x)^3}dx = \frac{\pi}{2}\frac{a'(2a^2+b^2)-3abb'}{(a^2-b^2)^{\frac{5}{2}}},$$

etc.

Generally,

$$\int_0^\pi \frac{dx}{(a+b\cos x)^n} = \frac{(-1)^{n-1}}{(n-1)!}\pi\frac{\partial^{n-1}}{\partial a^{n-1}}\frac{1}{\sqrt{a^2-b^2}},$$

$$\int_0^\pi \frac{\cos^{n-1}x\,dx}{(a+b\cos x)^n} = \frac{(-1)^{n-1}}{(n-1)!}\pi\frac{\partial^{n-1}}{\partial b^{n-1}}\frac{1}{\sqrt{a^2-b^2}}.$$

Ex. 2. Clearly $$\int e^{ax}\,dx = \frac{e^{ax}}{a};$$

$$\therefore \int x^n e^{ax}\,dx = \left(\frac{\partial}{\partial a}\right)^n\frac{e^{ax}}{a}.$$

Also $$\int x^n e^{ax}\,dx = \frac{1}{D}e^{ax}x^n = e^{ax}\frac{1}{D+a}x^n, \quad \left(D\equiv\frac{\partial}{\partial x}\right),$$

$$= \frac{e^{ax}}{a}\left(1-\frac{D}{a}+\frac{D^2}{a^2}-\dots\right)x^n.$$

[*Int. Calc. for Beginners*, Art. 213.]

Show that these results are identical.

Ex. 3. Starting with $$\int_0^\infty e^{-ax}\,dx = \frac{1}{a},$$

we have $$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}},$$

by n differentiations with respect to a. [See Art. 1897, Vol. II.]

Ex. 4. From such integrals as

$$\int\frac{dx}{(px+q)\sqrt{ax^2+2bx+c}}\quad\text{or}\quad\int\frac{dx}{(a_1x^2+2b_1x+c_1)\sqrt{a_2x^2+2b_2x+c_2}}$$

we can deduce

$$(1)\int\frac{dx}{(px+q)^n\sqrt{ax^2+2bx+c}},\quad\text{or}\quad(2)\int\frac{dx}{(px+q)(ax^2+2bx+c)^{\frac{3}{2}}},$$

or $$(3)\int\frac{dx}{(a_1x^2+2b_1x+c_1)^n\sqrt{a_2x^2+2b_2x+c_2}},$$

or $$(4)\int\frac{dx}{(a_1x^2+2b_1x+c_1)(a_2x^2+2b_2x+c_2)^{\frac{3}{2}}},$$

by respectively differentiating the first $n-1$ times with regard to q,
or once with regard to c,
or the second $n-1$ times with regard to c_1,
or once with regard to c_2,

when once the primary integral has been found (Chap. VIII.), and this will often be more convenient than the employment of a reduction formula. Differentiation with regard to other letters, p, a, b, a_1, b_1, a_2 or b_2, will give other integrals.

For example, by Art. 276 (supposing $bp>aq$, a and p positive),

$$\int\frac{dx}{(ax+b)(px+q)^{\frac{1}{2}}}=\frac{2}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\sqrt{\frac{px+q}{ax+b}}.$$

Therefore

$$\int\frac{dx}{(ax+b)^n(px+q)^{\frac{1}{2}}}=\frac{2(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial b^{n-1}}\left[\frac{1}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\sqrt{\frac{px+q}{ax+b}}\right]$$

and

$$\int\frac{x^{n-1}\,dx}{(ax+b)^n(px+q)^{\frac{1}{2}}}=\frac{2(-1)^n}{(n-1)!}\frac{\partial^{n-1}}{\partial a^{n-1}}\left[\frac{1}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\sqrt{\frac{px+q}{ax+b}}\right],$$

etc.

Ex. 5. If $Q=\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$, prove that

$$\int_0^\infty\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}-\frac{1}{\lambda}\right)\frac{\sqrt{\lambda}\,d\lambda}{Q}=0.$$

We have $$\frac{2dQ}{Q}=\frac{d\lambda}{a^2+\lambda}+\frac{d\lambda}{b^2+\lambda}+\frac{d\lambda}{c^2+\lambda};$$

$\therefore$ the integral in question is

$$\int_0^\infty\left(\frac{2dQ}{Q\,d\lambda}-\frac{1}{\lambda}\right)\frac{\sqrt{\lambda}\,d\lambda}{Q}$$

$$=-2\int_0^\infty\frac{d}{d\lambda}\left(\frac{\sqrt{\lambda}}{Q}\right)d\lambda=-2\left[\frac{\sqrt{\lambda}}{Q}\right]_0^\infty=0.$$

Similarly

$$\int_0^\infty\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)\frac{d\lambda}{Q}=\int_0^\infty\frac{2dQ}{Q^2}=-2\left[\frac{1}{Q}\right]_0^\infty=\frac{2}{abc}.$$

If $I=\int_0^\infty \frac{d\lambda}{Q}$, we have $\frac{\partial I}{\partial a^2}=-\frac{1}{2}\int_0^\infty \frac{1}{a^2+\lambda}\frac{d\lambda}{Q}$,

and the above equation may be written

$$\frac{\partial I}{\partial a^2}+\frac{\partial I}{\partial b^2}+\frac{\partial I}{\partial c^2}=-\frac{1}{abc}.$$

For several useful illustrations of such integrals, which occur in problems on the attraction of ellipsoidal shells, see *Analytical Statics*, by E. J. Routh, vol. II., pp. 100–101.

364. Differentiation of a Multiple Integral with regard to an Involved Constant.

It will be sufficient to take the case of a multiple integral of the second order.

Consider $$I\equiv\int_{x_0}^{x_1}dx\int_{y_0}^{y_1}dy\,\phi(x, y, c),$$

where c, x_0, x_1, y_0, y_1 are all functions of some quantity t but not involving x or y.

Let $$\int\phi(x, y, c)\,dy=F(x, y, c),$$

where x is regarded as a constant in this integration, so that

$$\frac{\partial F(x, y, c)}{\partial y}\equiv\phi(x, y, c);$$

then $\int_{y_0}^{y_1}\phi(x, y, c)\,dy=F(x, y_1, c)-F(x, y_0, c)\equiv v$, say.

Then $$I=\int_{x_0}^{x_1}v\,dx.$$

Differentiating by the rule of Art. 355,

$$\frac{dI}{dt}=\int_{x_0}^{x_1}\frac{\partial v}{\partial t}dx+\frac{dx_1}{dt}v_1-\frac{dx_0}{dt}v_0,$$

where v_0 and v_1 are the values of v when x receives the values x_0 and x_1 respectively.

Also $$\frac{\partial v}{\partial t}=\int_{y_0}^{y_1}\frac{\partial\phi}{\partial t}dy+\frac{dy_1}{dt}\phi(x, y_1, c)-\frac{dy_0}{dt}\phi(x, y_0, c).$$

Thus, substituting for $\frac{\partial v}{dt}$, we have

$$\frac{dI}{dt}=\int_{x_0}^{x_1}dx\int_{y_0}^{y_1}dy\frac{\partial\phi}{\partial t}+\int_{x_0}^{x_1}\left[\frac{dy_1}{dt}\phi(x, y_1, c)-\frac{dy_0}{dt}\phi(x, y_0, c)\right]dx$$
$$+\frac{dx_1}{dt}\int_{y_0}^{y_1}\phi(x_1, y, c)\,dy-\frac{dx_0}{dt}\int_{y_0}^{y_1}\phi(x_0, y, c)\,dy.$$

This may be written in the more compact form

$$\frac{dI}{dt}=\int_{x_0}^{x_1} dx\int_{y_0}^{y_1} dy\frac{\partial\phi}{\partial t}+\int_{x_0}^{x_1} dx\left[\frac{dy}{dt}\phi(x, y, c)\right]_{y_0}^{y_1}$$

$$+\left[\frac{dx}{dt}\int_{y_0}^{y_1}\phi(x, y, c)dy\right]_{x_0}^{x_1}.$$

A similar process may be applied in cases of Multiple Integrals of a higher order. It is to be understood that all limitations with regard to the nature of ϕ, and the range of integration, which correspond to those described in Art. 355 for the case of a single variable, are supposed to be assumed.

365. Remainder after n terms of Taylor's Series expressed as a Definite Integral.

Let $f(x)$ be a function of x which is finite and continuous throughout the range of values of x, from $x=a$ to $x=a+h$, as also all its differential coefficients as far as $f^{(n)}(x)$.

Let $x=a+h-z$ be an intermediate value of x, ($z<h$).

Considering the integral $\int_0^h f'(a+h-z)\,dz$, we may

(1) integrate directly as $\Big[-f(a+h-z)\Big]_0^h=f(a+h)-f(a)$,

or (2) apply the rule of continued integration by parts (Art. 95), viz.

$$\left[zf'(a+h-z)+\frac{z^2}{2!}f''(a+h-z)+\frac{z^3}{3!}f'''(a+h-z)+\dots\right.$$

$$\left.+\frac{z^{n-1}}{(n-1)!}f^{(n-1)}(a+h-z)\right]_0^h$$

$$+\int_0^h\frac{z^{n-1}}{(n-1)!}f^{(n)}(a+h-z)\,dz,$$

i.e.

$$hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\dots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$

$$+\int_0^h\frac{z^{n-1}}{(n-1)!}f^{(n)}(a+h-z)\,dz,$$

i.e. $$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\dots$$

$$+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\int_0^h\frac{z^{n-1}}{(n-1)!}f^{(n)}(a+h-z)\,dz.$$

Hence the remainder after n terms is

$$R_n \equiv \frac{1}{(n-1)!}\int_0^h z^{n-1} f^{(n)}(a+h-z)\,dz.$$

By theorem IX., Art. 331, this is equal to

$$\frac{1}{(n-1)!} f^{(n)}(a+h-\zeta)\int_0^h z^{n-1}\,dz, \quad i.e. \ \frac{h^n}{n!} f^n(a+h-\zeta)$$

for some value of ζ lying between $\zeta=0$ and $\zeta=h$, which may be written $\zeta=(1-\theta)h$ where θ is a positive proper fraction.

Hence $R_n=\dfrac{h^n}{n!}f^{(n)}(a+\theta h)$, which is Lagrange's form of remainder (see *Diff. Calc.*, Art. 130).

366. Remainders after $(n+1)$ terms in Lagrange's Theorem and in Laplace's Extension, expressed by means of a Definite Integral.

It is easy to find an expression for the remainder after $(n+1)$ terms in Laplace's extension of Lagrange's theorem (*Diff. Calc.*, Art. 518).

Lagrange's theorem states that if $z=y+x\phi(z)$ and u be any function of z, say $f(z)$, then the expansion of u in powers of x is

$$u \equiv f(z) = f(y) + x\phi(y)f'(y) + \frac{x^2}{2!}\frac{d}{dy}[\{\phi(y)\}^2 f'(y)] + \dots + \frac{x^n}{n!}\frac{d^{n-1}}{dy^{n-1}}[\{\phi(y)\}^n f'(y)] + \dots,$$

and Laplace's extension states that if $z=F\{y+x\phi(z)\}$,

$$f(z) = f\{F(y)\} + x\phi\{F(y)\}\frac{df\{F(y)\}}{dy} + \frac{x^2}{2!}\frac{d}{dy}\left[\overline{\phi\{F(y)\}}|^2\frac{df\{F(y)\}}{dy}\right] + \dots + \frac{x^n}{n!}\frac{d^{n-1}}{dy^{n-1}}\left[\overline{\phi\{F(y)\}}|^n\frac{df\{F(y)\}}{dy}\right] + \dots,$$

and contains the former as a particular case.

Take then $z=F\{y+x\phi(z)\}$, and consider the integral

$$I_n = \int_y^{F^{-1}(z)} [y+x\phi\{F(t)\}-t]^n f'[F(t)]F'(t)\,dt,$$

where $$f'(v) \equiv \frac{df(v)}{dv}.$$

We shall write $\phi F t$ for $\phi\{F(t)\}$, etc., to avoid the multiplicity of brackets.

Putting $n=0$, we have

$$I_0=\int_y^{F^{-1}(z)}\frac{d}{dt}(fFt)\,dt=\Big[fFt\Big]_y^{F^{-1}(z)}=f(z)-fF(y).$$

Again, differentiating I_n with regard to y (Art. 355),

$$\begin{aligned}\frac{dI_n}{dy}&=n\int_y^{F^{-1}(z)}[y+x\phi Ft-t]^{n-1}(f'Ft)(F't)\,dt\\&\qquad\qquad-[x\phi Fy]^n(f'Fy)(F'y)\\&=nI_{n-1}-x^n(\phi Fy)^n\frac{d}{dy}(fFy);\end{aligned}$$

$$\therefore\ I_{n-1}=\frac{x^n}{n}(\phi Fy)^n\frac{d}{dy}(fFy)+\frac{1}{n}\frac{d}{dy}I_n.$$

Putting $n=1, 2, 3, \ldots$ successively in this result,

$$\begin{aligned}I_0&=x\ \ (\phi Fy)\ \ \frac{d}{dy}(fFy)+\frac{d}{dy}I_1,\\I_1&=\frac{x^2}{2}\ (\phi Fy)^2\frac{d}{dy}(fFy)+\frac{1}{2}\frac{d}{dy}I_2,\\I_2&=\frac{x^3}{3}\ (\phi Fy)^3\frac{d}{dy}(fFy)+\frac{1}{3}\frac{d}{dy}I_3,\\I_3&=\frac{x^4}{4}\ (\phi Fy)^4\frac{d}{dy}(fFy)+\frac{1}{4}\frac{d}{dy}I_4,\end{aligned}$$

etc.;

whence

$$\begin{aligned}f(z)-fFy&=x(\phi Fy)\frac{d}{dy}(fFy)+\frac{d}{dy}I_1\\&=x(\phi Fy)\frac{d}{dy}(fFy)+\frac{x^2}{2!}\frac{d}{dy}\left[\overline{\phi Fy}|^2\frac{d}{dy}\overline{fFy}|\right]+\frac{1}{2!}\frac{d^2I_2}{dy^2}\\&=\text{etc.},\end{aligned}$$

and $$f(z)=fFy+x(\phi Fy)\frac{d}{dy}(fFy)+\frac{x^2}{2!}\frac{d}{dy}\left[\overline{\phi Fy}|^2\frac{d}{dy}\overline{fFy}|\right]+\ldots$$
$$+\frac{x^n}{n!}\frac{d^{n-1}}{dy^{n-1}}\left[\overline{\phi Fy}|^n\frac{d}{dy}\overline{fFy}|\right]+\frac{1}{n!}\frac{d^n}{dy^n}I_n.$$

The remainder sought is therefore

$$R_{n+1}\equiv\frac{1}{n!}\left(\frac{d}{dy}\right)^n\int_y^{F^{-1}(z)}[y+x\phi\{F(t)\}-t]^n f'\{F(t)\}F'(t)\,dt.$$

This includes, as a particular case, the remainder after $(n+1)$ terms in Lagrange's theorem, when $z=y+x\phi(z)$, viz.

$$R_{n+1}\equiv\frac{1}{n!}\left(\frac{d}{dy}\right)^n\int_y^z[y+x\phi(t)-t]^n f'(t)\,dt,$$

cited by Professor Williamson (*Encyclopaedia Britannica*, "Infinitesimal Calculus," § 151) as due to M. Popoff (*Comptes Rendus*, 1861), the demonstration of which by M. Zolotareff, quoted in the *Encyclopaedia Britannica*, is similar to the above.

GENERAL EXAMPLES.

1. Prove that

$$\frac{d}{da}\int_{a^q}^{a^p} a^m x^n\,dx = a^{m-1}\left[\left(\frac{m}{n+1}+p\right)a^{(n+1)p}-\left(\frac{m}{n+1}+q\right)a^{(n+1)q}\right],$$

and verify the result by performing the integration first.

2. If A be the area bounded by a parabola and its latus rectum $(4a)$, prove

(1) by differentiating the integral $4\int_0^a\sqrt{ax}\,dx$ with regard to a,

(2) by first integrating and then differentiating with regard to a, that

$$\frac{dA}{da}=\frac{16a}{3}.$$

3. Apply the method of Art. 355 to prove that

$$\frac{d}{dc}\int_{\frac{c}{2}}^{\frac{c\sqrt{3}}{2}}\sqrt{c^2-x^2}\,dx=\tfrac{1}{6}\pi c,$$

and explain geometrically each step of the process.

Obtain the same result by first integrating and then differentiating the result with regard to c; and also geometrically.

4. Show that if $$u=\int_0^\infty e^{-ax^3-bx^3}\,dx,$$

then $$3ab\frac{\partial^2 u}{\partial b^2}-3a\frac{\partial u}{\partial b}-2b^2\frac{\partial u}{\partial a}=1,$$

provided a be positive. [TRINITY, 1888.]

5. Show that $$\frac{d^n}{dc^n}\int_{-c}^{c}f(x+c)\,dx=2^n f^{(n-1)}(2c).$$ [α, 1883.]

6. If $f(x+c)=f(x)$ for all values of x, show that

$$\int_0^c f(y+az)\,dy$$

is independent of z. [α, 1887.]

7. Prove that

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + \beta^2 \sin^2 x)^{n+1}} = \frac{\pi}{2^{n+1}} \frac{(-1)^n}{\underline{|n}} \left(\frac{1}{a}\frac{\partial}{\partial a} + \frac{1}{\beta}\frac{\partial}{\partial \beta}\right)^n (a\beta)^{-1}.$$

8. Prove that if $u = (a^n + b^n)\int_\beta^a F\{(a-b)x\}\, dx$,

where F denotes any function, β and a being independent of a and b, and n being a positive integer, then

$$\left\{(a^n + b^n)\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)^n - 2\,\underline{|n}\right\} u = 0.$$

[OXFORD, 1886.]

9. If

$$u = \int_0^c \phi(x, t)\, dt,$$

where c is a function of u and x, prove that

$$\frac{du}{dx} = \frac{\int_0^c \frac{\partial \phi}{\partial x} dt + \frac{\partial c}{\partial x} \phi(x, c)}{1 - \frac{\partial c}{\partial u} \phi(x, c)}.$$

[δ, 1885.]

10. If

$$u = \int_a^\beta \phi(x, y)\, dy,$$

where a and β are functions of x and u, prove that

$$\frac{du}{dx} = \frac{\int_a^\beta \frac{\partial \phi}{\partial x} dy + \phi(x, \beta) \frac{\partial \beta}{\partial x} - \phi(x, a) \frac{\partial a}{\partial x}}{1 - \phi(x, \beta) \frac{\partial \beta}{\partial u} + \phi(x, a) \frac{\partial a}{\partial u}}.$$

11. Comment upon the application of the rule of Art. 355 to the case

$$\frac{d}{da}\int_{-a}^a \frac{\phi(x)\, dx}{\sqrt{a^2 - x^2}}.$$

Prove that in this case the true result is

$$\frac{1}{a}\int_{-a}^a \frac{x\phi'(x)\, dx}{\sqrt{a^2 - x^2}}.$$

12. If

$$u = \int_{f(a)}^{F(a)} \phi(\theta, a)\, d\theta,$$

we have $$\frac{du}{da} = \int_{f(a)}^{F(a)} \frac{\partial \phi(\theta, a)}{\partial a} d\theta + \phi\{F(a), a\} \frac{dF}{da} - \phi\{f(a), a\} \frac{df}{da}.$$

Do you consider that this formula fails in the case in which

$$F(a) = a, \quad f(a) = 0 \quad \text{and} \quad \phi(\theta, a) = \frac{1}{\sqrt{\cos\theta - \cos a}}?$$

If so, to what extent and in what respect?

Prove that in this case

$$\frac{du}{da}=\frac{\sin a}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\,d\theta}{\left(1-\sin^2\frac{a}{2}\sin^2\theta\right)^{\frac{3}{2}}}.$$

Make any remarks that occur to you as to the reasons for the peculiar form which the general formula assumes in this case.

[ϵ, 1884.]

13. Show that the equation

$$\frac{d}{dx}\int_0^1 x^3 e^{x^2\left(1-\frac{1}{t}\right)}\frac{dt}{t^2}=\int_0^1\frac{\partial}{\partial x}\left[x^3 e^{x^2\left(1-\frac{1}{t}\right)}\frac{1}{t^2}\right]dt$$

ceases to hold for $x=0$. [MATH. TRIPOS, 1897.]

14. Find a curve in which the abscissa of the centroid of the area of that portion bounded by the curve, the coordinate axes and an ordinate is proportional to the abscissa of the bounding ordinate.

[COLLEGES, 1878.]

15. A vessel in the form of a right circular cylinder with vertical axis and a flat horizontal base is filled to varying depths with liquid of varying density. If the depth of the centre of gravity of the liquid be always $\frac{1}{n}$ of the immersed portion of the axis, show that the density varies as $(\text{depth})^{\frac{2-n}{n-1}}$.

16. Find the general equation of all solids of revolution for which the distance from the vertex of the centroid of a segment made by a plane perpendicular to the axis, is proportional to the height of the segment. [TODHUNTER, *Integral Calculus*, p. 198.]

17. Find the form of the curve for which the area bounded by the curve, the coordinate axes and an ordinate is such that the moments of inertia of this area about the coordinate axes are in a constant ratio.

18. A body moves from rest at a distance a towards a centre of attraction varying inversely as the distance. Show that the time of describing the space between βa and $\beta^n a$ will be a maximum when

$$\beta n^{\frac{1}{2(n-1)}}=1$$

$$\left[\text{It may be assumed that } \left(\frac{dx}{dt}\right)^2\propto\log\frac{a}{x}.\right]$$

[TAIT AND STEELE, *Dynamics of a Particle*.]

19. Find the density of a parabolic plate as a function of the abscissa in order that the distance of the centroid from the vertex may vary as the square root of the length of the plate. [α, 1881.]

20. Find the equation of the curve such that the area included by the ordinate at any point, the axis of x and the curve is in a constant ratio to the area included by the ordinate, the axis of x and the tangent. [MATH. TRIPOS, 1882.]

21. Prove that

$$\frac{d}{da}\int_0^a \frac{F(x)\,dx}{\sqrt{a-x}} = \int_0^a \frac{F(x) + 2x\dfrac{dF(x)}{dx}}{2a\sqrt{a-x}}\,dx.$$

Under what circumstances will $\int_0^a \frac{F(x)\,dx}{\sqrt{a-x}}$ be independent of a?

[TODHUNTER, *Int. Calc.*]

22. If $$\tan\frac{\pi}{8}\sin\phi = x\sqrt{\frac{1-x^2}{1+x^2}} = \sin\psi,$$

verify that

$$\int_0^x \frac{dx}{\sqrt{1-x^8}} = \frac{1}{2\sqrt{2}}\int_0^\phi \frac{d\phi}{\sqrt{1-\tan^2\frac{\pi}{8}\sin^2\phi}} + \sin^2\frac{\pi}{8}\int_0^\psi \frac{d\psi}{\sqrt{1-\tan^2\frac{\pi}{8}\sin^2\psi}}.$$

[MATH. TRIPOS, 1896.]

23. Prove that

$$b^2\int_0^\infty \frac{\cos bx}{a^2+x^2}\,dx + 6\int_0^\infty \frac{\cos bx}{(a^2+x^2)^2}\,dx - 8a^2\int_0^\infty \frac{\cos bx}{(a^2+x^2)^3}\,dx = 0.$$

[ϵ, 1884.]

24. Verify that

$$y = A\int_0^1 v^{\beta-1}(1-v)^{\gamma-\beta-1}(1-xv)^{-\alpha}\,dv$$

satisfies the differential equation of the hypergeometric series, viz.

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0,$$

when $\beta > 0$ and $\gamma > \beta$.

25. If $$u = \int_0^\pi e^{qx\cos\theta}\{A + B\log(x\sin^2\theta)\}\,d\theta,$$

verify that $$x\frac{d^2u}{dx^2} + \frac{du}{dx} - q^2xu = 0.$$

26. Prove that $y = \frac{1}{\pi}\int_0^\pi (x + \cos\phi\sqrt{x^2-1})^n \, d\phi$

satisfies the equation

$$\frac{d}{dx}\left[(x^2-1)\frac{dy}{dx}\right] = n(n+1)y.$$

[St. John's, 1892.]

27. Show that the differential equation

$$\frac{d^2u}{dx^2} + u = \frac{a}{x} + \frac{b}{x^2}$$

is satisfied by $$u = \int_0^\infty \frac{a\sin z + b\cos z}{x+z}\, dz.$$

Write down the complete solution. [St. John's, 1883.]

28. If $$y = \int_0^\pi \sqrt{x} e^{nx\cos\theta} \cos\{\sqrt{7}\log(\sqrt{x}\sin\theta) + a\} d\theta,$$

prove that $$\frac{d^2y}{dx^2} = \left(n^2 - \frac{2}{x^2}\right) y.$$

[St. John's, 1889.]

29. Prove that

$$\int_0^\pi (\cosh x - \sinh x \cos\phi)^{\frac{2n-1}{2}} d\phi = \int_0^\pi \frac{d\phi}{(\cosh x - \sinh x\cos\phi)^{\frac{2n+1}{2}}}.$$

Prove also that if

$$P = \int_0^\pi (\cosh x - \sinh x\cos\phi)^{\frac{1}{2}} \, d\phi,$$

$$\frac{dP}{dx} = \frac{1}{2}\int_0^\pi \frac{\cos\phi \, d\phi}{(\cosh x - \sinh x\cos\phi)^{\frac{3}{2}}}.$$

[a, 1886.]

30. If $\frac{y}{x} = \int_0^{\frac{\pi}{2}} \cos(mx^n \sin\phi)\cos^{\frac{1}{n}}\phi \, d\phi$, prove that y satisfies the equation

$$\frac{d^2y}{dx^2} + m^2n^2x^{2n-2}y = 0.$$

[a, 1886.]

31. Verify that

$$y = \int e^{ux} V[A + B\log\{U_1(a_2 + b_2x)\}]\, du,$$

where $U_1 = b_2u^2 + b_1u + b_0$, $\log VU_1 = \int \frac{a_2u^2 + a_1u + a_0}{U_1} du$,

and the limits are given by $e^{ux}VU_1 = 0$, satisfies the equation

$$(a_2 + b_2x)\frac{d^2y}{dx^2} + (a_1 + b_1x)\frac{dy}{dx} + (a_0 + b_0x)y = 0,$$

provided $a_1b_2 - a_2b_1 = b_2^2$. [Spitzer, *Crelle*, vol. liv.]

32. Verify that if x be positive

$$u = C_1 \int_{-q}^{q} e^{xt}(t^2 - q^2)^{\frac{a}{2}-1}\,dt + C_2 \int_{-\infty}^{-q} e^{xt}(t^2 - q^2)^{\frac{a}{2}-1}\,dt,$$

and if x be negative

$$u = C_1 \int_{-q}^{q} e^{xt}(t^2 - q^2)^{\frac{a}{2}-1}\,dt + C_2 \int_{q}^{\infty} e^{xt}(t^2 - q^2)^{\frac{a}{2}-1}\,dt$$

solves the differential equation

$$x\frac{d^2u}{dx^2} + a\frac{du}{dx} - q^2xu = 0.$$

[PETZVAL.]

33. Prove that

$$\int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \sin\theta \operatorname{arc\,cos}\frac{\cos a}{\sin\theta}\,d\theta = \frac{\pi}{2}(1 - \cos a).$$

[TRINITY, 1886.]

34. Prove that

$$\int_0^1 \log\frac{1+ax}{1-ax}\,\frac{dx}{x\sqrt{1-x^2}} = \pi\sin^{-1}a.$$

[OXFORD, 1888.]

35. Establish the known result

$$\int_0^\infty \frac{\log x}{(a+x)^2}\,dx = \frac{\log a}{a},$$

and hence prove that when n is a positive integer

$$(\alpha)\quad \int_0^\infty \frac{\log x}{(a+x)^{n+2}}\,dx = \frac{1}{(n+1)a^{n+1}}\left\{\log a - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right\},$$

$$(\beta)\quad \int_0^\infty \frac{(\log x)^2}{(a+x)^{n+2}}\,dx = \frac{1}{(n+1)a^{n+1}}\left\{\left(\frac{\pi^2}{3} - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \dots - \frac{1}{n^2}\right)\right.$$
$$\left. + \left(\log a - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right)^2\right\}.$$

[MATH. TRIPOS, 1883.]

36. If the operator Δ, applied to a function of α, has the effect of changing α to $\alpha + 1$, and subtracting the original function, show that

$$\Delta\int_a^b \phi(x, \alpha)\,dx = \int_a^b \Delta\phi(x, \alpha)\,dx,$$

where a and b are independent of α.

Prove that

$$\int_0^\infty e^{-\alpha x}(e^{-x} - 1)^n\,dx = \frac{(-1)^n}{\alpha(\alpha+1)\dots(\alpha+n)}.$$

[BERTRAND, *C.I.*, p. 183.]

37. Given $u=\int_0^\infty \frac{\cos ax}{1+x^2}dx$, differentiating twice we have

$$\frac{d^2u}{da^2}=-\int_0^\infty \frac{x^2\cos ax}{1+x^2}dx.$$

But this is indeterminate when x is infinite. Discuss the validity of the differentiation. [BERTRAND, *Cal. Int.*, p. 181.]

38. Is it true that

$$\int_0^1\left[\int_0^1 \frac{a^2-x^2}{(a^2+x^2)^2}dx\right]da=\int_0^1\left[\int_0^1 \frac{a^2-x^2}{(a^2+x^2)^2}da\right]dx\,?$$

If not, why not? [See Art. 1899, Vol. II.]
Evaluate each side separately and compare the results.

[BERTRAND, *Cal. Int.*, p. 187.]

39. If $P+\iota Q=\phi(x+\iota y)$, show that in general

$$\int_a^b\int_a^\beta \frac{\partial Q}{\partial y}dx\,dy=\int_a^\beta\int_a^b \frac{\partial P}{\partial x}dy\,dx$$

and

$$\int_a^b\int_a^\beta \frac{\partial P}{\partial y}dx\,dy=-\int_a^\beta\int_a^b \frac{\partial Q}{\partial x}dy\,dx.$$

Examine the case $\phi(x+\iota y)=e^{-(x+\iota y)^2}$, taking $a=0$, $\alpha=0$ and $b=\infty$.

[BERTRAND.]

40. If $f(x)=(2-x)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}}(1-x\sin^2\theta)^{-\frac{1}{2}}d\theta$, show that

$$\frac{df(x)}{dx}=\frac{1}{2}(2-x)^{-\frac{1}{2}}\int_0^{\frac{\pi}{4}}\cos 2\theta\left[(1-x\cos^2\theta)^{-\frac{3}{2}}-(1-x\sin^2\theta)^{-\frac{3}{2}}\right]d\theta.$$

Hence show that as x increases from 0 to 1, $f(x)$ increases from $\frac{\pi}{\sqrt{2}}$ to ∞.

[C. S., 1898.]

41. Prove that

$$\int_0^u du\int_0^u du\int_0^u du\ldots\int_0^u du\,f(u)=\frac{1}{(n-1)!}\int_0^u (u-z)^{n-1}f(z)\,dz,$$

there being n integration signs in the left member of the equality.

[R. P.]

42. Show that

$$\frac{d^2}{dc^2}\left\{\int_0^c\int_0^c \phi(x+y+c)\,dx\,dy\right\}=9\phi(3c)-8\phi(2c)+\phi(c).$$

[OXF. II. P., 1890.]

43. Show that the quartic function

$$Q \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

can, in general, be expressed in three different ways as the sum of two squares $P^2 + R^2$, where

$$P \equiv a^{-\frac{3}{2}}[(ax+b)^2 + 3(ac - b^2) - 2\lambda]$$

and $$R \equiv a^{-\frac{3}{2}}\lambda^{-\frac{1}{2}}[2(ax+b)\lambda + a^2d - 3abc + 2b^3],$$

λ having any one of three determinate values λ_1, λ_2, λ_3.

Verify the evaluation of the integral $\int \frac{dx}{Q}$ in the form

$$\frac{a^2}{4}\left[(\lambda_2 - \lambda_3)\lambda_1^{\frac{1}{2}}\tan^{-1}\frac{R_1}{P_1} + (\lambda_3 - \lambda_1)\lambda_2^{\frac{1}{2}}\tan^{-1}\frac{R_2}{P_2}\right.$$
$$\left.+ (\lambda_1 - \lambda_2)\lambda_3^{\frac{1}{2}}\tan^{-1}\frac{R_3}{P_3}\right]\Big/\Lambda,$$

where $$\Lambda = (\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_2).$$

[MATH. TRIP., 1897.]

44. Show that

$$\int_0^a x^{n-1}(1 - a + x)^{-n-r}dx = \frac{(n-1)!}{(n+r-1)!}\left(\frac{d}{da}\right)^{r-1}\frac{a^{n+r-1}}{1-a}.$$

[I. C. S., 1892.]

CHAPTER XI.

PRELIMINARY TO INTEGRATION OF $\int \frac{M}{N}\frac{dx}{\sqrt{Q}}$, WHERE Q IS A RATIONAL QUARTIC. DEFINITIONS OF ELLIPTIC FUNCTIONS. ELEMENTARY CONSIDERATIONS.

367. In many problems of both pure and applied mathematics, such as the investigation of the length of an arc of an ellipse, or of a lemniscate, or the time of a finite oscillation of an ordinary simple circular pendulum, integrals occur in which the integrand contains a square root of an algebraic function of higher degree than the second.

Now the integral $$\int \frac{dx}{\sqrt{Q}},$$
where Q is the general biquadratic function
$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4,$$
cannot *in general* be integrated by means of the circular, inverse circular, or inverse hyperbolic functions, though it has been seen that for particular values of the coefficients this may be possible; for no such function is known which will, on differentiation, give rise to the general expression $\frac{1}{\sqrt{Q}}$ as its differential coefficient.

Hence, in discussing such an integral as this, we are in a position similar to that which would have occurred if we had required the integral $\int \frac{dx}{\sqrt{a+bx+cx^2}}$ before the inverse circular or inverse hyperbolic functions had been discovered. The integration even of the case $\int \frac{dx}{\sqrt{1-x^2}}$ would then have pre-

sented a difficulty. And the necessity for the consideration of such an integral would have formed a suitable starting-point for the investigation of such functions as would have $\frac{1}{\sqrt{1-x^2}}$, or, more generally, $\frac{1}{\sqrt{a+bx+cx^2}}$ for their differential coefficients.

And the whole theory of such functions could have been built up from this starting-point.

368. For instance, let $F(x)=\int_0^x \frac{dw}{\sqrt{1-w^2}}$.

Then $F(0)=0$.

Let x and y be two variables connected by the equation

$$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}=0,$$

i.e.
$$F'(x)\,dx+F'(y)\,dy=0.$$

The integral is $F(x)+F(y)=\text{constant}=F(z)$, say, where z is the value of y when x vanishes.

But multiplying by $\sqrt{1-x^2}\sqrt{1-y^2}$,

$$dx\sqrt{1-y^2}+dy\sqrt{1-x^2}=0,$$

and we can integrate this by parts, viz.

$$x\sqrt{1-y^2}+\int x\frac{y}{\sqrt{1-y^2}}\,dy+y\sqrt{1-x^2}+\int y\frac{x}{\sqrt{1-x^2}}\,dx=\text{constant}=C,$$

i.e.
$$x\sqrt{1-y^2}+y\sqrt{1-x^2}+\int xy\left(\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}\right)=C,$$

and the part under the integration sign vanishes.

Hence, $x\sqrt{1-y^2}+y\sqrt{1-x^2}=z$, say, where z is the value of y if x vanishes.

Hence we have the addition equation

$$F(x)+F(y)=F\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right),$$

and if we then choose to write $\sin^{-1}$ (a supposed unknown symbol) for F, we should have

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right),$$

or writing $\sin^{-1}x=\theta$ and $\sin^{-1}y=\phi$,

$$\sin(\theta+\phi)=\sin\theta\sqrt{1-\sin^2\phi}+\sin\phi\sqrt{1-\sin^2\theta},$$

and we should thus have arrived at one of the fundamental propositions of trigonometry, and could have built up the general theory.

Such is actually our position with regard to the integration of $\int \frac{dx}{\sqrt{Q}}$, or, more generally, $\int \frac{M}{N} \frac{dx}{\sqrt{Q}}$, where M and N are rational integral algebraic functions of x, and Q is a rational integral algebraic polynomial of degree higher than the second, say the quartic

$$Q \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4,$$

and the absence of knowledge of any function which, upon differentiation, would give a general result of this kind long barred the progress of geometers.

369. It was natural that after having exhausted the discussion of integrations which could be expressed algebraically or by means of logarithms, or by inverse circular functions, that is in terms of arcs of a circle, that investigators should turn their attention to such expressions as could be integrated by means of arcs of an ellipse or a hyperbola. Thus Colin Maclaurin, in his *Fluxions*, vol. ii., Art. 799, of date 1742, discusses "the fluent of $\frac{\dot{x}\sqrt{x}}{2\sqrt{xx-1}}$," or as it would now be written $\frac{1}{2}\int \frac{\sqrt{x}\,dx}{\sqrt{x^2-1}}$, *i.e.* $\frac{1}{2}\int \frac{x\,dx}{\sqrt{x(x^2-1)}}$, which he expresses as the arc of a rectangular hyperbola of semi-axis unity, viz. drawing a tangent at the vertex A of the hyperbola, centre C, and a circle with the same centre and radius x cutting the tangent at the point M, then letting the bisector of $A\hat{C}M$ cut the hyperbola at E, arc $AE = \frac{1}{2}\int \frac{x\,dx}{\sqrt{x(x^2-1)}}$, which we leave to the student to verify.

370. The real starting-point of the general theory of such integrals, which have been termed Elliptic Integrals, from their intimate connexion with that curve, may be taken to be Fagnano's discovery,* that upon every ellipse or hyperbola it is possible to assign in an infinite number of ways two

* Fagnano, *Produzioni matematiche*, tom. ii.

arcs whose difference is equal to an algebraic expression, and that the lemniscate "jouit de cette singulière propriété, que ses arcs peuvent être multipliés ou divisés algébriquement, comme les arcs de cercle, quoique chacun d'eux soit une transcendante d'un ordre supérieur."*

371. Definitions. Various mathematicians, Euler,† Lagrange,‡ Landen§ and others, turned their attention to this matter, and much progress was made. But the chief advance was due to the investigations of Legendre, first in his *Mémoires sur les Transcendantes Elliptiques*, 1793, and, after a long interval, in his *Exercices de Calcul Intégral*, 1811. In this last work he treated the general reduction of the integral

$$\int \frac{P\,dx}{\sqrt{Q}},$$

where P is any rational function whatever of x, and Q is the quartic function

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4,$$

showing that in all cases the integration may be made to depend upon that of three fundamental integrals, viz.

$$\left.\begin{aligned} F(\theta, k) &= \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^\theta \frac{d\theta}{\Delta}, \\ E(\theta, k) &= \int_0^\theta \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^\theta \Delta\,d\theta, \\ \Pi(\theta, k, n) &= \int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \\ &= \int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\Delta}, \end{aligned}\right\} \text{ where } \Delta=\sqrt{1-k^2\sin^2\theta},$$

which he calls the "Elliptic Integrals of the First, Second and Third kind respectively," k being a real constant quantity less than unity, called the *modulus*, and n any constant whatever.

372. Legendre in a footnote, (pages 18, 19) of the *Exercices* suggested names for these functions, but it does not appear that the names were generally adopted, except as to the initial letter E and Π still used for the second and third. He remarks:

"Ces fonctions réunissent un si grand nombre de propriétés, que

* Legendre, *Exercices de Calcul Intégral*, 1811.

† Euler, *Novi. Com. Petrop.*, tom. vi. et vii.

‡ *Mém. de Turin*, tom. iv.

§ *Math. Memoirs*, by John Landen, 1780.

quand elles seront plus généralement connues, on jugera sans doute nécessaire de leur imposer un nom particulier, et de désigner la fonction de c et ϕ égale à $\int \frac{d\phi}{\Delta}$, comme on désigne l'arc dont le sinus est x, ou le nombre dont le logarithme est y. Il semble qu'on caractériserait assez bien la fonction F en lui donnant le nom de *Nome*, parce que cette fonction a la propriété de régler tout ce qui concerne la comparaison des fonctions elliptiques. Peut-être conviendrait-il en même temps de donner les noms d'*Epinome* et de *Paranome* aux fonctions E et Π que constituent les deux autres espèces."

373. Legendre established addition formulae for each of these functions analogous to the trigonometrical formulae for $\sin(\theta \pm \phi)$, $\cos(\theta \pm \phi)$, whence their whole theory may be deduced, as for the ordinary circular functions of trigonometry, and their numerical values calculated and tabulated for definite values of k and n. This having been done, they are available for numerical use, as in the case of the circular and inverse circular functions.

374. All three of Legendre's standard forms are comprehended in the one formula

$$H, \text{ or } \Big[H\Big]_0^\theta = \int_0^\theta \frac{A + B\sin^2\theta}{1+n\sin^2\theta}\,\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The cases are

$$\begin{array}{llll} A=1, & B=0, & n=0, & H \equiv F(\theta, k), \\ A=1, & B=-k^2, & n=0, & H \equiv E(\theta, k), \\ A=1, & B=0, & & H \equiv \Pi(\theta, k, n). \end{array}$$

375. **The "Complete Values." The Real Periodicity.**

The function $\dfrac{A+B\sin^2\theta}{1+n\sin^2\theta}\cdot\dfrac{1}{\sqrt{1-k^2\sin^2\theta}}$

obviously goes through all its values four times, as θ increases from 0 to 2π, and then repeats the same cycle. The values in the second quadrant are merely repetitions of those in the first, passed through in the reverse order.

It is clear then that

$$\Big[H\Big]_0^{\frac{\pi}{2}} = \Big[H\Big]_{\frac{\pi}{2}}^{\pi} = \Big[H\Big]_{\pi}^{\frac{3\pi}{2}} = \Big[H\Big]_{\frac{3\pi}{2}}^{2\pi} = \text{etc.},$$

and that
$$\Big[H\Big]_0^{\theta} = \Big[H\Big]_{\pi-\theta}^{\pi}.$$

We may call $\Big[H\Big]_0^{\frac{\pi}{2}}$ the quarter period of the integral H.

In the case of the first elliptic integral, this "complete" integral $\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ is denoted by F_1 or K, and called the *real quarter period of* $F(\theta, k)$.

Similarly, E_1 and Π_1 are written for the "complete" integral of the second and third kinds respectively, *i.e.* when the limits are 0 and $\frac{\pi}{2}$, and E_1, Π_1 are the respective quarter periods of $E(\theta, k)$ and $\Pi(\theta, k, n)$.

Thus
$$\int_\theta^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\left(\int_0^{\frac{\pi}{2}}-\int_0^\theta\right)\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$
$$=K-F$$

$\left[\text{analogous to } \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x\right]$.

In this respect these integrals resemble the length of the arc of an ellipse, or of any oval symmetrical about two perpendicular axes. In fact, as will be presently shown, one of them, E, represents the length of an arc of an ellipse measured from the end of the minor axis. And it was this particular fact that led Legendre to style them Elliptic functions.

It will be noticed that the "complete" values are not numerical until the values of k, n are assigned, but are functions of k and n.

376. It is not the object of the present chapter to discuss elliptic functions at length, nor to establish the mode of reduction of $\int\frac{Pdx}{\sqrt{Q}}$ to one of the above canonical forms. These matters, as well as the addition formulae, will be postponed for later treatment. The present chapter must be regarded as an introductory description of such functions, so that the student will gradually grow accustomed to their use in cases that may appear in treating of the rectification of ellipses and other curves.

377. The Jacobian Notation.

In the integral $u=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ it is usual to call *the superior limit* θ the amplitude of u, and write it as

$$\theta=\operatorname{am} u,$$

and in accordance with the usual notation for inverse functions

$$u=\operatorname{am}^{-1}\theta.$$

Thus $$\operatorname{am}^{-1}\theta \equiv \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

If $x=\sin\theta$, we have $x=\sin\operatorname{am} u$, which is abbreviated into $x=\operatorname{sn} u$ and $u=\operatorname{sn}^{-1}x$.

Similarly, $\sqrt{1-x^2}=\cos\theta=\cos\operatorname{am} u$, abbreviated to $\operatorname{cn} u$;

$$\frac{x}{\sqrt{1-x^2}}=\tan\theta=\tan\operatorname{am} u, \text{ abbreviated to } \operatorname{tn} u.$$

The quantity $\sqrt{1-k^2\sin^2\theta}$, which we have called Δ, may be written $\Delta(\theta)$, (mod. k), or $\Delta(\theta, k)$ when it is necessary to put θ, k in evidence;

$$\therefore\ \sqrt{1-k^2\sin^2\theta}=\Delta\operatorname{am} u,$$

which is further abbreviated to $\operatorname{dn} u$.

Thus $$\operatorname{dn} u\equiv\Delta\operatorname{am} u\equiv\Delta\theta\equiv\sqrt{1-k^2\sin^2\theta}.$$

The names of these expressions, $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, are spoken as spelt, *i.e.* each letter read off.

378. Differentiation.

From the integral itself $\frac{du}{d\theta}=\frac{1}{\sqrt{1-k^2\sin^2\theta}}=\frac{1}{\operatorname{dn} u}$

Hence we can differentiate each of these functions.

Thus

$$\frac{d}{du}\operatorname{sn} u=\frac{d}{du}\sin\theta=\cos\theta\frac{d\theta}{du}=\operatorname{cn} u\operatorname{dn} u.$$

$$\frac{d}{du}\operatorname{cn} u=\frac{d}{du}\cos\theta=-\sin\theta\frac{d\theta}{du}=-\operatorname{sn} u\operatorname{dn} u,$$

$$\frac{d}{du}\operatorname{dn} u=\frac{d}{du}\sqrt{1-k^2\sin^2\theta}=-\frac{k^2\sin\theta\cos\theta\, d\theta}{\sqrt{1-k^2\sin^2\theta}\,du}=-k^2\operatorname{sn} u\operatorname{cn} u.$$

It follows that any expression involving such functions may be differentiated by the ordinary rules of differentiation.

379. **Integration.**

Conversely, we can integrate various forms involving such functions.

Thus
$$\int \mathrm{cn}\, u\, \mathrm{dn}\, u\, du = \mathrm{sn}\, u,$$
$$\int \mathrm{sn}\, u\, \mathrm{dn}\, u\, du = -\mathrm{cn}\, u,$$
$$\int \mathrm{sn}\, u\, \mathrm{cn}\, u\, du = -\frac{1}{k^2}\mathrm{dn}\, u.$$

380. The **elementary transformations** are merely those of ordinary trigonometry for single angles.

Thus
$$\mathrm{cn}^2 u = \cos^2\theta = 1 - \sin^2\theta = 1 - \mathrm{sn}^2 u,$$
$$\mathrm{sn}^2 u = \sin^2\theta = 1 - \cos^2\theta = 1 - \mathrm{cn}^2 u,$$
$$\mathrm{dn}^2 u = 1 - k^2 \sin^2\theta = 1 - k^2\, \mathrm{sn}^2 u,$$
$$\mathrm{tn}\, u = \frac{\mathrm{sn}\, u}{\mathrm{cn}\, u}, \quad \mathrm{ctn}\, u \equiv \cot \mathrm{am}\, u = \frac{\mathrm{cn}\, u}{\mathrm{sn}\, u} = \frac{1}{\mathrm{tn}\, u},$$
$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1,$$
$$\mathrm{dn}^2 u + k^2\, \mathrm{sn}^2 u = 1,$$
etc.

381. If $x = \sin\theta$, $F = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$,

which exhibits the quartic nature of the radical.

The equation $u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ may then be written as

$$x = \mathrm{sn}\, u, \text{ (mod. } k); \quad \text{or as } \mathrm{sn}(u, k);$$

and
$$u = \mathrm{sn}^{-1} x, \text{ (mod. } k); \quad \text{or } \mathrm{sn}^{-1}(x, k).$$

382. The earlier authors treating of this subject, Legendre, Euler and others, regarded the direct integral u as the function to be studied, and θ as its inverse.

The course followed by all later writers, Abel, Clifford, Ferrers, Cayley, Greenhill and others, is to regard θ as the direct function and u as its inverse.

383. The inverse nature of u is expressed in calling it $\mathrm{am}^{-1}\theta$, and this is in conformity with the simple case where $k=0$, viz.

$$u_1 \equiv \int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x,$$

whilst
$$u_2 \equiv \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \mathrm{sn}^{-1}(x, k).$$

384. **Complementary Modulus.**

It is desirable to introduce a new quantity k' such that

$$k^2 + k'^2 = 1;$$

k' is called the complementary modulus

385. **Transformations.**

Each of the functions, $\mathrm{sn}\, u$, $\mathrm{cn}\, u$, $\mathrm{dn}\, u$, $\mathrm{tn}\, u$, can be expressed in terms of the others.

If $\mathrm{sn}\, u = x$,
$$\left.\begin{aligned} \mathrm{cn}\, u &= \sqrt{1-x^2} = \sqrt{1-\mathrm{sn}^2 u} \\ \mathrm{dn}\, u &= \sqrt{1-k^2x^2} = \sqrt{1-k^2\,\mathrm{sn}^2 u} \\ \mathrm{tn}\, u &= \frac{x}{\sqrt{1-x^2}} = \frac{\mathrm{sn}\, u}{\mathrm{cn}\, u} = \frac{\mathrm{sn}\, u}{\sqrt{1-\mathrm{sn}^2 u}} \end{aligned}\right\}$$

If $\mathrm{cn}\, u = x$,
$$\begin{aligned} \mathrm{sn}\, u &= \sqrt{1-x^2} &&= \sqrt{1-\mathrm{cn}^2 u}, \\ \mathrm{dn}\, u &= \sqrt{1-k^2(1-x^2)} &&= \sqrt{k'^2+k^2\,\mathrm{cn}^2 u}, \\ \mathrm{tn}\, u &= \frac{\sqrt{1-x^2}}{x} &&= \frac{\sqrt{1-\mathrm{cn}^2 u}}{\mathrm{cn}\, u}. \end{aligned}$$

If $\mathrm{dn}\, u = x$,
$$\begin{aligned} \mathrm{sn}\, u &= \frac{\sqrt{1-x^2}}{k} &&= \frac{\sqrt{1-\mathrm{dn}^2 u}}{k}, \\ \mathrm{cn}\, u &= \frac{\sqrt{k^2-1+x^2}}{k} &&= \frac{\sqrt{\mathrm{dn}^2 u - k'^2}}{k}, \\ \mathrm{tn}\, u &= \frac{\sqrt{1-x^2}}{\sqrt{x^2-k'^2}} &&= \sqrt{\frac{1-\mathrm{dn}^2 u}{\mathrm{dn}^2 u - k'^2}}. \end{aligned}$$

If $\mathrm{tn}\, u = x$,
$$\begin{aligned} \mathrm{sn}\, u &= \frac{x}{\sqrt{1+x^2}} &&= \frac{\mathrm{tn}\, u}{\sqrt{1+\mathrm{tn}^2 u}}, \\ \mathrm{cn}\, u &= \frac{1}{\sqrt{1+x^2}} &&= \frac{1}{\sqrt{1+\mathrm{tn}^2 u}}, \\ \mathrm{dn}\, u &= \frac{\sqrt{1+k'^2x^2}}{\sqrt{1+x^2}} &&= \sqrt{\frac{1+k'^2\,\mathrm{tn}^2 u}{1+\mathrm{tn}^2 u}}. \end{aligned}$$

386. Inverse Notation.

With the inverse notation the same formulae would be written

$$\mathrm{sn}^{-1}x=\mathrm{cn}^{-1}\sqrt{1-x^2}=\mathrm{dn}^{-1}\sqrt{1-k^2x^2}=\mathrm{tn}^{-1}\frac{x}{\sqrt{1-x^2}},$$

$$\mathrm{cn}^{-1}x=\mathrm{sn}^{-1}\sqrt{1-x^2}=\mathrm{dn}^{-1}\sqrt{k'^2+k^2x^2}=\mathrm{tn}^{-1}\frac{\sqrt{1-x^2}}{x},$$

$$\mathrm{dn}^{-1}x=\mathrm{sn}^{-1}\frac{\sqrt{1-x^2}}{k}=\mathrm{cn}^{-1}\frac{\sqrt{x^2-k'^2}}{k}=\mathrm{tn}^{-1}\frac{\sqrt{1-x^2}}{\sqrt{x^2-k'^2}},$$

$$\mathrm{tn}^{-1}x=\mathrm{sn}^{-1}\frac{x}{\sqrt{1+x^2}}=\mathrm{cn}^{-1}\frac{1}{\sqrt{1+x^2}}=\mathrm{dn}^{-1}\frac{\sqrt{1+k'^2x^2}}{\sqrt{1+x^2}}.$$

387. Ex. $\mathrm{cn}^{-1}\left(\sqrt{\frac{\cos 2\theta+\cos 2\beta}{1+\cos 2\beta}},\ \sqrt{\frac{1+\cos 2\beta}{2}}\right)$

$$=\mathrm{sn}^{-1}\left(\sqrt{1-\frac{\cos 2\theta+\cos 2\beta}{1+\cos 2\beta}},\ \cos\beta\right)$$

$$=\mathrm{sn}^{-1}\left(\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\beta}},\ \cos\beta\right)$$

$$=\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\beta},\ \cos\beta\right).$$

Similarly

$$\mathrm{cn}^{-1}\left(\sqrt{\frac{\cos 2\theta-\cos 2\beta}{1-\cos 2\beta}},\ \sqrt{\frac{1-\cos 2\beta}{2}}\right)=\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\beta},\ \sin\beta\right).$$

388. Illustrative Examples of Reduction to the Legendrian Form.

1. Consider $I\equiv\int_0^x\frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}}\quad(x<b<a).$

Let $x=b\sin\theta$,

$$I=\int_0^\theta\frac{b\cos\theta\,d\theta}{\sqrt{(a^2-b^2\sin^2\theta)\,b^2\cos^2\theta}}$$

$$=\frac{1}{a}\int_0^\theta\frac{d\theta}{\sqrt{1-\frac{b^2}{a^2}\sin^2\theta}}=\frac{1}{a}F\left(\theta,\ \frac{b}{a}\right).$$

$\theta=\mathrm{am}\,(aI)$,

$x=b\sin\theta=b\,\mathrm{sn}\,(aI)$; mod. $\frac{b}{a}$,

$I=\frac{1}{a}\,\mathrm{sn}^{-1}\left(\frac{x}{b},\ \frac{b}{a}\right).$

2. Consider the case $I=\int_x^1\frac{dx}{\sqrt{1-x^4}}.$

Put $x=\cos\theta$,

$$I=\int_\theta^0 \frac{-\sin\theta\, d\theta}{\sin\theta\sqrt{1+\cos^2\theta}}$$

$$=\int_0^\theta \frac{d\theta}{\sqrt{2-\sin^2\theta}}=\frac{1}{\sqrt{2}}\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$$

$$=\frac{1}{\sqrt{2}}F\left(\theta,\ \frac{1}{\sqrt{2}}\right);$$

$$\theta=\operatorname{am}(I\sqrt{2});\ \text{mod. } \frac{1}{\sqrt{2}},$$

$$x=\operatorname{cn} I\sqrt{2},$$

$$I=\frac{1}{\sqrt{2}}\operatorname{cn}^{-1}\left(x,\ \frac{1}{\sqrt{2}}\right).$$

3. Consider

$$I=\int_x^\infty \frac{dx}{\sqrt{4(x-a)(x-b)(x-c)}},\quad \text{where } a<b<c<x.$$

Let $x-a=(c-a)\operatorname{cosec}^2\theta$.

Then
$$I=\int_\theta^0 \frac{-(c-a)\,2\operatorname{cosec}^2\theta\cot\theta\, d\theta}{\sqrt{4(c-a)\operatorname{cosec}^2\theta\{(c-a)\operatorname{cosec}^2\theta-(b-a)\}\{(c-a)\cot^2\theta\}}}$$

$$=\int_0^\theta \frac{d\theta}{\sqrt{(c-a)-(b-a)\sin^2\theta}}$$

$$=\frac{1}{\sqrt{c-a}}\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{b-a}{c-a}\sin^2\theta}}$$

$$=\frac{1}{\sqrt{c-a}}F\left(\theta,\ \sqrt{\frac{b-a}{c-a}}\right),$$

$$\theta=\operatorname{am}(\sqrt{c-a}\,I);\ \text{mod. } \sqrt{\frac{b-a}{c-a}},$$

$$\sqrt{\frac{c-a}{x-a}}=\sin\theta=\operatorname{sn}\ (\sqrt{c-a}\,I);$$

$$\therefore\ I=\frac{1}{\sqrt{c-a}}\operatorname{sn}^{-1}\left(\sqrt{\frac{c-a}{x-a}},\ \sqrt{\frac{b-a}{c-a}}\right).$$

4. Consider the case

$$I=\int_x^1 \frac{dx}{\sqrt{(1-x^2)(x+\lambda)}},\quad (\lambda<1).$$

Put $x+\lambda=(1+\lambda)\cos^2\phi$.

Thus,

$$I=\int_\phi^0 \frac{-(1+\lambda)\,2\cos\phi\sin\phi\, d\phi}{\sqrt{\{1-\lambda+(1+\lambda)\cos^2\phi\}\{(1+\lambda)-(1+\lambda)\cos^2\phi\}(1+\lambda)\cos^2\phi}}$$

$$=2\int_0^\phi \frac{d\phi}{\sqrt{2-(1+\lambda)\sin^2\phi}}$$

$$=\sqrt{2}\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{1+\lambda}{2}\sin^2\phi}}=\sqrt{2}\,F\left(\phi,\ \sqrt{\frac{1+\lambda}{2}}\right);$$

$$\therefore \quad \phi = \mathrm{am}\left(\frac{I}{\sqrt{2}}\right), \quad \cos\phi = \mathrm{cn}\left(\frac{I}{\sqrt{2}}\right),$$

$$I = \sqrt{2}\,\mathrm{cn}^{-1}\left(\sqrt{\frac{x+\lambda}{1+\lambda}},\ \sqrt{\frac{1+\lambda}{2}}\right).$$

If $x = \cos 2\theta$ and $\lambda = \cos 2\beta$,

$$I = \sqrt{2}\,\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\beta},\ \cos\beta)\right). \quad \text{(Art. 387.)}$$

Similarly,

$$\int_x^1 \frac{dx}{\sqrt{(1-x^2)(x-\lambda)}} = \sqrt{2}\,\mathrm{cn}^{-1}\left(\sqrt{\frac{x-\lambda}{1-\lambda}},\ \sqrt{\frac{1-\lambda}{2}}\right)$$

$$= \sqrt{2}\,\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\beta},\ \sin\beta\right).$$

These integrals are useful in the rectification of a Cassinian oval.

5. Consider the integration

$$I = \int_0^x \sqrt{\frac{a^2-x^2}{c^2-x^2}}\,dx, \quad x < c < a.$$

Putting $x = c\sin\theta$,

$$I = \int_0^\theta \sqrt{a^2 - c^2\sin^2\theta}\,d\theta = a\int_0^\theta \sqrt{1 - \frac{c^2}{a^2}\sin^2\theta}\,d\theta$$

$$= aE\left(\theta,\ \frac{c}{a}\right).$$

6. Consider the integration

$$I = \int_c^x \sqrt{\frac{x^2-a^2}{x^2-c^2}}\,dx, \quad \text{where } x > c > a.$$

Here we may put

$$\frac{x^2-c^2}{x^2-a^2} = \sin^2\omega.$$

Then
$$x = \frac{\sqrt{c^2-a^2\sin^2\omega}}{\cos\omega},$$

and
$$\frac{dx}{d\omega} = \frac{(c^2-a^2)\sin\omega}{\cos^2\omega\sqrt{c^2-a^2\sin^2\omega}};$$

$$\therefore \quad I = \int_0^\omega \sec^2\omega \frac{c^2-a^2}{\sqrt{c^2-a^2\sin^2\omega}}\,d\omega$$

$$= \int_0^\omega \sec^2\omega \frac{c^2-a^2\sin^2\omega - a^2\cos^2\omega}{\sqrt{c^2-a^2\sin^2\omega}}\,d\omega$$

$$= \int_0^\omega \sec^2\omega\sqrt{c^2-a^2\sin^2\omega}\,d\omega - \int_0^\omega \frac{a^2\,d\omega}{\sqrt{c^2-a^2\sin^2\omega}}$$

$$= \tan\omega\sqrt{c^2-a^2\sin^2\omega} + \int_0^\omega \frac{a^2\sin^2\omega - a^2}{\sqrt{c^2-a^2\sin^2\omega}}\,d\omega$$

$$= \tan\omega\sqrt{c^2-a^2\sin^2\omega} + \int_0^\omega \frac{(c^2-a^2)-(c^2-a^2\sin^2\omega)}{\sqrt{c^2-a^2\sin^2\omega}}\,d\omega$$

$$=\tan\omega\sqrt{c^2-a^2\sin^2\omega}+\frac{(c^2-a^2)}{c}\int_0^\omega\frac{d\omega}{\sqrt{1-\frac{a^2}{c^2}\sin^2\omega}}$$

$$-c\int_0^\omega\sqrt{1-\frac{a^2}{c^2}\sin^2\omega}\,d\omega$$

$$=\tan\omega\sqrt{c^2-a^2\sin^2\omega}+\frac{c^2-a^2}{c}F\left(\omega,\frac{a}{c}\right)-cE\left(\omega,\frac{a}{c}\right),$$

the integration needed in the rectification of a hyperbola.

7. Reduce the integral

$$I=\int_\theta^{\frac{\pi}{2}}\sqrt{\frac{a^2\cos^2\theta+b^2\sin^2\theta}{a^2\cos^2\theta+b^2\sin^2\theta+c^2}}\,d\theta,$$

to Legendrian form, taking $a>b$.

Write $b\tan\theta=a\cot\chi$.

Then
$$d\theta=-\frac{a}{b}\frac{\operatorname{cosec}^2\chi\,d\chi}{1+\frac{a^2}{b^2}\cot^2\chi}=-ab\frac{d\chi}{b^2\sin^2\chi+a^2\cos^2\chi}.$$

Hence

$$I=\int_\theta^{\frac{\pi}{2}}\frac{\sqrt{a^2+b^2\tan^2\theta}}{\sqrt{(a^2+c^2)+(b^2+c^2)\tan^2\theta}}\,d\theta$$

$$=-\int_\chi^0\frac{a\operatorname{cosec}\chi}{\sqrt{(a^2+c^2)+(b^2+c^2)\frac{a^2}{b^2}\cot^2\chi}}\,\frac{ab\,d\chi}{b^2\sin^2\chi+a^2\cos^2\chi}$$

$$=\int_0^\chi\frac{a^2b^2\,d\chi}{[a^2-(a^2-b^2)\sin^2\chi]\sqrt{(a^2+c^2)\,b^2\sin^2\chi+(b^2+c^2)\,a^2\cos^2\chi}}$$

$$=\int_0^\chi\frac{a^2b^2\,d\chi}{[a^2-(a^2-b^2)\sin^2\chi]\sqrt{(b^2+c^2)\,a^2-(a^2-b^2)\,c^2\sin^2\chi}}$$

$$=\frac{a^2b^2}{a^3\sqrt{(b^2+c^2)}}\int_0^\chi\frac{d\chi}{\left(1-\frac{a^2-b^2}{a^2}\sin^2\chi\right)\sqrt{1-\frac{a^2-b^2}{b^2+c^2}\frac{c^2}{a^2}\sin^2\chi}};$$

$$\therefore\ I=\frac{b^2}{a\sqrt{b^2+c^2}}\Pi\left(\chi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right),$$

an integral of the Third Species.

This integral is needed in the rectification and quadrature of a sphero-conic.

389. The Simple Pendulum. Dynamical illustration of the real periodicity of F.

Consider the *finite oscillation* of a simple circular pendulum. Let θ be the angular displacement of the rod from the vertical at time t, a the extreme value of θ, m the

mass of the bob, a the length of the rod. The line of zero-velocity in this case cuts the circle described by the bob at two points A, A' between which the bob oscillates.

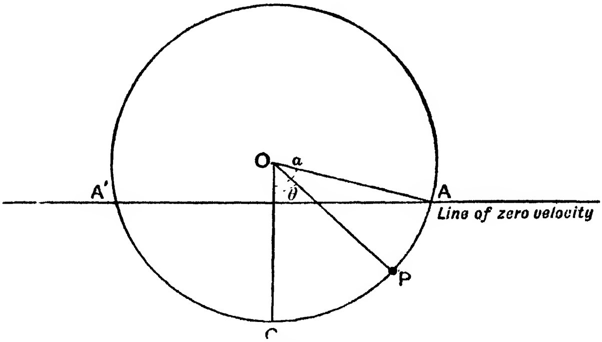


Fig. 38.

The energy equation is

$$\tfrac{1}{2} ma^2\dot{\theta}^2 = mg(a\cos\theta - a\cos\alpha)$$
$$= 2mga\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)$$

giving
$$t = \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_0^\theta \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}$$

t being measured from the instant at which the bob passes through its lowest position.

Let
$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi;$$

$$\therefore\ d\theta = \frac{2\sin\frac{\alpha}{2}\cos\phi\, d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}};$$

$$\therefore\ t = \sqrt{\frac{a}{g}}\int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}};$$

$$\therefore\ t = \sqrt{\frac{a}{g}}\,\mathrm{am}^{-1}\phi;\ \left(\mathrm{mod.}\ \sin\frac{\alpha}{2}\right),$$

$$i.e.\ \sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\,\mathrm{sn}\left(\sqrt{\frac{g}{a}}\,t\right).$$

When $\theta=\alpha$, and $\therefore\ \phi=\frac{\pi}{2}$, $\dot{\theta}=0$, and the time to this point, viz. T, is given by

$$T=\sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\phi}}=\sqrt{\frac{a}{g}}\,F_1(\phi);\ \left(\text{mod.}\sin\frac{\alpha}{2}\right),$$

and is the quarter period of the whole time of a complete oscillation. Writing K for F_1 it appears that the function

$$\int_0^{\phi}\frac{d\phi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\phi}}$$

is periodic and has a real period $4K$. Thus F_1 or K is called the "quarter period of the integral F," viz.,

$$K=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},\quad\text{where } k=\sin\frac{\alpha}{2}.$$

For an indefinitely small oscillation α is infinitesimal and $T=\frac{\pi}{2}\sqrt{\frac{a}{g}}$, the ordinary formula for a small oscillation.

390. **Complete Revolutions.**

Case of the pendulum making complete revolutions.

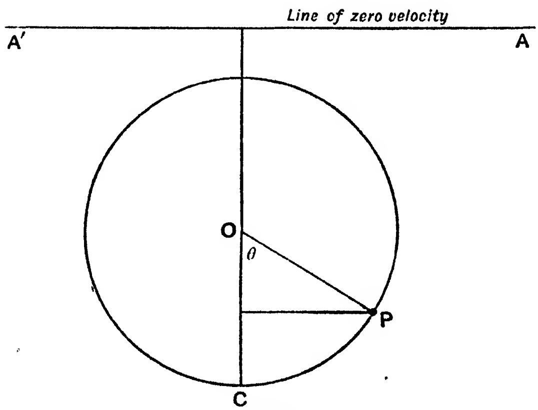


Fig. 39.

In the case when the line of zero velocity is at a height h ($>2a$) above the lowest point and does not cut the circle

described by the bob of the pendulum, the velocity of the bob is not exhausted when it arrives at the highest point of its path. The rod then makes complete revolutions and does not oscillate. In this case the energy equation is

$$\tfrac{1}{2}ma^2\dot{\theta}^2 = mg\left[h - a\overline{1-\cos\theta}\right];$$

$$\therefore\ \dot{\theta}^2 = \frac{2g}{a}\left(\frac{h}{a} - 2\sin^2\frac{\theta}{2}\right)$$

$$= \frac{2gh}{a^2}\left(1 - \frac{2a}{h}\sin^2\frac{\theta}{2}\right),$$

and
$$\frac{\sqrt{2gh}}{a}\,.\,t = \int_0^\theta \frac{d\theta}{\sqrt{1 - \frac{2a}{h}\sin^2\frac{\theta}{2}}}.$$

Let
$$\theta = 2\phi,$$

$$\frac{\sqrt{2gh}}{2a}t = \int_0^\phi \frac{d\phi}{\sqrt{1 - \frac{2a}{h}\sin^2\phi}}, \quad \left(\frac{2a}{h} < 1\right)$$

$$= F\left(\phi, \sqrt{\frac{2a}{h}}\right).$$

The time of a half revolution is given by $\phi = \frac{\pi}{2}$,

and
$$T = \frac{2a}{\sqrt{2gh}}F_1;\ \text{mod. } \sqrt{\frac{2a}{h}},$$

$$\frac{\theta}{2} = \phi = \operatorname{am}\frac{\sqrt{2gh}}{2a}t, \quad \sin\frac{\theta}{2} = \operatorname{sn}\frac{\sqrt{2gh}}{2a}t,$$

$$t = \frac{2a}{\sqrt{2gh}}\operatorname{sn}^{-1}\left(\sin\frac{\theta}{2}, \sqrt{\frac{2a}{h}}\right).$$

391. Legendre's Formulae.

Legendre gives (Exercises, p. 199) a list of results connecting various integrals at once by elementary means with the first two standard integrals of Art. 371, viz.

$$\int_0^\theta \frac{d\theta}{\Delta} = F(\theta, k), \quad \int_0^\theta \Delta\, d\theta = E(\theta, k).$$

These we may usefully reproduce for reference, and they will furnish a useful set of examples for the student to verify.

EXAMPLES (LEGENDRE).

Prove the following twelve results:

1. $$\int_0^\theta \frac{d\theta}{\Delta^3} = \frac{1}{k'^2} E(\theta, k) - \frac{k^2}{k'^2} \frac{\sin\theta\cos\theta}{\Delta}.$$

[Putting $P = \dfrac{\sin\theta\cos\theta}{\Delta}$ and differentiating, we obtain, after a little reduction, $k^2 \dfrac{dP}{d\theta} = \Delta - \dfrac{k'^2}{\Delta^3}$, then integrating we obtain the result stated.]

2. $$\int_0^\theta \frac{\sin^2\theta\, d\theta}{\Delta} = \frac{1}{k^2}(F - E).$$

3. $$\int_0^\theta \frac{\cos^2\theta\, d\theta}{\Delta} = \frac{1}{k^2}(E - k'^2 F).$$

4. $$\int_0^\theta \frac{\sec^2\theta\, d\theta}{\Delta} = \frac{1}{k'^2}(\Delta \tan\theta + k'^2 F - E).$$

5. $$\int_0^\theta \frac{\tan^2\theta\, d\theta}{\Delta} = \frac{1}{k'^2}(\Delta \tan\theta - E).$$

6. $$\int_0^\theta \frac{\tan^2\frac{\theta}{2}\, d\theta}{\Delta} = 2\Delta \tan\frac{\theta}{2} + F - 2E.$$

7. $$\int_0^\theta \frac{\sec^2\frac{\theta}{2}\, d\theta}{\Delta} = 2\Delta \tan\frac{\theta}{2} + 2F - 2E.$$

8. $$\int_0^\theta \Delta \sec^2\theta\, d\theta = \Delta \tan\theta + F - E.$$

9. $$\int_0^\theta \Delta \tan^2\theta\, d\theta = \Delta \tan\theta + F - 2E.$$

10. $$\int_0^\theta \Delta^3 d\theta = \frac{k^2}{3} \Delta \sin\theta\cos\theta + \frac{4-2k^2}{3} E - \frac{k'^2}{3} F.$$

11. $$\int_0^\theta \Delta \sin^2\theta\, d\theta = -\tfrac{1}{3}\Delta \sin\theta\cos\theta + \frac{2k^2-1}{3k^2} E + \frac{k'^2}{3k^2} F.$$

12. $$\int_0^\theta \Delta \cos^2\theta\, d\theta = \tfrac{1}{3}\Delta \sin\theta\cos\theta + \frac{1+k^2}{3k^2} E - \frac{k'^2}{3k^2} F.$$

392. Further discussion of Elliptic integrals is reserved till Chapter XXXI. Enough has been written to explain their nature, and the student will be able to employ the notation when wanted in the intervening chapters.

EXAMPLES.

1. By putting $x = \dfrac{1-\sin\theta}{1+\sin\theta}$, shew that

$$u = \int_x^1 \frac{dx}{\sqrt{x(1+6x+x^2)}} = \frac{1}{\sqrt{2}}\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{1}{\sqrt{2}}F\left(\theta, \frac{1}{\sqrt{2}}\right);$$

and that

$$x = \frac{1-\operatorname{sn}(u\sqrt{2})}{1+\operatorname{sn}(u\sqrt{2})}, \left(\text{mod. } \frac{1}{\sqrt{2}}\right); \quad i.e.\ u = \frac{1}{\sqrt{2}}\operatorname{sn}^{-1}\left(\frac{1-x}{1+x}, \frac{1}{\sqrt{2}}\right).$$

2. Prove that

$$F_1(\theta, k) = \frac{\pi}{2}\left(1 + \frac{1^2}{2^2}k^2 + \frac{1^2 . 3^2}{2^2 . 4^2}k^4 + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2}k^6 + \dots\right);$$

and that $\quad F_1(\theta, \frac{1}{10}) = 1{\cdot}574745$ very nearly.

3. Prove that

$$E_1(\theta, k) = \frac{\pi}{2}\left(1 - \frac{1}{2^2}k^2 - \frac{1^2 . 3}{2^2 . 4^2}k^4 - \frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2}k^6 - \dots\right).$$

4. Prove that

$$\Pi_1(\theta, k, n) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}k^2 - n\right)\frac{1}{2} + \left(\frac{1 . 3}{2 . 4}k^4 - \frac{1}{2}k^2 n + n^2\right)\frac{1 . 3}{2 . 4}\right.$$
$$+ \left(\frac{1 . 3 . 5}{2 . 4 . 6}k^6 - \frac{1 . 3}{2 . 4}k^4 n + \frac{1}{2}k^2 n^2 - n^3\right)\frac{1 . 3 . 5}{2 . 4 . 6}$$
$$\left. + \dots\right] \text{ if } n \text{ be } < 1.$$

5. Establish the truth of

(a) $$\left(\operatorname{sn} u + \frac{1}{\operatorname{cn} u}\right)^2 + \left(\operatorname{cn} u + \frac{1}{\operatorname{sn} u}\right)^2 = \left(1 + \frac{1}{\operatorname{sn} u \operatorname{cn} u}\right)^2.$$

(b) $$\frac{\dfrac{\operatorname{cn} u}{\operatorname{sn} u} - \dfrac{\operatorname{sn} u}{\operatorname{cn} u}}{\operatorname{cn} u + \operatorname{sn} u} = \frac{1}{\operatorname{sn} u} - \frac{1}{\operatorname{cn} u}.$$

(c) $$\left(\frac{1}{\operatorname{sn} u} - \operatorname{sn} u\right)\left(\frac{1}{\operatorname{cn} u} - \operatorname{cn} u\right)\left(\frac{\operatorname{sn} u}{\operatorname{cn} u} + \frac{\operatorname{cn} u}{\operatorname{sn} u}\right) = 1.$$

(d) $$\frac{1-\operatorname{cn} u}{1+\operatorname{cn} u} = \left(\frac{1}{\operatorname{sn} u} - \frac{\operatorname{cn} u}{\operatorname{sn} u}\right)^2.$$

6. Prove that

(1) $$\operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u = k'^2,$$

(2) $$\frac{1}{\operatorname{cn}^2 u} = 1 + \operatorname{tn}^2 u,$$

(3) $$\frac{1}{\operatorname{sn}^2 u} = 1 + \frac{1}{\operatorname{tn}^2 u}.$$

7. Prove that

(1) $\dfrac{d}{du}\,\mathrm{sn}^2 u = 2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u,$

(2) $\displaystyle\int \mathrm{sn}^p u\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,du = \frac{\mathrm{sn}^{p+1} u}{p+1},$

(3) $\displaystyle\int \frac{\mathrm{dn}\,u}{a+b\,\mathrm{cn}\,u}\,du = \frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\left(\frac{a\,\mathrm{cn}\,u+b}{a+b\,\mathrm{cn}\,u}\right), \quad a > b.$

8. Prove

$$2\,\mathrm{sn}\,u\,\mathrm{cn}\,v = \sin(\mathrm{am}\,u + \mathrm{am}\,v) + \sin(\mathrm{am}\,u - \mathrm{am}\,v),$$
$$2\,\mathrm{cn}\,u\,\mathrm{cn}\,v = \cos(\mathrm{am}\,u + \mathrm{am}\,v) + \cos(\mathrm{am}\,u - \mathrm{am}\,v).$$

9. By putting $x = a\cos\theta$, show that

$$\int_x^a \frac{dx}{\sqrt{a^4-x^4}} = \frac{1}{a\sqrt{2}}\,\mathrm{cn}^{-1}\left(\frac{x}{a}, \frac{1}{\sqrt{2}}\right).$$

10. Prove $\displaystyle\int_a^x \frac{dx}{\sqrt{x^4-a^4}} = \frac{1}{a\sqrt{2}}\,\mathrm{cn}^{-1}\left(\frac{a}{x}, \frac{1}{\sqrt{2}}\right).$

11. By putting $x = a\sqrt{\dfrac{1+z}{1-z}}$, show that

$$\int_x^\infty \frac{dx}{\sqrt{a^4+x^4}} = \frac{1}{2a}\,\mathrm{cn}^{-1}\left(\frac{x^2-a^2}{x^2+a^2}, \frac{1}{\sqrt{2}}\right).$$

12. Prove that

$$\int_x^\infty \frac{dx}{\sqrt{a^4+2a^2x^2\cos 2\alpha + x^4}} = \frac{1}{2a}\,\mathrm{cn}^{-1}\left(\frac{x^2-a^2}{x^2+a^2}, \sin\alpha\right).$$

13. Prove that

$$\mathrm{sn}\,K = 1,\ \mathrm{cn}\,K = 0,\ \mathrm{dn}\,K = k',\ \mathrm{tn}\,K = \infty.$$

14. Prove that

(1) $\dfrac{d}{du}(\mathrm{sn}\,u + \mathrm{cn}\,u)^n = n(\mathrm{sn}\,u + \mathrm{cn}\,u)^{n-1}(\mathrm{cn}\,u - \mathrm{sn}\,u)\,\mathrm{dn}\,u,$

(2) $\displaystyle\int (k^2\,\mathrm{sn}\,u + \mathrm{cn}\,u)^n (k^2\,\mathrm{cn}\,u - \mathrm{sn}\,u)\,\mathrm{dn}\,u\,du = \frac{(k^2\,\mathrm{sn}\,u + \mathrm{cn}\,u)^{n+1}}{(n+1)}.$

15. Draw graphs of $y = \Delta\theta$ and $y = \dfrac{1}{\Delta\theta}$, showing that the former consists of an undulating curve lying entirely below the line $y = 1$ and the other of an undulating line lying entirely above the line $y = 1$. Take the cases $k^2 = \frac{1}{2}$ and $k^2 = \frac{1}{4}$.

Show that the areas bounded by these curves, the x-axis, the y-axis and any ordinate at a point whose abscissa is θ represent $E(\theta)$ and $F(\theta)$ completely. Examine what happens in the limiting cases $k = 0$ and $k = 1$.

16. Show that the complete elliptic integrals of the First and Second Species may be expressed as

$$F_1 = \frac{\pi}{2} f(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2),$$

$$E_1 = \frac{\pi}{2} f(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2),$$

where $f(a, b, c, x)$ is the hypergeometric series

$$1 + \frac{a \cdot b}{1 \cdot c} x + \frac{a\,\overline{a+1}\,b\,\overline{b+1}}{1 \cdot 2\,c\,\overline{c+1}} x^2 + \dots .$$

17. Show by differentiating $F(\theta, k)$ and $E(\theta, k)$ with regard to k

$$\left.\begin{aligned} &(1)\ \frac{dE}{dk} = \frac{1}{k}(E - F), \\ &(2)\ \frac{dF}{dk} = \frac{1}{kk'^2}(E - k'^2 F) - \frac{k}{k'^2}\frac{\sin\theta\cos\theta}{\Delta}. \end{aligned}\right\}$$

Hence, eliminating E and F alternately, show that

$$\left.\begin{aligned} &(1 - k^2)\frac{d^2F}{dk^2} + \frac{1 - 3k^2}{k}\frac{dF}{dk} - F + \frac{\sin\theta\cos\theta}{\Delta^3} = 0, \\ &(1 - k^2)\frac{d^2E}{dk^2} + \frac{1 - k^2}{k}\frac{dE}{dk} + E - \frac{\sin\theta\cos\theta}{\Delta} = 0, \end{aligned}\right\}$$

and for the complete functions F_1, E_1

$$\left.\begin{aligned} &(1 - k^2)\frac{d^2F_1}{dk^2} + \frac{1 - 3k^2}{k}\frac{dF_1}{dk} - F_1 = 0, \\ &(1 - k^2)\frac{d^2E_1}{dk^2} + \frac{1 - k^2}{k}\frac{dE_1}{dk} + E_1 = 0. \end{aligned}\right\}$$

CHAPTER XII.

QUADRATURE (I).

PLANE SURFACES, CARTESIAN AND POLAR EQUATIONS.

393. The process of finding the area bounded by any defined contour line is termed **Quadrature**, or, which amounts to the same thing, Quadrature is the investigation of the *size of a square* which shall have the same area as that of the region under consideration.

The closed contour may consist of a single curve or of a system of several arcs of different curves or straight lines.

As we shall, in most cases, have to form some rough idea of the shape of the curves under discussion so as to be able properly to assign the limits of integration, the student should be familiar with the rules of procedure adopted in the tracing of curves for the various systems of coordinates by which they may be defined, Cartesians, Polars, etc., and for such information may be referred to the author's treatise on the Differential Calculus, Chap. XII.

394. It has been already shown (Art. 11) that the area bounded by a curve whose equation is $y=\phi(x)$, any pair of ordinates, $x=a$ and $x=b$ and the x-axis, may be considered as the limit of the sum of an infinite number of inscribed rectangles; and that the expression for the area is

$$\int_a^b y\,dx, \quad \text{or} \quad \int_a^b \phi(x)\,dx;$$

and it was assumed that $\phi(x)$ is a finite and continuous function of x, which does not change sign between these limits. In the same way the area bounded by the curve, two given abscissae, $y=c$ and $y=d$, and the y-axis is $\int_c^d x\,dy$.

If the angle between the coordinate axes were ω instead of 90°, we should have the expressions

$$\sin\omega\int_a^b y\,dx, \quad \text{or} \quad \sin\omega\int_c^d x\,dy$$

for the area.

395. Again, if the area desired be bounded by two given curves $y=\phi(x)$ and $y=\psi(x)$, and two given ordinates $x=a$ and $x=b$, it will be clear by similar reasoning that this area

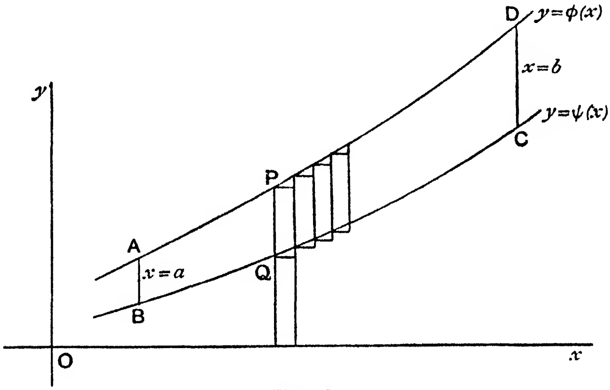


Fig. 40.

may be also considered as the limit of the sum of a series of rectangles constructed as indicated in the figure. If PQ be the portion of any of the ordinates intercepted between the curves, and δx the breadth of the elementary rectangle of which PQ is a side, the expression for the area will accordingly be

$$Lt_{\delta x=0}\sum_{x=a}^{x=b} PQ\,\delta x, \quad \text{or} \quad \int_a^b [\phi(x)-\psi(x)]\,dx,$$

where the same assumption is made as before as to $\phi(x)$ and $\psi(x)$ being finite and continuous from $x=a$ to $x=b$, and, moreover, $\phi(x)-\psi(x)$ must retain the same sign throughout the integration, *i.e.* the curves must not cross each other, and $\phi(x)$ has been assumed $>\psi(x)$ throughout.

396. **Case when the Coordinates are expressed in terms of a Parameter.**

We have regarded x as the independent variable. If this is not so the formula can be modified to suit the circumstances.

Suppose the curve defined by the equations

$$x=\phi(t), \quad y=\psi(t),$$

and that the values of t corresponding to the initial and final ordinates are t_1 and t_2.

Then $y\,\delta x=\psi(t)\,\phi'(t)\,\delta t$ to the first order, and in the limit

$$\int_a^b y\,dx=\int_{t_1}^{t_2}\psi(t)\,\phi'(t)\,dt,$$

it being supposed that the integrand remains finite and continuous throughout, and that as t changes continuously, increasing from the value t_1 to the value t_2, the point $(x, 0)$ also travels continuously along the x-axis from $(a, 0)$ to $(b, 0)$ without going over any part of its course more than once, and always in the same direction of increase of x.

397. **Case where the Arc is the Parameter.**

If the arc of the curve be the independent variable, being measured from some definite point on the curve, then at a point at which the gradient of the tangent is ψ, we have $dx=\cos\psi\,ds$, and we may write the expression $\int y\,dx$ as

$$\int y\frac{dx}{ds}ds, \quad \text{or} \quad \int y\cos\psi\,ds,$$

the limits of the integration with regard to s being the values of s corresponding to the beginning and end of the arc, and supposing that $y\cos\psi$ does not change sign.

In the same way we may write $\int x\,dy$ as

$$\int x\frac{dy}{ds}ds, \quad \text{or} \quad \int x\sin\psi\,ds.$$

398. **Area expressed by a Line Integral round the Contour.**

Let the formulae $\int y\cos\psi\,ds$, $\int x\sin\psi\,ds$ be applied to the evaluation of the area of a closed curve consisting of a single oval.

Let us suppose s measured from any point on the curve in such a direction that a person travelling along it in the direction of an increase of s has the area sought always to his left. Let ψ be the angle the tangent makes with the positive direction of the x-axis. Let $APBQ$ be the oval in question, and let

AL, BN be the tangents parallel to the y-axis. In the arc APB in the figure, ψ is changing from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, and $\cos\psi$ is positive. In the arc BQA ψ is changing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, and $\cos\psi$ is negative. Integrating then $\int y\cos\psi\, ds$ from A to B, through P, we obtain the area $ALMNBPA$ taken

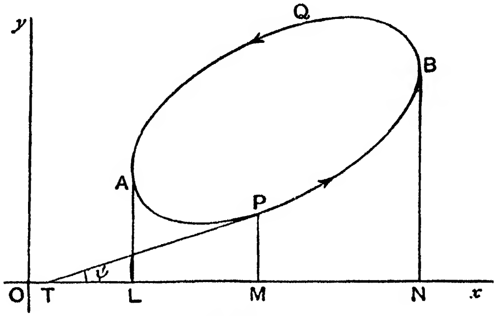


Fig. 41.

positively, whilst integration from B to A, through Q, obtains the area $BQALMNB$ taken negatively. Hence, to obtain the whole area, it is necessary to take our formula as $-\int y\cos\psi\, ds$ in integration round the whole perimeter in the counter-clockwise direction.

In the same way and under the same circumstances the area will also be given by $+\int x\sin\psi\, ds$.

This is the conventional mode of measuring s. If we measured in a clockwise direction the signs would both be reversed.

399. **Precautions.**

If the curve cuts itself once, having a node, as in the case of a lemniscate, it will be clear, from an inspection of the accompanying figure, that, in travelling completely round the whole curve, the directions in which the two loops are travelled round in continuously progressing in the direction of the increase of s, are one clockwise and the other counter-clockwise, and therefore, in conducting the integration completely round we get the difference of the areas of the two

loops with either formula, and in the case of equality of the loops the total line-integral of $x \sin \psi$, or of $y \cos \psi$, round the complete curve will be zero. If we require the absolute area

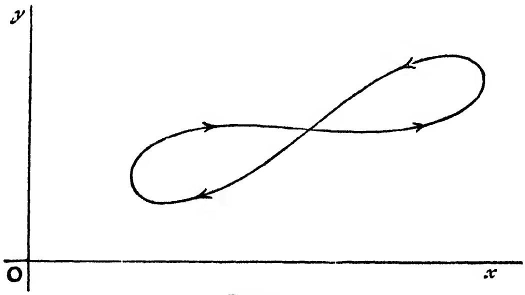


Fig. 42.

enclosed we must therefore treat each loop separately and add the positive results.

If in travelling continuously round the perimeter of the closed curve there be several nodes and several loops, we shall see in the same way that the total line-integral of $x \sin \psi$ or of $y \cos \psi$, will give the difference of the areas of the odd and even loops.

400. The student should examine the truth of the result in

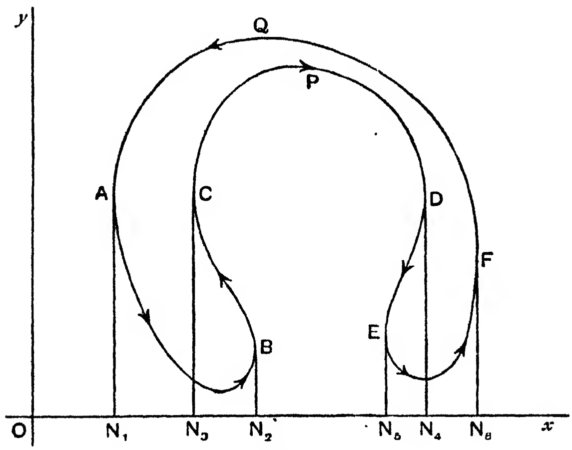


Fig. 43.

figures of other shapes—say a horseshoe-shaped closed curve, such as shown in Fig. 43.

Let $ABCDEF$ be the points at which the tangents are parallel to the y-axis, then if AN_1, BN_2, etc., be the ordinates, the integral

$-\int y \cos\psi\, ds$ yields

$$-\text{area } AN_1N_2B + \text{area } BCN_3N_2 - \text{area } CN_3N_4D$$
$$+ \text{area } DEN_5N_4 - \text{area } EN_5N_6F + \text{area } FQAN_1N_6F,$$

i.e. the closed area $ABCPDEFQA$.

401. If y be continuous, but $\dfrac{dy}{dx}$ discontinuous at points on the boundary of the figure, as at $ABCD$ in Fig. 44, the integration must be conducted along each of the portions into

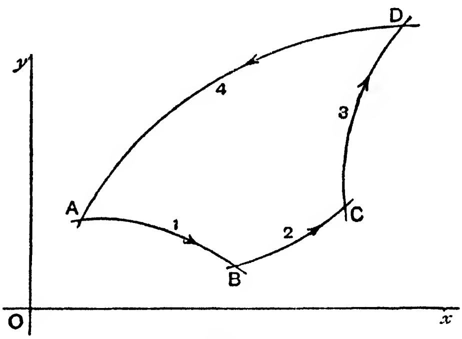


Fig. 44.

which the perimeter is divided by the discontinuities, but the same rule holds, as before, viz.

$$\text{area } ABCD = -\int_A^B y_1 \cos\psi_1\, ds_1 - \int_B^C y_2 \cos\psi_2\, ds_2$$
$$-\int_C^D y_3 \cos\psi_3\, ds_3 - \int_D^A y_4 \cos\psi_4\, ds_4,$$

or

$$= +\int_A^B x_1 \sin\psi_1\, ds_1 + \int_B^C x_2 \sin\psi_2\, ds_2$$
$$+\int_C^D x_3 \sin\psi_3\, ds_3 + \int_D^A x_4 \sin\psi_4\, ds_4,$$

suffixes denoting the several portions along which the integration is conducted, and s_1, s_2, s_3, etc., always being measured

"in the same sense" along the perimeter. Here the limits of the integrals are denoted by the points A, B, C ... of the perimeter successively arrived at in a continuous progress round it.

402. If $\phi(x)$ has an infinite ordinate between a and b, say at $x=c$, it has been explained that the infinity can be excluded by taking

$$\int_a^b \phi(x)\,dx \quad \text{to mean } Lt_{\epsilon=0}\left[\int_a^{c-\epsilon}\phi(x)\,dx+\int_{c+\epsilon}^b \phi(x)\,dx\right].$$

As, however, $\phi(x)$ will, in general, change sign in passing through an infinite value and the graph reappear from infinity at the opposite end of the asymptote, it will be desirable to consider the areas on opposite sides of the asymptote separately, and, after evaluation, add the *positive results* together. This is of course the same precaution we have had to take in Art. 395, in stipulating that $\phi(x)$ does not change sign between the limits, which would mean that part of the curve was above the x-axis and part below, so that carelessness in this respect would lead to a result which would represent the difference of the two portions of the area required instead of their sum.

403. **Illustrative Examples.**

1. Find the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$, the ordinates $x=c$, $x=d$ and the x-axis.

Here

$$\text{Area}=\int_c^d \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_c^d$$

$$=\frac{b}{2a}\left[d\sqrt{a^2-d^2}-c\sqrt{a^2-c^2}+a^2\left(\sin^{-1}\frac{d}{a}-\sin^{-1}\frac{c}{a}\right)\right],$$

a result obtainable without integration by reduction of the ordinates of the auxiliary circle in the ratio $b:a$.

For a quadrant of the ellipse, we put $d=a$ and $c=0$, and the above expression becomes $\frac{b}{2a}\cdot a^2\cdot\frac{\pi}{2}$ or $\frac{\pi ab}{4}$ giving πab for the area of the whole ellipse.

2. Find the area which lies in the first quadrant and is bounded by the circle $x^2+y^2=2ax$ and the parabola $y^2=ax$.

The curves touch at the origin and cut again at (a, a).

The limits for x are therefore from $x=0$ to $x=a$.

The area sought is therefore

$$\int_0^a \{\sqrt{2ax-x^2}-\sqrt{ax}\}\,dx.$$

Putting $x=a(1-\cos\theta)$ in the first

$$\int_0^a \sqrt{2ax-x^2}\,dx=\int_0^{\frac{\pi}{2}} a^2\sin^2\theta\,d\theta=a^2\frac{1}{2}\frac{\pi}{2}=\frac{\pi a^2}{4},$$

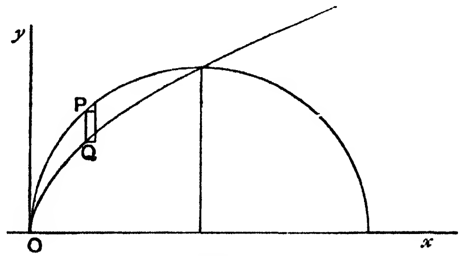


Fig. 45.

as of course might have been written down, being a quadrant of a circle of radius a; and

$$\int_0^a \sqrt{ax}\,dx=\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a=\tfrac{2}{3}a^2.$$

Thus the area required is $\qquad a^2\left(\frac{\pi}{4}-\frac{2}{3}\right).$

3. Find the area

(1) of the loop of the curve

$$x(x^2+y^2)=a(x^2-y^2),$$

(2) of the portion bounded by the curve and its asymptote.

Here $$y^2=x^2\frac{a-x}{a+x}.$$

To trace this curve, we observe

(1) It is symmetrical about the x-axis.

(2) No real part exists for points at which $x>a$ or $<-a$.

(3) It has an asymptote $x+a=0$.

(4) It goes through the origin, and the tangents there are $y=\pm x$.

(5) It crosses the x-axis when $x=a$, and at this point $\frac{dy}{dx}$ is infinite.

(6) The shape of the curve is therefore that shown in the figure (Fig. 46).

Hence, for the loop the limits of integration are 0 and a, and then double the result so as to include the portion below the x-axis.

For the portion between the curve and the asymptote, the limits are $x=-a$ to $x=0$ and double as before.

For the loop we therefore have,

$$\text{Area} = 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx.$$

For the portion between the curve and the asymptote we have,

$$\text{Area} = -2\int_{-a}^0 x\sqrt{\frac{a-x}{a+x}}\,dx.$$

The meaning of the negative sign is this: In choosing the + sign before the radical in $y=x\sqrt{\frac{a-x}{a+x}}$, we are tracing the portion of the curve *below the x-axis* on the left of the origin and *above the x-axis* on

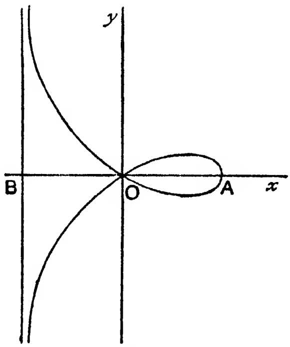


Fig. 46.

the right of the origin. Hence, y being negative between the limits $-a$ and 0, it is to be expected that we should obtain a negative result if we evaluate the expression,

$$Lt\sum_{x=-a}^{x=0} y\,dx.$$

Therefore we prefix the $-$ before the radical before integration to ensure a positive result.

To integrate $\int x\sqrt{\frac{a-x}{a+x}}\,dx$, put $x=a\cos\theta$ and $\therefore\ dx=-a\sin\theta\,d\theta$.

Thus $$\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx = -\int_{\frac{\pi}{2}}^0 a\cos\theta\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}\,a\sin\theta\,d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}(\cos\theta-\cos^2\theta)\,d\theta$$

$$= a^2\left(1-\frac{1}{2}\frac{\pi}{2}\right)=\left(1-\frac{\pi}{4}\right)a^2.$$

And $$\text{Area of loop} = 2a^2\left(1-\frac{\pi}{4}\right).$$

Again, $\int_{-a}^{0} x\sqrt{\frac{a-x}{a+x}}\,dx = -\int_{\pi}^{\frac{\pi}{2}} a\cos\theta\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}\,a\sin\theta\,d\theta$

$$= a^2\int_{\frac{\pi}{2}}^{\pi}(\cos\theta-\cos^2\theta)\,d\theta$$

$$= -a^2\left(1+\frac{\pi}{4}\right),$$

and the area between the asymptote and the curve

$$= 2a^2\left(1+\frac{\pi}{4}\right).$$

With regard to the latter portion of this example, it is to be observed that the greatest ordinate is an infinite one. In Arts. 11 and 394 it was assumed that every ordinate was finite. Is then the result obtained for the area bounded by the curve and the asymptote rigorously true?

It will be noted that the factor $(a+x)^{\frac{1}{2}}$ which occurs in the denominator and gives rise to the infinite value of y has an index <1 and positive. Hence (Art. 348) we infer that the principal value of the integral is finite.

Let us examine the case more closely, and integrate between $-a+\epsilon$ and 0, where ϵ is some small positive quantity, so as to exclude the infinite ordinate at the point where $x=-a$.

We have as before

$$\int_{-a+\epsilon}^{0} x\sqrt{\frac{a-x}{a+x}}\,dx = a^2\int_{\frac{\pi}{2}}^{\pi-\delta}(\cos\theta-\cos^2\theta)\,d\theta,$$

where $-a+\epsilon = a\cos(\pi-\delta)$, so that δ is a small positive angle, viz.

$$\cos^{-1}\left(1-\frac{\epsilon}{a}\right).$$

This integral is then

$$a^2\left[\sin\theta-\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_{\frac{\pi}{2}}^{\pi-\delta} = a^2\left[(\sin\delta-1)-\left(\frac{\pi-\delta}{2}-\frac{\pi}{4}\right)+\frac{\sin 2\delta}{4}\right]$$

$$= a^2\left[-1-\frac{\pi}{4}+\frac{\delta}{2}+\sin\delta+\frac{\sin 2\delta}{4}\right],$$

and approaches indefinitely closely to the former result

$$-a^2\left(1+\frac{\pi}{4}\right),$$

when δ is made to diminish without limit to zero.

4. Prove that the whole area of the curve

$$x^4-2ax^2y+a^2(x^2+y^2)=a^4 \text{ is } \pi a^2.$$

Here, solving for y,

$$y=\frac{x^2}{a}\pm\sqrt{a^2-x^2}$$

$$=y_1\pm y_2,$$

where y_1 is the ordinate of a parabola and y_2 that of a circle of radius a.

The area of a strip parallel to the y-axis and of breadth δx is

$$[(y_1+y_2)-(y_1-y_2)]\,\delta x = 2y_2\,\delta x,$$

and the total area of the curve is $2\int_{-a}^{a} y_2\,dx$, *i.e.* the same as that of the circle, $=\pi a^2$.

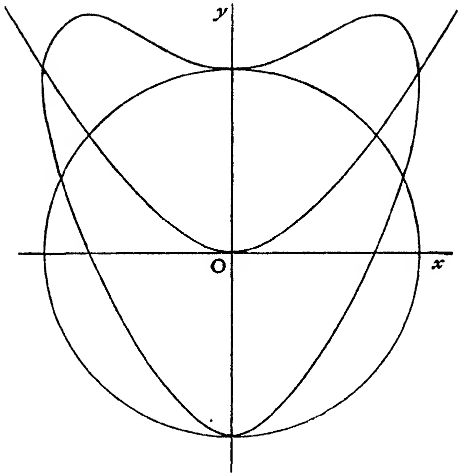


Fig. 47.

404. The last example will suggest to the student that if the curve $y=\phi(x)\pm\sqrt{a^2-x^2}$ be drawn, it may be regarded as constructed by means of two curves, viz.

$$y_1=\phi(x) \quad \text{and} \quad y_2=\sqrt{a^2-x^2},$$

the latter being a circle and the ordinates of the resultant curve being the sum or difference of y_1 and y_2, viz.

$$y=y_1\pm y_2,$$

and as in the parabola and circle of Ex. 4, the closed curve formed will be divisible into strips of length $(y_1+y_2)-(y_1-y_2)$ and breadth δx, and therefore of area $2y_2\,\delta x$.

Hence the area in any such case is $2\int_{-a}^{a} y_2\,dx=\pi a^2$, and is the same as that of the circle.

This curve, if written in rational form, is

$$x^2+y^2+[\phi(x)]^2-a^2=2y\phi(x),$$

$\phi(x)$ being supposed rational. And the areas of all such curves are $=\pi a^2$.

Similarly, for curves of form

$$x^2+y^2+[\phi(y)]^2-a^2=2x\phi(y),$$

which are clearly to be constructed as

$$x=\phi(y)\pm\sqrt{a^2-y^2},$$

and consist of closed curves of area πa^2; or more generally still, if $y^2=f(x)$ be a closed curve whose area is A, then another curve can be constructed from it of form

$$y=\phi(x)\pm\sqrt{f(x)},$$

i.e.

$$y^2-2y\phi(x)+[\phi(x)]^2-f(x)=0$$

whose area is also A.

For the areas of corresponding elementary strips parallel to the y-axis are for the original curve and the derived curve respectively,

$$2\sqrt{f(x)}\,\delta x \quad\text{and}\quad [\{\phi(x)+\sqrt{f(x)}\}-\{\phi(x)-\sqrt{f(x)}\}]\,\delta x,$$

which are equal, and therefore their sums are equal also. Similarly for

$$x^2-2x\phi(y)+[\phi(y)]^2-f(y)=0.$$

405. In Art. 395 it is shown that the area between the two curves $y=\phi(x)$ and $y=\psi(x)$ and a pair of ordinates $x=a, x=b$ is

$$\int_a^b [\phi(x)-\psi(x)]\,dx.$$

It may be that $y=\phi(x)$ and $y=\psi(x)$ are different branches of *the same curve*. This is really what happens in the various cases considered in the last article.

406. Ex. Consider the case of an ellipse

$$ax^2+2hxy+by^2=1,\quad h^2<ab.$$

If y_1, y_2 are the ordinates for any abscissa x,

$$y_1+y_2=-\frac{2h}{b}x,$$

$$y_1y_2=\frac{ax^2}{b}-\frac{1}{b};$$

$\therefore$ the length of the strip is

$$y_1-y_2=2\sqrt{\frac{1}{b}-\frac{ab-h^2}{b^2}x^2}=2\,\frac{\sqrt{ab-h^2}}{b}\sqrt{\frac{b}{ab-h^2}-x^2}.$$

And the area is

$$\int_{x_1}^{x_2} (y - y_2)\, dx, \text{ between ordinates } x_1 \text{ and } x_2,$$

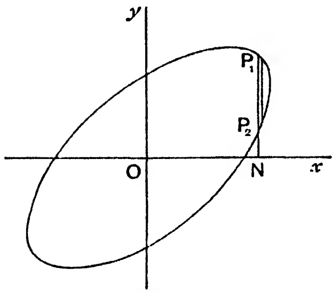


Fig. 48.

or for the whole ellipse

$$\frac{\sqrt{ab-h^2}}{b} \times \text{area of circle of radius } \frac{\sqrt{b}}{\sqrt{ab-h^2}}, \text{ i.e. } \frac{\pi}{\sqrt{ab-h^2}}.$$

Examples.

1. Obtain the area bounded by a parabola and its latus rectum. A series of ordinates are drawn between the vertex and the latus rectum, parallel to the latter, viz. $x=\left(\frac{r}{n}\right)^{\frac{2}{3}}a$, where $r=1, 2, 3, \ldots n-1$. Show that they divide the aforementioned area into n equal parts.

2. Obtain the areas bounded by the curve, the x-axis, and the specified ordinates in the following cases:

(a) The catenary $y=c\cosh\frac{x}{c}$, from $x=0$ to $x=h$.

(b) The logarithmic curve $y=e^x$, from $x=0$ to $x=h$.

(c) The logarithmic curve $y=\log_e x$, from $x=1$ to $x=h$ $(h>1)$

(d) The ellipse $y=\frac{b}{a}\sqrt{a^2-x^2}$, from $x=\sqrt{a^2-b^2}$ to $x=a$.

(e) The hyperbola $xy=k^2$, from $x=a$ to $x=b$, a and b both >0; first, if the hyperbola be rectangular, second, if the angle between the asymptotes be ω.

(f) The curve $y=xe^{x^2}$, from $x=0$ to $x=h$.

3. Obtain the area (1) bounded by the parabolas $y^2=4ax$, $x^2=4ay$;
(2) bounded by the parabolas $y^2=4ax$, $x^2=4by$.

In what ratio is this area divided by the common chord in each case?

4. Find the areas of the portions into which the ellipse $x^2/a^2+y^2/b^2=1$ is divided (1) by the straight line $y=c$;

(2) by the two straight lines $y=c$, $x=d$, supposed to cut within the ellipse.

5. Trace the curve $x^2y^2=a^2(y^2-x^2)$, and find the whole area included between the curve and its asymptotes.

6. Find the area between the curve $y^2(a+x)=(a-x)^3$ and its asymptote.

7. Find the area of the loop of the curve

$$y^2x+(x+a)^2(x+2a)=0.$$

8. Two curves in which $y \propto x^m$ and two in which $y \propto x^n$ form a quadrilateral; show that its area is

$$\frac{m \sim n}{(m+1)(n+1)}(x_1y_1-x_2y_2+x_3y_3-x_4y_4),$$

where (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are the coordinates of the corners taken in order. [TRINITY, 1891.]

9. By means of the integral $\int y\,dx$ taken round the contour of the triangle formed by the intersecting lines,

$$y=a_1x+b_1,$$
$$y=a_2x+b_2,$$
$$y=a_3x+b_3,$$

show that they enclose the area

$$\frac{(b_1-b_3)^2}{2(a_1-a_3)}+\frac{(b_2-b_1)^2}{2(a_2-a_1)}+\frac{(b_3-b_2)^2}{2(a_3-a_2)}.$$

[SMITH'S PRIZE, 1876.]

10. A four-sided figure is formed by the three parabolas,

$$y^2-9ax+81a^2=0,$$
$$y^2-4ax+16a^2=0,$$
$$y^2-ax+a^2=0,$$

and the axis of x. Prove that its area is $12a^2$, and is equal to the area enclosed by the chords of the area. [COLLEGES α, 1886.]

11. Find the curvilinear area enclosed between the parabola $y^2=4ax$ and its evolute. [OXF. I. P., 1889.]

12. Show that the area cut off from a semi-cubical parabola by a tangent is divided by the tangent at the cusp in the ratio 64 : 17. [OXFORD II. P., 1889.]

13. (i) Find the area of a loop of the curve

$$ay^2=x^2(a-x).$$ [I. C. S., 1882.]

(ii) Find the whole area of the curve

$$a^2y^2=a^2x^2-x^4.$$ [I. C. S., 1881.]

14. Trace the curve $a^2x^2=y^3(2a-y)$, and prove that its area is equal to that of the circle whose radius is a. [I. C. S., 1887 AND 1890.]

15. Trace the curve $a^4y^2=x^5(2a-x)$, and prove that its area is to that of the circle of radius a as 5 to 4.

16. Find the area of the curve

$$y(x^2+1)=x^3-1 \text{ from } x=0 \text{ to } x=1.$$

[ST. JOHN'S, 1881.]

17. (i) Find the area between $y^2=\dfrac{x^3}{a-x}$ and its asymptote.

(ii) Show that the whole area between

$$y^2(x-a)(b-x)=c^2x^2$$

and its asymptote is $\pi c(a+b)$. [OX. II. P., 1903.]

(iii) Show that the area between the curve

$$y^2x=a^3-a^2x$$

and its asymptote is that of a circle of radius a. [ST. JOHN'S, 1889.]

18. Find the area between the axis of x, the hyperbola $x^2/a^2-y^2/b^2=1$, and the line $y=x\tan\alpha$, where

$$\frac{b}{a}>\tan\alpha>0.$$

[OX. I. P., 1901.]

If A be the vertex, O the centre, and P any point on the hyperbola $x^2/a^2-y^2/b^2=1$, prove that

$$x=a\cosh\frac{2S}{ab},\quad y=b\sinh\frac{2S}{ab},$$

where S is the sectorial area AOP [MATH. TRIPOS, 1885.]

19. Find by integration the area lying on the same side of the axis of x as the positive part of the axis of y, and which is contained by the lines

$$y^2=4ax,$$
$$x^2+y^2=2ax,$$
$$x=y+2a.$$

Express the area both when x is the independent variable and when y is the independent variable. [COLLEGES, 1882.]

20. Prove that the area of the loop of

$$a(x-y)(x-2y)=y^3 \text{ is } \frac{a^2}{60}.$$

[COLL. β, 1891.]

21. Find the areas of the two regions of space bounded by the straight line $y=c$, and the curves whose equations are

$$x^2+y^2=c^2,$$
$$y^2+4x^2=4c^2.$$

[I. C. S., 1891.]

22. Prove that the area contained between the curve

$$(x+3a)(x^2+y^2)=4a^3$$

and its asymptote is $3a^2\sqrt{3}$. [OXF. I. P., 1901.]

23. Prove that the area of the curve

$$x^4 - 3ax^3 + a^2(2x^2 + y^2) = 0$$

is $\frac{3}{8}\pi a^2$. [MATH. TRIP., 1893.]

24. Find the area of one loop of the curve

$$y^4 - y^2 + x^2 = 0. \qquad \text{[COLLEGES } \alpha\text{, 1885.]}$$

25. Through the cusp of the evolute of a parabola, a line is drawn perpendicular to the axis. Show that it divides the area between the parabola and the evolute in the ratio 17 : 5. [C. S., 1896.]

26. Show that the ordinate $x = a$ divides the area between $y^2(2a - x) = x^3$ and its asymptote into two parts in the ratio

$$3\pi - 8 : 3\pi + 8. \qquad \text{[MATH. TRIP. I., 1912.]}$$

407. **Sectorial Areas. Polar Coordinates.**

When the area to be found is bounded by a curve $r = f(\theta)$ and two radii vectores drawn from the origin in given directions, we may divide the area into elementary sectors with the same small angle $\delta\theta$, as shown in the figure. Let the

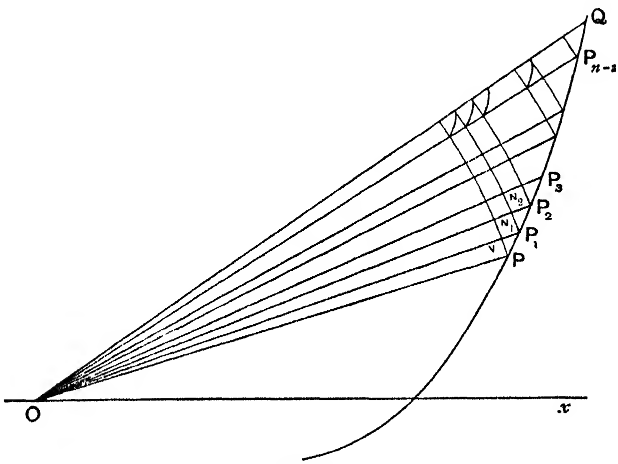


Fig. 49.

area to be found be bounded by the arc PQ and the radii vectores OP, OQ. Draw radii vectores OP_1, OP_2, ... OP_{n-1} at equal angular intervals, so that

$$P\hat{O}P_1 = P_1\hat{O}P_2 = \ldots = P_{n-1}\hat{O}Q = \delta\theta.$$

Then by drawing with centre O the successive circular arcs PN, P_1N_1, P_2N_2, etc., it may be at once seen that the limit of the sum of the circular sectors OPN, OP_1N_1, OP_2N_2, etc.

is the area required. For the remaining elements PNP_1, $P_1N_1P_2$, $P_2N_2P_3$, etc., may be made rotate about O so as to occupy new positions on the greatest sector, say $OP_{n-1}Q$, as indicated in the figure Their sum is plainly less than this sector; and in the limit when the angle of this sector is indefinitely diminished its area also diminishes without limit, provided the radius vector OQ is finite.

Now the area of a circular sector is

$$\tfrac{1}{2}(\text{radius})^2 \times \text{circular measure of angle of sector.}$$

Thus the area required $= \frac{1}{2}\text{Lt}\,\Sigma\, r^2\,\delta\theta$, the summation being conducted for such values of θ as lie between $\theta = x\hat{O}P$ and $\theta = x\hat{O}P_{n-1}$ *i.e.*, $x\hat{O}Q$ in the limit, Ox being the initial line.

In the notation of the integral calculus, if $x\hat{O}P = \alpha$ and $x\hat{O}Q = \beta$, this will be expressed as

$$\tfrac{1}{2}\int_\alpha^\beta r^2\,d\theta \quad \text{or} \quad \tfrac{1}{2}\int_\alpha^\beta [f(\theta)]^2\,d\theta.$$

It is assumed that $f(\theta)$ is finite and continuous from $\theta = \alpha$ to $\theta = \beta$ inclusive.

408. If the curve consist of a closed oval and the origin be within it, the limits of integration to find the whole area are 0 and 2π, viz. the extent to which a radius vector must rotate about O to sweep out the whole area (Fig. 50).

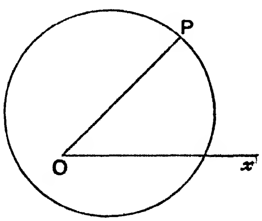


Fig. 50.

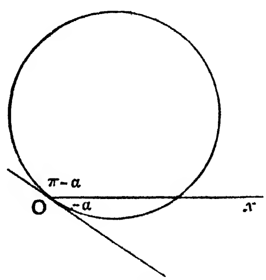


Fig. 51.

If the origin be on the perimeter of the oval, and if it be not a singular point, the limits will be from $-\alpha$ to $+\pi-\alpha$ if the tangent at the origin makes an angle $-\alpha$ with the x-axis as shown in Fig. 51.

In this case, if the initial line be an axis of symmetry, it is sufficient to integrate from 0 to $\frac{\pi}{2}$ and double the result (Fig. 52)

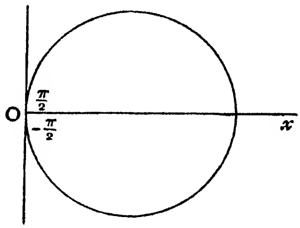


Fig. 52.

If there be a loop and the origin be a singular point on the curve at which the tangents make an angle $2a$ with each

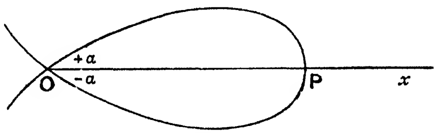


Fig. 53.

other, and if the initial line be an axis of symmetry, the limits for the area of the loop will be 0 and a and double the result (Fig. 53).

409. Another Expression for an Area.

Let (x, y) be the Cartesian coordinates of any point P on a curve, $(x+\delta x,\ y+\delta y)$ those of an adjacent point Q. Let

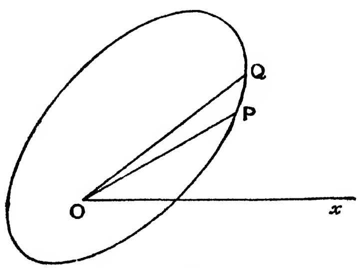


Fig. 54.

(r, θ), $(r+\delta r,\ \theta+\delta\theta)$ be the corresponding polar coordinates. Also, we shall suppose that, in travelling along the curve from P to Q on an infinitesimal arc PQ, the direction of rotation of

the radius vector OP is counter-clockwise, and that the area to be considered is on the *left hand to a person travelling in this direction* (Fig. 54).

Then, to the first order of infinitesimals,

$$\begin{aligned}\tfrac{1}{2}r^2\delta\theta &= \text{sectorial area } OPQ \\ &= \tfrac{1}{2}\begin{vmatrix} x, & y, & 1 \\ x+\delta x, & y+\delta y, & 1 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \tfrac{1}{2}(x\,\delta y - y\,\delta x).\end{aligned}$$

Hence, another expression for the area of a sectorial portion of a curve bounded by a definite portion of an arc is

$$\frac{1}{2}\int(x\,dy-y\,dx) \quad \text{or} \quad \frac{1}{2}\int\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)ds,$$

the limits being the initial and final values of s, corresponding to the portion of the sectorial area to be found.

Obviously we might take any other independent variable, say t, and supposing the curve expressed as

$$x=f(t), \quad y=F(t),$$

and that the values of t, corresponding to the beginning and end of the arc, are t_1 and t_2 respectively,

$$\text{sectorial area}=\frac{1}{2}\int_{t_1}^{t_2}\{f(t)\,F'(t)-f'(t)\,F(t)\}\,dt.$$

If the curve be a closed curve and the origin lies within it, the limits for θ are 0 and 2π, and

$$\text{area}=\frac{1}{2}\int_0^{2\pi}r^2\,d\theta.$$

In the same case, if we take the formula

$$\frac{1}{2}\int(x\,dy-y\,dx) \quad \text{or} \quad \frac{1}{2}\int[f(t)\,F'(t)-f'(t)\,F(t)]\,dt,$$

the limits for t must be such that the point (x, y) travels once, and once only, completely round the curve.

410. If the origin lies outside the curve, as the current point P travels round the curve, we obtain sectorial elements such as OP_1Q_1 (Fig. 55), including portions of space such as OP_2Q_2,

shown in the figure, which lie outside the curve. These portions are, however, ultimately removed from the whole integral

$$\frac{1}{2}\int(x\,dy - y\,dx),$$

when the point P travels over the element P_2Q_2, for the

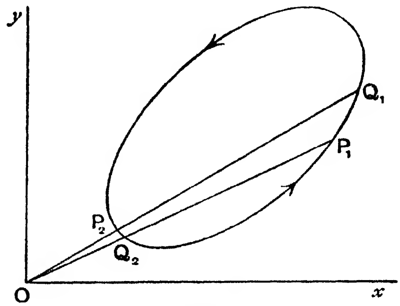


Fig. 55.

sectorial element OP_2Q_2 is reckoned negatively as θ is decreasing and $\delta\theta$ is negative.

411. Precautions.

If the curve cross itself as in Fig. 56, the expression

$$\frac{1}{2}\int(x\,dy - y\,dx),$$

taken round the whole perimeter, no longer represents the *sum* of the areas of the several regions. For draw two contiguous radii vectores OP_1, OQ_1, cutting the curve again at Q_2, P_3, Q_4 and P_2, Q_3, P_4 respectively. Then, in travelling round the curve continuously through the complete perimeter, we obtain positive elements such as OP_1Q_1 and OP_3Q_3, and negative elements such as OP_2Q_2 and OP_4Q_4.

Now, taking all these elements positively,

$$OP_1Q_1 - OP_2Q_2 + OP_3Q_3 - OP_4Q_4$$
$$= \text{quadrilateral } P_1Q_1P_4Q_4 - \text{quadrilateral } P_2Q_2P_3Q_3,$$

and in integrating for the whole curve we therefore obtain the difference of the two regions instead of their sum.

Similarly, if the curve cuts itself more than once, the integral $\frac{1}{2}\int(x\,dy - y\,dx)$ gives the difference of the sum of

the odd regions and the sum of the even regions. Thus, to obtain the absolute area bounded by such a curve, we must take *our limits for each area separately* and obtain the *absolute area of each region*, and then *add together the results*. It is

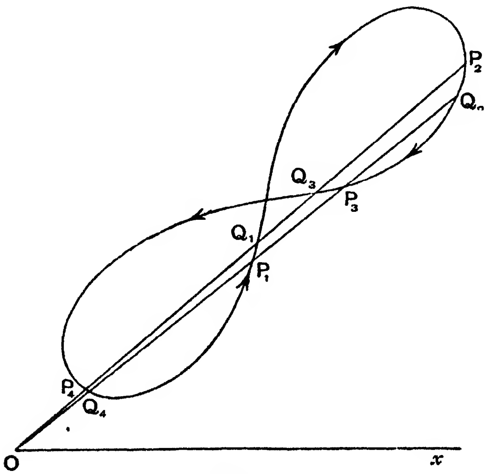


Fig. 56.

obvious that in curves consisting of several equal regions, or loops, it will be sufficient to ascertain the area of any one, and then to multiply that area by the number of the loops.

412. **Another Form.**

If we write $\frac{y}{x}=v$, we have

$$x\,dy-y\,dx=x^2\,dv,$$

and accordingly we may transform the formula

$$\frac{1}{2}\int(x\,dy-y\,dx) \text{ into } \frac{1}{2}\int x^2\,dv.$$

This is equivalent to a choice of new coordinates, of which one is the Cartesian abscissa and the other, viz. v, is the tangent of the polar angle θ.

In using the formula, x is to be expressed in terms of v, and the limits of the integration so chosen that the current point (x, y) travels from the beginning to the end of the arc, *i.e.* if α, β be the limits for θ, $\tan\alpha$ and $\tan\beta$ will be the limits for v.

In using this formula, however, *care must be taken not to integrate through an infinite value of v.* It must be remembered that $v=\tan\theta$ and becomes infinite when $\theta=\frac{\pi}{2}$, or any odd multiple of $\frac{\pi}{2}$.

413. For example, if we apply this method to the area of an ellipse $x^2/a^2+y^2/b^2=1$, putting $y/x=v$, we have

$$x^2\left(\frac{1}{a^2}+\frac{v^2}{b^2}\right)=1,$$

and
$$\text{Area}=\frac{1}{2}\int x^2\,dv=\frac{1}{2}\int\frac{b^2\,dv}{\frac{b^2}{a^2}+v^2}=\left[\frac{ab}{2}\tan^{-1}\frac{av}{b}\right]$$

between properly chosen limits. Now, in the first quadrant v varies from 0 to ∞. Hence the area of a quadrant $=\frac{ab}{2}\cdot\frac{\pi}{2}=\frac{\pi ab}{4}$, and therefore the area of the ellipse $=\pi ab$.

It will be noted that the formula

$$\text{Area}=\frac{1}{2}\int(x\,dy-y\,dx),\quad \textit{i.e.}\quad \frac{1}{2}\int\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)ds,$$

is equivalent to half the sum of $\int x\frac{dy}{ds}\,ds$ and $-\int y\frac{dx}{ds}\,ds$, each of which has been shown to represent the area when the integration follows the complete perimeter.

414. It may also be worth the student's notice to remark that the problem of finding the area bounded by $y=\phi(x)$, the x-axis, and a pair of ordinates $x=a$, $x=b$, viz. $A=\int_a^b\phi(x)\,dx$, is manifestly the same as that of finding the mass of a rod of small section but of line density $\phi(x)$, of length $b-a$, and of any shape if x be measured along the rod. For the mass of a length δx of the rod is $\phi(x)\,\delta x$, the limit of the sum of such expressions being required, when δx is indefinitely diminished, between limits $x=a$ and $x=b$, that is $\int_a^b\phi(x)\,dx$.

415. Illustrative Examples.

1. Obtain the area of the semicircle bounded by $r=a\cos\theta$ and the initial line.

Here the radius vector sweeps over the angular interval from

$$\theta=0\quad\text{to}\quad\theta=\frac{\pi}{2}.$$

Hence the area is

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta=\frac{a^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi a^2}{8}, \quad \textit{i.e.}\ \frac{1}{2}\pi\,(\text{radius})^2.$$

2. Find the area of the lemniscate $r^2=a^2\cos 2\theta$.

Here the axis is a line of symmetry; the tangents at the origin are $\theta=\pm\frac{\pi}{4}$.

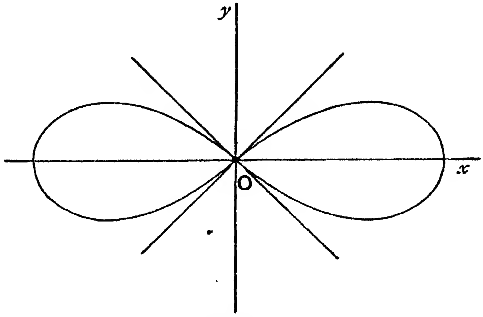


Fig. 57.

The area is therefore

$$4\times\frac{1}{2}a^2\int_0^{\frac{\pi}{4}}\cos 2\theta\,d\theta=2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}=a^2.$$

3. Find the area of the pedal of an ellipse with regard to the centre

With the usual axes and notation, the equation of the pedal is

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta,$$

and
$$\text{Area}=4\times\frac{1}{2}\int_0^{\frac{\pi}{2}}(a^2\cos^2\theta+b^2\sin^2\theta)\,d\theta=\pi\frac{a^2+b^2}{2}.$$

4. Find the area of one loop of the curve $r=a\sin 3\theta$.

The curve consists of three equal loops, as indicated in the figure

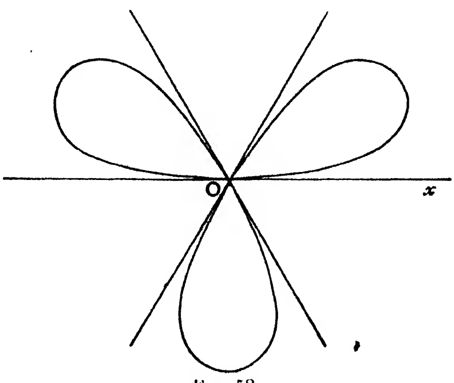


Fig. 58.

The proper limits for the integration extending over the first loop are $\theta=0$ and $\theta=\frac{\pi}{3}$, for these are two successive values of θ for which r vanishes :

$$\therefore \text{ Area of loop} = \frac{1}{2}\int_0^{\frac{\pi}{3}} a^2 \sin^2 3\theta \, d\theta$$

$$= \frac{a^2}{6}\int_0^{\pi} \sin^2\phi \, d\phi, \quad \text{where } 3\theta=\phi,$$

$$= \frac{a^2}{3}\int_0^{\frac{\pi}{2}} \sin^2\phi \, d\phi = \frac{a^2}{3}\frac{1}{2}\frac{\pi}{2} = \frac{\pi a^2}{12}.$$

The total area of the three loops is therefore $\frac{\pi a^2}{4}$.

5. Find the area of the curve

$$x = a\cos^3 t,$$
$$y = b\sin^3 t.$$

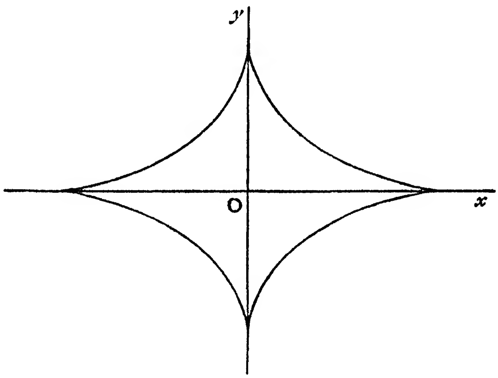


Fig. 59.

Upon elimination of t, we have $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$, and the shape is shown in the figure. There is symmetry about both axes, and the area

$$= 4\int_0^a y\,dx = 4\int_{\frac{\pi}{2}}^0 b\sin^3 t(-3a\cos^2 t\sin t)\,dt$$

$$= 12ab\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt$$

$$= 12ab\,\frac{\Gamma(\frac{5}{2})\,(\Gamma(\frac{3}{2})}{2\Gamma(4)} = 12ab\,\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2.3.2.1}$$

$$= \tfrac{3}{8}\pi ab\ ;$$

or we may use the formula

$$\frac{1}{2}\int\Big[F'(t)f(t) - f'(t)F(t)\Big]dt,$$

which gives

$$4 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} (a\cos^4 t \,.\, 3b\sin^2 t + b\sin^4 t \,.\, 3a\cos^2 t)\, dt$$

$$= 6ab \int_0^{\frac{\pi}{2}} (\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)\, dt$$

$$= 6ab \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t\, dt$$

$$= 6ab\, \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{2\Gamma 3} = \tfrac{3}{8}\pi ab, \text{ as before.}$$

6. Find the area of the loop of the curve

$$x^5 + y^5 - 5ax^2y^2 = 0.$$

(1) There is symmetry about the line $y = x$.

(2) There is an asymptote $x + y = a$.

(3) By Newton's rule, the form at the origin is that of two semicubical parabolas $y^3 = 5ax^2$, $x^3 = 5ay^2$.

The shape is then as shown in Fig. 60.

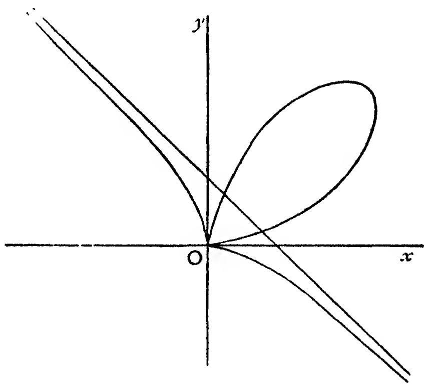


Fig. 60.

The polar equation is

$$r = 5a\,\frac{\sin^2\theta\cos^2\theta}{\sin^5\theta + \cos^5\theta}.$$

As there is symmetry about $\theta = \frac{\pi}{4}$, we may take limits 0 to $\frac{\pi}{4}$ and double.

$$\text{Area of loop} = 2 \,.\, \tfrac{1}{2} \,.\, 25a^2 \int_0^{\frac{\pi}{4}} \frac{\sin^4\theta\cos^4\theta\, d\theta}{(\sin^5\theta + \cos^5\theta)^2},$$

or, putting $\tan\theta = t$,

$$\text{Area} = 25a^2 \int_0^1 \frac{t^4\, dt}{(1+t^5)^2}$$

$$= 5a^2\left[-\frac{1}{1+t^5}\right]_0^1 = 5a^2\left[-\tfrac{1}{2} + 1\right] = \tfrac{5}{2}a^2.$$

Otherwise; this curve is unicursal; and we may write (putting $y=tx$)

$$x=\frac{5at^2}{1+t^5},\quad y=\frac{5at^3}{1+t^5},$$

and integrate $\frac{1}{2}\int\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$, with limits 0 and ∞, which gives

$$\frac{(5a)^2}{2}\int_0^\infty \frac{t^2(3t^2-2t^7)-t^3(2t-3t^6)}{(1+t^5)^3}\,dt=\frac{25a^2}{2}\int_0^\infty \frac{t^4}{(1+t^5)^2}\,dt=\frac{5a^2}{2},$$

as before.

EXAMPLES.

Find the areas bounded by

1. $r^2=a^2\cos^2\theta-b^2\sin^2\theta$, the central pedal of a hyperbola.

2. One loop of $r=a\sin 4\theta$. Also state the total area.

3. One loop of $r=a\sin 5\theta$. Also state the total area.

4. One loop of $r=a\sin n\theta$.
Give the total area in the cases, (i) n even; (ii) n odd.

5. The portion of $r=ae^{\theta\cot\alpha}$ bounded by the radii vectores

$$\theta=\beta,\quad \theta=\beta+\gamma\quad (\gamma<2\pi).$$

6. Any sector of $r^{\frac{1}{2}}\theta=a^{\frac{1}{2}}$ $\quad(\theta=\alpha$ to $\theta=\beta)$.

7. Any sector of the reciprocal spiral $r\theta=a$ $\quad(\theta=\alpha$ to $\theta=\beta)$.

8. The cardioide $r=a(1-\cos\theta)$.

9. The Limaçon $r=a+b\cos\theta$, (i) if $a>b$; (ii) if $a<b$ obtain the two areas of outer and inner portions.

10. Find the area included between the two loops of the curve

$$r=a(2\cos\theta+\sqrt{3}).$$ [Oxf. I. P., 1889.]

11. Prove that the area in the positive quadrant of the curve

$$(x^2+y^2)^5=(a^2x^3+b^2y^3)^2 \text{ is } \tfrac{1}{3}(a^2+b^2).$$ [γ, 1899.]

12. Find the area of the closed part of the Folium

$$r=\frac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}.$$ [I. C. S., 1884.]

13. Show that the area of a loop of the curve

$$ax^{2n+1}-b^2x^ny^n+cy^{2n+1}=0$$

is $\dfrac{1}{2(2n+1)}\dfrac{b^4}{ac}$, a and c being positive. [Colleges, 1881.]

14. Trace the curve whose equation is

$$r^4=a^4\sec\theta\tan\theta,$$

and find the area between the curve and any pair of radii vectores drawn from the pole. [Trinity, 1882.]

15. Trace the lemniscate $r^2 = a^2 \cos 2\theta$ and its first positive pedal, and show that the area of a loop of the latter is double the area of a loop of the former.

Find the areas of each of the two small lozenge-shaped portions common to the two loops of the pedal.

16. Show that the area contained between the curve

$$r = a \cos 5\theta$$

and the circle $r = a$ is three-fourths of the area of the circle.

[OXF. I. P., 1888.]

17. Find the area between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote. [ST. JOHN'S, 1881.]

18. Prove that the area of the curve

$$r^2(2c^2\cos^2\theta - 2ac\sin\theta\cos\theta + a^2\sin^2\theta) = a^2c^2$$

is equal to πac. [I. C. S., 1879.]

19. Find the area of the curve

$$r = 3a\cos\theta + a\cos 3\theta.$$ [MATH. TRIP., 1882.]

20. Find the area of the loop of the curve

$$r^2 = a^2\theta\cos\theta$$

between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

GENERAL PROBLEMS ON QUADRATURE. (CARTESIANS AND POLARS.)

1. Find the area bounded by

$$x^2 + y^2 = 4a^2, \quad x^2 + y^2 = 2ay \quad \text{and} \quad x = a.$$ [H. C. S.]

Also the area of the loop of the curve

$$by^2 = x^2(a - x)$$

(a and b both positive). [I. C. S., 1882.]

2. Find the whole area of the curve

$$y^2 = x^2\frac{a^2 - x^2}{a^2 + x^2}.$$ [I. C. S., 1885; COLLEGES, 1892.]

3. A parabola $y^2 = ax$ cuts the hyperbola $x^2 - y^2 = 2a^2$ at the points P, Q; and the tangent at P to the hyperbola cuts the parabola again at R. Find the area of the curvilinear triangle PQR.

4. Find the area included between one of the branches of the curve $x^2y^2 = a^2(x^2 + y^2)$ and its asymptotes.

Find the whole area of the curve

$$x^4 + y^4 = a^2(x^2 + y^2).$$ [COLLEGES α, 1887.]

5. Trace the curve $a^2y^2 = x^3(2a - x)$, and prove that its area is equal to that of the circle whose radius is a. [I. C. S., 1887.]

6. Prove that the whole area of

$$(x^2 + a^2)y^2 + 3a^3y + 2a^4 = 0$$

is

$$(3 - 2\sqrt{2})\pi a^2.$$

[COLLEGES β, 1891.]

7. Find the area of the loops of the curve

$$y^4 - x^4 - a^2y^2 + b^2x^2 = 0 \text{ when } b^2 > a^2.$$

[OXFORD I. P., 1902.]

8. Find the area bounded by the cycloid

$$x = a(\theta + \sin\theta),$$
$$y = a(1 - \cos\theta),$$

and the straight line joining two consecutive cusps.

9. Show that the coordinates of a point P on the Folium of Descartes $x^3 + y^3 = axy$ can be expressed as

$$x = \frac{at}{1+t^3}, \quad y = \frac{at^2}{1+t^3}.$$

Show that as t varies from 0 to ∞ P traces out a closed loop, and that its area is $\frac{a^2}{6}$. [COLLEGES, 1896.]

10. Prove that the area of either loop of the curve

$$x^5 + y^5 - 5a^2x^2y = 0$$

is

$$\frac{2\pi a^2}{\sqrt{10 + 2\sqrt{5}}}.$$

[γ, 1893.]

11. Show that in that part of the curve $(x + y - 3c)xy + c^3 = 0$ for which x is positive, the area between the curve, the axis of x, and the ordinate which touches the curve is $\frac{1}{2}c^2$. [ST. JOHN'S, 1886.]

12. Trace the curve $y^4 + x^3y = a^2x^2$, and show that the area of the segment which lies between the axis of y and the straight line whose equation is $y = x$ is $\frac{1}{6}a^2 \log 2$.

[COLLEGES ϵ, 1883.]

13. Pairs of ordinates of the hyperbola $xy = a^2$ are determined by the condition that the area included by any pair, the curve, and the x-axis is constant; show that the lengths of any such pair are in a constant ratio. [OXFORD I. P., 1888.]

14. Show that the area between the curve

$$x(x^2+y^2-a^2)+\frac{2}{9}a^3\sqrt{3}=0$$

and its asymptote is πa^2. [ST. JOHN'S, 1892.]

15. Show that the area between the inner branch of the curve

$$(x^2+y^2-a^2)^2=\tfrac{1}{4}x^2(x^2+y^2)$$

and the positive parts of the two axes is $\pi a^2/3\sqrt{3}$. [ST. JOHN'S, 1888.]

16. Prove that the whole area of the epicycloid generated by a point on a circle of radius $\frac{a}{4}$ rolling on a fixed circle of radius a is to the area of the fixed circle in the ratio of 15 to 8.

17. Find the whole area of the curve whose equation is
$(x^2+y^2)(x+y+a)(x+y-a)+x^2y^2=0$. [COLLEGES, 1886.]

18. Find the area of a loop of the curve

$$x^4+y^4=2a^2xy.$$ [OXFORD I. P., 1888.]

19. Find the area cut off from an ellipse by a focal chord. [COLLEGES α, 1883.]

20. Prove that the areas cut off by the equiangular spiral $r=ae^{\theta\cot\alpha}$ from the space bounded by any two fixed lines through the pole are in geometrical progression. [OXFORD I. P., 1900.]

21. Find the area of the curve $r=a\theta e^{b\theta}$ enclosed between two given radii vectores and two successive branches of the curve. [TRINITY, 1881.]

22. Find the area of the loop of the curve $r=a\theta\cos\theta$ between $\theta=0$ and $\theta=\frac{\pi}{2}$. [OXFORD II. P., 1890.]

23. Find the area of the curve

$$(r-a\cos\theta)^2=a^2\cos 2\theta.$$ [COLLEGES α, 1887.]

24. Show that the area of the loop of the folium $x^3+y^3=3axy$ is divided by the parabola $y^2=ax$ in the ratio $5:4$.

In what ratio does the line $x+y=2a$ cut the loop in the above folium. [OXFORD I. P., 1889.]

25. Find the area included between the axis of y and the curve

$$y^2+2y-2x(y+1)=x^4-3x^3+3,$$

the curve being supposed to stop at the node. [ST. JOHN'S, 1884.]

26. Determine by integration the area of the ellipse

$$x^2 + xy + y^2 = 1.$$

27. (i) Find the whole area enclosed by the hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$ [OXFORD I. P., 1888.]

(ii) Prove that the area of the locus of intersection of pairs of tangents at right angles for this curve is $\frac{1}{4}\pi a^2$. [MATH. TRIPOS, 1888.]

28. Prove that the locus of the points of bisection of the intercepts on the normals of a cycloid between the cycloid and its base divides the area between the cycloid and its base into two parts in the ratio $7:5$. [OXFORD II. P., 1886.]

29. Trace the curve $x^{2n+1} + y^{2n+1} = (2n+1)ax^n y^n$, when n is even, and when n is odd, n being a positive integer; and prove that the area of the loop is $(2n+1)\frac{a^2}{2}$. Prove that this is also the area between the infinite branches of the curve and the asymptote. [ST. JOHN'S, 1882.]

30. Find the whole area contained between the curve

$$x^2(x^2+y^2) = a^2(y^2 - x^2)$$

and its asymptotes. [OXFORD I. P., 1887.]

31. Find the area bounded by the circle $x = a\cos\theta$, $y = a\sin\theta$ and the hyperbola $x = b\cosh u$, $y = b\sinh u$; that area being taken which lies within the circle and on the convex side of the hyperbola, and b being less than a. [TRINITY, 1888.]

32. (a) Show that in the Archimedean Spiral $r = a\theta$, if A_1, A_2, A_3, A_4, ... be the areas of the inner loop and the successive heart-shaped figures formed by the convolutions of the curve

$$A_1 = \frac{\pi^2 a^2}{4}, \quad A_{n+1} = 2n\pi^2 a^2.$$

(b) In the Reciprocal Spiral $r\theta = a$, if A_1, A_2, A_3 ... be the areas of the successive closed loops,

$$A_n = \frac{4a^2}{\pi}\frac{1}{4n^2-1}.$$

33. Find the area of the loop of the curve

$$(x+y)(x^2+y^2) = 2axy.$$ [OXFORD I. P., 1890.]

34. At all points of the first negative pedal of the curve $r = \cosh(m\theta\cot\alpha)$ lines are drawn making a constant angle α with

the tangent. Show that the area bounded by any pair of such lines, the curve enveloped and the first negative pedal is

$$A\{1+(m^2-1)\cos^2\alpha\},$$

where A is the area of the corresponding portion of the first negative pedal bounded by radii vectores from the pole.

[COLLEGES α, 1891.]

35. Find the area of that portion of the loop of the curve

$$r^2=p\cos\theta+q\sin\theta,$$

which is not enclosed by the curve

$$r^2=b+a\cos\theta.$$

If a family of such curves be taken (by varying p and q), such that this area is constant, show that the envelope of the system is a curve whose equation is

$$r^2=c+a\cos\theta.$$ [COLLEGES β, 1889.]

36. Show that the whole area enclosed by the outer line of the curve $r^{\frac{2}{3}}=a^{\frac{2}{3}}\cos\frac{2}{3}\theta$ is $\frac{9}{8}a^2\sqrt{3}$. [COLLEGES, 1876.]

37. In a hyperbola, C is the centre, A the end of the transverse axis and P any point (x, y) on the same branch of the curve as A; prove that twice the area of the sector CAP is

$$ab\log\left(\frac{x}{a}+\frac{y}{b}\right).$$

38. Show that the area contained between a hyperbola, any tangent and a line parallel to the asymptote which bisects the part of the tangent intercepted between the curve and the asymptote

$$=\frac{ab}{2}(\log 2-\tfrac{5}{8}),$$

and is constant. [TRINITY, 1886.]

39. Prove that the area of the curve

$$x=\frac{ap}{(1+p^2)^2},\quad y=\frac{1}{2}\frac{ap^2(1-p^2)}{(1+p^2)^2}$$

is $\frac{1}{32}\pi a^2$. [MATH. TRIPOS, 1882.]

40. Show that the area cut off from the ellipse

$$ax^2+2hxy+by^2=1$$

by the line $lx+my=1$ is

$$\alpha\beta(\theta-\sin\theta\cos\theta),$$

where α, β are the semiaxes of the ellipse and

$$\cos\theta=\frac{\sqrt{ab-h^2}}{\sqrt{am^2+bl^2-2hlm}}.$$ [COLLEGES, 1892.]

41. Trace the curve whose equation is

$$x(x^{2n}+y^{2n})=a^2y^{2n-1},$$

and prove that the area between the curve, the axis of x and a tangent parallel to the axis of y is

$$\frac{a^2}{4n}(2n-1-\log 2n).$$

[ST. JOHN'S, 1885.]

42. Show that in the curve

$$r^2=\sec 2\theta \log(2\cos^2\theta)$$

the area between the curve and the lines $\theta=\pm\frac{1}{4}\pi$ is $(\frac{1}{4}\pi)^2$.

[ST. JOHN'S, 1886.]

43. Find in integral form, and completely, the area enclosed between two confocal conics and two given radii from the centre.

[TRINITY, 1881.]

44. Prove that the area of each of the two equal and similar pieces of the ellipse $x^2/a^2+y^2/b^2=1$ which are cut off by the hyperbola $x^2/\alpha^2-y^2/\beta^2=1$ $(\alpha<a)$ is

$$ab\sin^{-1}\frac{\beta(a^2-\alpha^2)^{\frac{1}{2}}}{(a^2\beta^2+\alpha^2b^2)^{\frac{1}{2}}}-\alpha\beta\sinh^{-1}\frac{b(a^2-\alpha^2)^{\frac{1}{2}}}{(a^2\beta^2+\alpha^2b^2)^{\frac{1}{2}}}.$$

[ST. JOHN'S, 1887.]

45. Prove that the areas of the two loops of the curve

$$r^2-2ar\cos\theta-8ar+9a^2=0$$

are $$(32\pi+24\sqrt{3})a^2 \quad\text{and}\quad (16\pi-24\sqrt{3})a^2.$$

[MATH. TRIPOS, 1875.]

46. The area between two tangents to the same convolution of an equiangular spiral at right angles to one another, and the curve, is

$$pp'+\tfrac{1}{2}(p^2-p'^2)\cot 2\gamma,$$

where p, p' are the perpendiculars from the pole on the tangents and γ is the angle of the spiral.

[COLLEGES, 1882.]

47. A circle with centre at the origin cuts the loop of the Folium $x^3+y^3-3axy=0$. If the angle subtended at the origin by the common chord equals

$$2\tan^{-1}\frac{2^{\frac{1}{3}}-1}{2^{\frac{1}{3}}+1},$$

prove that the area between the loop and the circle is

$$\frac{a^2}{2}\left[1-(2^{\frac{2}{3}}+2^{\frac{4}{3}})\tan^{-1}\frac{2^{\frac{2}{3}}-1}{2^{\frac{4}{3}}}\right].$$

[COLLEGES, 1885.]

48. The centre of a circle of constant radius a moves along a fixed straight line AB in its plane, and from A a fixed point in the line a tangent AP is drawn to the circle. Show that the area included between the locus of P and its asymptotes is πa^2.

[MATH. TRIPOS, 1882.]

49. Show that the curve

$$r = a\left(\frac{1}{2}\sqrt{3} + \cos\frac{\theta}{2}\right)$$

has three loops, whose areas are

$$a^2(\tfrac{5}{4}\pi + 2\sqrt{3}), \quad a^2(\tfrac{5}{8}\pi - \tfrac{5}{4}\sqrt{3}), \quad a^2(\tfrac{5}{12}\pi - \tfrac{3}{4}\sqrt{3})$$

respectively. [COLLEGES, 1892.]

50. Show that the area of the Cassinian

$$r^4 - 2a^2r^2\cos 2\theta + a^4 = b^4$$

is
$$2\int_0^{\frac{\pi}{2}} \sqrt{b^4 - a^4\sin^2\phi}\, d\phi, \quad \text{provided } b > a,$$

but is
$$2\int_0^{\frac{\pi}{2}} \frac{b^4\cos^2\phi\, d\phi}{\sqrt{a^4 - b^4\sin^2\phi}}, \quad \text{when } a > b.$$

[MATH. TRIPOS, 1883.]

51. Prove that the area of the first negative pedal of an ellipse with respect to the focus is

$$\frac{\pi a^2(2 - 3e^2)}{2\sqrt{1 - e^2}}, \quad (e < \tfrac{1}{2}),$$

where a and e are the semi major axis and the eccentricity of the ellipse. [COLLEGES, 1892.]

How do you interpret this result if $e < \frac{1}{2}$?

52. Find the area of the curve whose Cartesian equation is

$$a^2(y - x)^2 = (a + x)^3(a - x).$$

[MATH. TRIPOS, 1896.]

53. Find the value of $\int_0^1 v_x\, dx$, v_x being the real root of the cubic

$$v_x^3 + v_x^2 v_1 + v_x v_1^2 - \frac{c}{x} = 0.$$

[COLLEGES, 1872; R. P.]

54. Find the area in the first quadrant bounded by the axes of coordinates and the curve

$$\sinh^{-1}\frac{x}{a} + \sinh^{-1}\frac{y}{b} = c,$$

taking a, b, c all positive. [I. C. S., 1897.]

55. Trace the whole curve

$$x^2y^2 = c^2(a-x)(x-b),$$

where $0 < b < a$, and find its whole area. [I. C. S., 1898.]

56. It is given that the abscissa ON and ordinate NP of a point on any arch of a cycloidal arc are $a(\theta - \sin\theta)$ and $a(1-\cos\theta)$. NP is produced to K so that $NK = 2a$, and the rectangle $ONKA$ is completed. Prove that the area included by ON, NP and the arc OP never differs from three-fourths of $ONKA$ by more than $\frac{3a^2}{8}\sqrt{3}$; and find for what positions of P the difference vanishes.

[I. C. S., 1912.]

57. Trace on squared centimetre paper the curves

$$x^4 + y^4 = 4a^2xy,$$
$$x^4 + y^4 = 4ax^2y,$$

taking $a = 10$ cm., and estimate the area of a loop of each curve.

Prove that $\displaystyle\int_0^\infty \frac{t^2}{(1+t^4)^2}\,dt = \frac{1}{4}\int_0^\infty \frac{t^2}{1+t^4}\,dt = \frac{\pi}{8\sqrt{2}}$,

and hence calculate the area of a loop of the second curve. Find also the area of a loop of the first curve. Give each area to the nearest square centimetre when a is 10 centimetres.

[C. S., 1913.]

58. Obtain the area contained between the two curves

$$r^2\cos 2\theta = 4a^2\cos^4\theta \quad\text{and}\quad r^2\cos 2\theta = a^2.$$

[Oxf. I. P., 1912.]

59. Show that the area of the loop of the curve

$$x^7 + y^7 = ax^3y^3$$

is equal to $a^2/14$. [Oxf. I. P., 1914.]

60. Prove by any method that the area of the ellipse

$$\{a(x-2)+3y\}^2 + 4(x+1)(x-2) = 0$$

is independent of a, and find the area.

Prove also that the straight line $y = x$ divides the ellipse $x^2 + 3y^2 = 6y$ into two areas which are in the ratio

$$4\pi - 3\sqrt{3} : 8\pi + 3\sqrt{3}.$$ [Oxf. I. P., 1916.]

61. Trace the curve

$$r\cos\theta = a\sin 3\theta,$$

and show that the area of a loop is

$$\tfrac{1}{8}a^2(9\sqrt{3} - 4\pi).$$ [Math. Trip. I., 1919.]

62. Show that the curve $r=a(2\cos\theta+\cos 3\theta)$ has three loops, the area of the larger loop being $\dfrac{10\pi+9\sqrt{3}}{12}a^2$, and the areas of the two smaller loops being $\dfrac{5\pi-9\sqrt{3}}{24}a^2$. [MATH. TRIP. I., 1916.]

63. Show that the coordinates of any point on the curve

$$y^2(a+x)=x^2(3a-x)$$

may be taken as

$$x=a\sin 3\theta/\sin\theta, \quad y=a\sin 3\theta/\cos\theta,$$

and prove that the area of the loop and the area between the curve and its asymptote are both equal to $3\sqrt{3}a^2$. [MATH. TRIP. I., 1915.]

64. Show that the area of the loop of the curve

$$(x^2+y^2)^2-4axy^2=0$$

in the positive quadrant is $\frac{1}{4}\pi a^2$. [MATH. TRIP. I., 1920.]

65. Having established Simpson's Rule, that if

$$y=y(x)\equiv a_0+a_1x+a_2x^2+a_3x^3,$$

then

$$\int_0^1 y\,dx=\tfrac{1}{6}\{y(0)+y(1)+4y(\tfrac{1}{2})\},$$

prove that if $y(x)$ also contains a term a_4x^4 the error in still using Simpson's Rule is

$$\frac{1}{120}a_4.$$

[MATH. TRIP. I., 1920.]

CHAPTER XIII.

QUADRATURE (II).

TANGENTIAL POLARS, PEDAL EQUATIONS AND PEDAL CURVES, INTRINSIC EQUATIONS, ETC.

416. **Other Expressions for an Area**

Many other expressions may be deduced for the area of a plane curve, or proved independently, specially adapted to the cases when the curve is defined by systems of coordinates other than Cartesians or Polars, or for regions bounded in a particular manner.

To avoid continual redefinition of the symbols used we may state that in the subsequent work the letters

$$x,\ y,\ r,\ \theta,\ s,\ p,\ \psi,\ \phi,\ \rho$$

have the meanings assigned to them throughout the treatment of Curvature in the author's *Differential Calculus*.

417. **The (p, s) formula.**

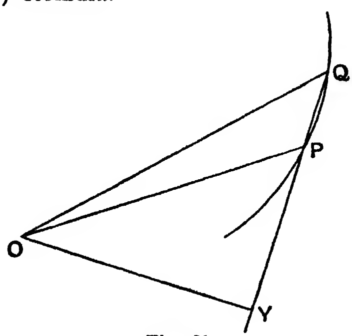


Fig. 61.

Let PQ be an element δs of a plane curve and OY the perpendicular from the pole upon the chord PQ. Then

$$\triangle OPQ = \tfrac{1}{2} OY \,.\, PQ,$$

and any sectorial area

$$= Lt\,\Sigma\triangle OPQ = \tfrac{1}{2} Lt\,\Sigma OY.PQ,$$

the summation being conducted along the whole bounding arc. In the notation of the Integral Calculus this is $\frac{1}{2}\int p\,ds$.

This might be deduced from the polar formula at once.

For $A = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int r^2 \frac{d\theta}{ds} ds = \frac{1}{2}\int r \sin\phi\, ds = \frac{1}{2}\int p\, ds,$

where ϕ is the angle between the tangent and the radius vector.

418. **Tangential-Polar Form** (p, ψ).

Again, since $\rho = \frac{ds}{d\psi} = p + \frac{d^2p}{d\psi^2}$,

we have $\text{Area} = \frac{1}{2}\int p\,ds = \frac{1}{2}\int p\rho\, d\psi = \frac{1}{2}\int p\left(p + \frac{d^2p}{d\psi^2}\right) d\psi,$

a form suitable for use when the Tangential-Polar (*i.e.* p, ψ) form of the equation to the curve is given.

This gives the sectorial area bounded by the curve and the initial and final radii vectores.

419. **Caution.**

In using the formula

$$A = \frac{1}{2}\int p\left(p + \frac{d^2p}{d\psi^2}\right) d\psi,$$

care should be taken not to integrate over a point, between the proposed limits, at which the integrand changes sign. If such points exist the whole integration is to be conducted in sections along each of which the sign of the integrand is permanent. The results for the several sections are then to be taken positively and added together. When a point of inflexion is passed $p + \frac{d^2p}{d\psi^2}$ passes through an infinite value and changes sign.

420. **The Case of a Closed Curve.**

When the curve is closed the formula admits of some simplification.

For integrating by parts

$$\int p\frac{d^2p}{d\psi^2}d\psi=\left[p\frac{dp}{d\psi}\right]-\int\left(\frac{dp}{d\psi}\right)^2 d\psi.$$

Hence $$\text{Area}=\frac{1}{2}\left[p\frac{dp}{d\psi}\right]+\frac{1}{2}\int\left\{p^2-\left(\frac{dp}{d\psi}\right)^2\right\}d\psi.$$

In integrating round the whole perimeter the term between square brackets, viz. $\frac{1}{2}\left[p\frac{dp}{d\psi}\right]$ disappears, for it resumes the same value as it originally had when we return to the starting-point after integrating round the contour of the curve. Hence, for a closed curve,

$$\text{Area}=\frac{1}{2}\int\left\{p^2-\left(\frac{dp}{d\psi}\right)^2\right\}d\psi.$$

421. Ex. 1. Let A_1CA_2 be one foil of the epicycloid $p=A\sin B\psi$ and OA_1 the initial line. Then p vanishes if $B\psi=0, \pi, 2\pi, \ldots$.

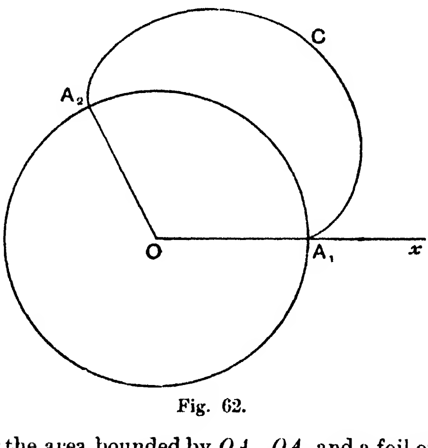


Fig. 62.

Therefore, for the area bounded by OA_1, OA_2 and a foil of the epicycloid, viz. the kite-shaped figure OA_1CA_2O in Fig. 62,

$$\text{Area}=\frac{1}{2}\int_0^{\frac{\pi}{B}}p\left(p+\frac{d^2p}{d\psi^2}\right)d\psi=\frac{1}{2}\int_0^{\frac{\pi}{B}}A\sin B\psi\{A\sin B\psi-AB^2\sin B\psi\}\,d\psi$$

$$=\frac{A^2(1-B^2)}{2}\int_0^{\frac{\pi}{B}}\sin^2 B\psi\,d\psi$$

$$=\frac{A^2(1-B^2)}{2}\cdot\frac{1}{B}\int_0^{\pi}\sin^2\phi\,d\phi,\quad \text{if } \phi=B\psi,$$

$$=\frac{\pi}{4}\frac{A^2}{B}(1-B^2).$$

Thus, for the whole cardioide, which is a one-cusped epicycloid formed as the path of a point attached to the circumference of a circle of radius a rolling upon an equal circle whose centre is at the origin O,

$$p = 3a \sin \frac{\psi}{3}. \qquad \text{(See } \textit{Diff. Calc.}\text{, p. 345.)}$$

And the area is

$$\frac{\pi}{4}(3a)^2 \times 3\left(1 - \frac{1}{9}\right) = 6\pi a^2.$$

Ex. 2. Otherwise, the cardioide $p = 3a \sin \frac{\psi}{3}$ is a "closed" curve.

Let us apply the second formula

$$\frac{1}{2}\int\left(p^2 - \left|\frac{dp}{d\psi}\right|^2\right) d\psi \text{ in this case.}$$

The whole area $= \frac{1}{2}\int\left(9a^2 \sin^2\frac{\psi}{3} - a^2\cos^2\frac{\psi}{3}\right) d\psi$ taken between limits

$$\psi = 0 \quad \text{and} \quad \psi = 3\pi.$$

Putting $\psi = 3\theta$, this becomes

$$\frac{3a^2}{2}\int_0^\pi (9 \sin^2\theta - \cos^2\theta)\, d\theta$$

$$= 3a^2\left(9\frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2}\right) = 6\pi a^2, \text{ as before.}$$

422. Pedal Curves.

If $p = f(\psi)$ be the tangential-polar equation of a given curve, $\delta\psi$ is the angle between the perpendiculars from the pole

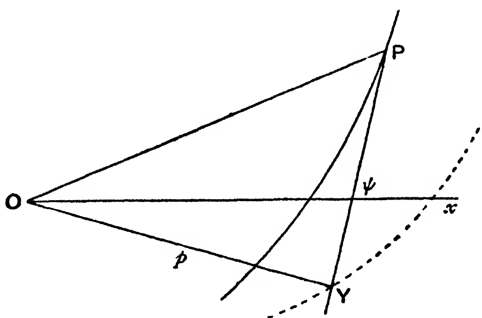


Fig. 63.

upon two contiguous tangents, and the area of the pedal curve may be expressed as

$$\text{Lt}\, \frac{1}{2}\Sigma OY^2\delta\psi = \frac{1}{2}\int OY^2 d\psi, \quad \textit{i.e.}\ \frac{1}{2}\int p^2 d\psi,$$

p, ψ being the polar coordinates of Y.

423. Ex. Find the area of the pedal of a circle with regard to a point on the circumference (*i.e.* the cardioide).

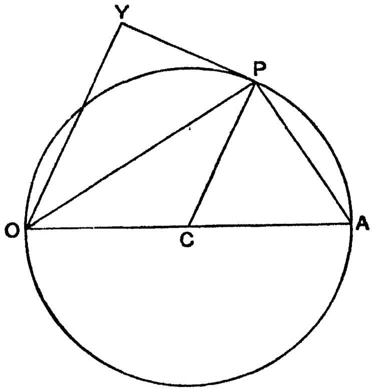


Fig. 64.

Here, if OY is the perpendicular on the tangent at P, and OA the diameter $=2c$, it is geometrically obvious that OP bisects the angle AOY. Hence calling $A\hat{O}Y$, ψ, we have for the tangential polar equation of the circle

$$p = OY = OP\cos\frac{\psi}{2} = OA\cos^2\frac{\psi}{2},$$

i.e.

$$p = 2c\cos^2\frac{\psi}{2}.$$

Hence Area $=\frac{1}{2}\int 4c^2\cos^4\frac{\psi}{2}\,d\psi$, where the limits are to be taken as 0 and π, and the result is to be doubled so as to include the lower portion of the pedal.

Then

$$A = 4c^2\int_0^\pi \cos^4\frac{\psi}{2}\,d\psi = 4c^2\,.\,2\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta = 8c^2\,\frac{3}{4}\,\frac{1}{2}\,.\,\frac{\pi}{2} = \frac{3}{2}\pi c^2.$$

424. Area bounded by a Curve, its Pedal and a Pair of Tangents.

Let P, Q be two contiguous points on a given curve; Y, Y' the corresponding points of the pedal for any origin O (Fig. 65).

Then since, with the usual notation, $PY=\dfrac{dp}{d\psi}$, the elementary triangle bounded by two contiguous tangents PY, QY', and the chord of the pedal YY', is to the first order of small quantities

$$\frac{1}{2}\left(\frac{dp}{d\psi}\right)^2\delta\psi.$$

Hence the area of any portion bounded by the two curves and a pair of tangents to the original curve may be expressed as

$$\frac{1}{2}\int\left(\frac{dp}{d\psi}\right)^2 d\psi,$$

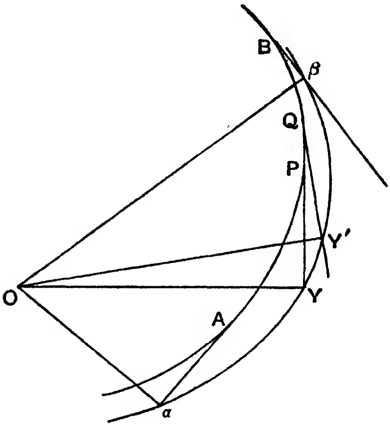


Fig. 65.

and is the same as the *corresponding portion of the pedal of the evolute,* for $PY =$ the perpendicular from O upon the normal at P (Fig. 66).

425. **Pedal of Evolute of a Closed Curve.**

In the case of a closed curve, then, the equation

$$\text{Area} = \frac{1}{2}\int\left\{p^2 - \left(\frac{dp}{d\psi}\right)^2\right\} d\psi$$

admits of two interpretations.

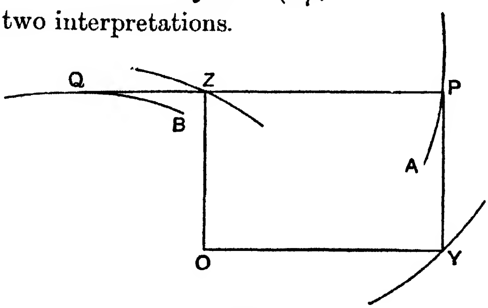


Fig. 66.

Let O be the pole, AP an arc of the *closed* oval, BQ an arc of the evolute, P, Q corresponding points on the curve and the

evolute, OY, OZ, perpendiculars from O on the tangent and normal at P.

Then the Y locus is the pedal to the curve, the Z locus is the pedal to the evolute. Hence the equation

$$\frac{1}{2}\int p^2 d\psi = \text{area of oval} + \frac{1}{2}\int\left(\frac{dp}{d\psi}\right)^2 d\psi$$

expresses

(A) That the area of the *pedal of the oval*
$=$ area of the *oval* $+$ the area of the *region between the oval and its pedal.*

(B) That the area of the *pedal of the oval*
$=$ area of the *oval* $+$ the area of the *pedal of the evolute.*

426. **Additional Results.**

Further, since

$$\text{area of pedal} = \text{area of oval} + \frac{1}{2}\int\left(\frac{dp}{d\psi}\right)^2 d\psi$$

and

$$\text{area of pedal} = \frac{1}{2}\int p^2 d\psi,$$

we have upon addition

$$2 \times \text{area of pedal} = \text{area of oval} + \frac{1}{2}\int\left\{p^2 + \left(\frac{dp}{d\psi}\right)^2\right\} d\psi$$

$$= \text{area of oval} + \frac{1}{2}\int r^2 d\psi$$

or

$$= \text{area of oval} + \frac{1}{2}\int \frac{r^2}{\rho}\, ds,$$

i.e. the area of the pedal of a closed curve with regard to any origin within it exceeds half the area of the curve by $\frac{1}{4}\int \frac{r^2}{\rho}\, ds$.

This result may be regarded as giving an interpretation for the integral

$$\int r^2 d\psi \quad \text{or} \quad \int \frac{r^2}{\rho}\, ds,$$

an expression which figures in the discussion of roulettes.

427. **Geometrical Proofs.**

These facts may be established by elementary geometry thus.

Let P_1, Q_1, Y_1, Z_1 be the contiguous positions to P, Q, Y, Z on the respective loci, and let YP, Y_1P_1 intersect at T and YP, OY_1 at N.

Then

$$\triangle OYP - \triangle OY_1P_1 = (\triangle OYN + \triangle ONP)$$
$$- (\triangle ONP + \triangle NY_1T + \text{quadrilateral } OPTP_1)$$
$$= \triangle OYN - \triangle NY_1T - \text{quadrilateral } OPTP_1$$
$$= \text{sectorial area } OYY_1 - \text{sectorial area } TYY_1$$
$$- \text{quadrilateral } OPTP_1.$$

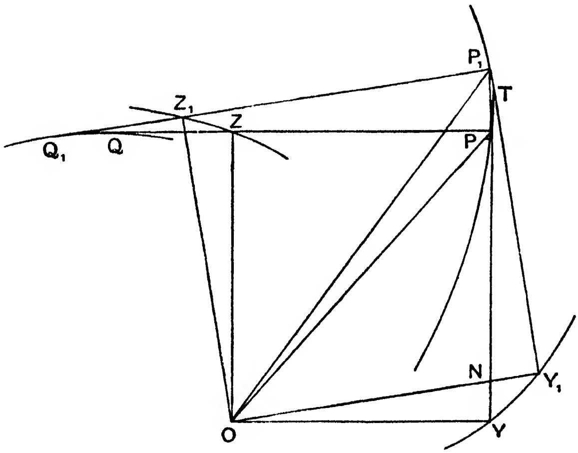


Fig. 67.

And summing for a closed oval,

$$\Sigma(\triangle OYP - \triangle OY_1P_1) = 0;$$

$$\therefore \ \Sigma OYY_1 = \Sigma TYY_1 + \Sigma OPTP_1,$$

and $$\triangle OZZ_1 = \triangle TY\dot{Y}_1 \text{ to the first order;}$$

$$\therefore \ \Sigma OYY_1 = \Sigma OZZ_1 + \text{area of oval}$$

or $$= \Sigma TYY' + \text{area of oval},$$

i.e. $\left.\begin{array}{c}\text{area of pedal of evolute, or area}\\ \text{between pedal and oval}\end{array}\right\} = \begin{array}{c}\text{area of pedal of oval}\\ -\text{area of oval.}\end{array}$

428. Ex. 1. As an illustration, consider the central pedal of the evolute of an ellipse.

Area of pedal of evolute = area of pedal of ellipse − area of ellipse

$$= \frac{\pi}{2}(a^2 + b^2) - \pi ab$$

$$= \frac{\pi}{2}(a - b)^2.$$

Ex. 2. The pedal of a circle of radius c and centre C with regard to a point O on the circumference is $r=c(1+\cos\theta)$, a cardioide. The evolute of the circle is a point, viz. the centre. As the current point P travels round the circumference of the circle once, the path of Z, the foot of

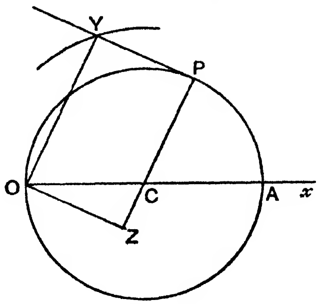


Fig. 68.

the perpendicular upon PC travels round its path (viz. a circle on OC for diameter) *twice*. The pedal of the evolute is therefore the *twice described* circle of radius $\frac{c}{2}$.

And

$$\text{area of cardioide} = \text{area of circle radius } c + 2\times\text{area of circle of radius } \frac{c}{2}$$
$$=\tfrac{3}{2}\pi c^2.$$

429. **Pedal Equation** (p, r).

When the relation between p, r is given, *i.e.* the pedal equation, we have

$$\text{Area} = \frac{1}{2}\int p\,ds = \frac{1}{2}\int p\frac{ds}{dr}\,dr = \frac{1}{2}\int p\sec\phi\,dr$$
$$=\frac{1}{2}\int\frac{rp}{\sqrt{r^2-p^2}}\,dr.$$

This again gives the sectorial area between the curve and a definite pair of radii vectores.

Again care is required in the use of the formula to avoid integration through a value of r for which $\sec\phi$ changes sign, *i.e.* when ϕ changes from acute to obtuse, as it will do at points where r has a maximum or minimum value. If such points occur, the integration must be conducted separately for each of the portions into which these points divide the perimeter and the results taken positively added together.

430. Ex. 1. In the equiangular spiral $p=r\sin\alpha$, and any sectorial area

$$=\frac{1}{2}\int_{r_1}^{r_2}\frac{r^2\sin\alpha}{r\cos\alpha}dr=\frac{1}{4}(r_2^2-r_1^2)\tan\alpha.$$

Ex. 2. Find the area of the lemniscate $p=\dfrac{r^3}{a^2}$.

Here
$$A=\frac{1}{2}\int\frac{r\dfrac{r^3}{a^2}dr}{\sqrt{r^2-\dfrac{r^6}{a^4}}}=\frac{1}{2}\int\frac{r^3}{\sqrt{a^4-r^4}}dr$$

$$=\tfrac{1}{2}[-\tfrac{1}{2}\sqrt{a^4-r^4}].$$

Taking limits from $r=0$ to $r=a$, we get a result $\dfrac{a^2}{4}$.

This gives the area of half a loop.

The whole area is four times this result, viz. $=a^2$.

Note, that if we integrated through the maximum without change of sign of the radical from $r=0$ to $r=0$ again, we should obtain a zero result—*i.e.* the *difference* of the two halves of the loop instead of the sum as desired.

431. **Area included between a Curve, two Radii of Curvature and the Evolute.**

In this case we take as our element of area the elementary

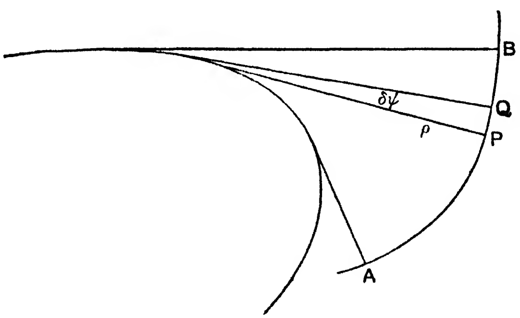


Fig. 69.

triangle contained by two contiguous radii of curvature and the infinitesimal arc δs of the curve.

To first order infinitesimals this is $\frac{1}{2}\rho^2\,\delta\psi$, using the same notation as before.

And area required

$$=Lt\,\Sigma\,\tfrac{1}{2}\rho^2\,\delta\psi,$$

i.e. $$=\frac{1}{2}\int\rho^2\,d\psi,\quad\text{or}\quad=\frac{1}{2}\int\rho\,ds,$$

or $$=\frac{1}{2}\int\left(p+\frac{d^2p}{d\psi^2}\right)ds,\quad\text{or}\quad\frac{1}{2}\int\left(p+\frac{d^2p}{d\psi^2}\right)^2d\psi,$$

or other forms adapted to the particular species of coordinates in use.

For instance, for Cartesians

$$=\frac{1}{2}\int\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\sqrt{1+y_1^2}\,dx$$

$$=\frac{1}{2}\int\frac{(1+y_1^2)^2}{y_2}\,dx,\quad\text{where } y_1=\frac{dy}{dx},\quad y_2=\frac{d^2y}{dx^2};$$

or for Polars

$$=\frac{1}{2}\int\frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}\sqrt{r^2+r_1^2}\,d\theta$$

$$=\frac{1}{2}\int\frac{(r^2+r_1^2)^2}{r^2+2r_1^2-rr_2}\,d\theta,\quad\text{where } r_1=\frac{dr}{d\theta},\quad r_2=\frac{d^2r}{d\theta^2},$$

etc.

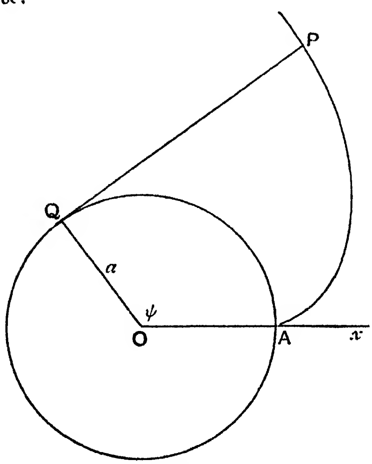


Fig. 70.

432. Ex. 1. The area between a circle, an involute and a tangent to the circle is (Fig. 70)

$$\frac{1}{2}\int_0^{\psi}(a\psi)^2\,d\psi=\frac{a^2\psi^3}{6}.$$

Ex. 2. The area between the tractrix and its asymptote is found in similar manner. The tractrix is described in *Diff. Calc.*, Art. 444. The portion of the tangent between the point of contact and the x-axis is of constant length c.

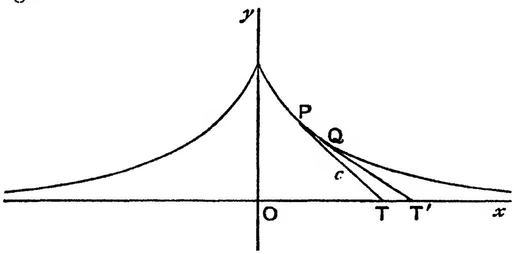


Fig. 71.

Taking two adjacent tangents and the axis of x as forming an elementary triangle (Fig. 71),

$$\text{Area} = 2 \cdot \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} c^2 d\psi$$

$$= \frac{\pi c^2}{2}.$$

433. Area swept by a "Tail."

In exactly the same way as in the last example we may find the area swept out by a "tail" of length varying according

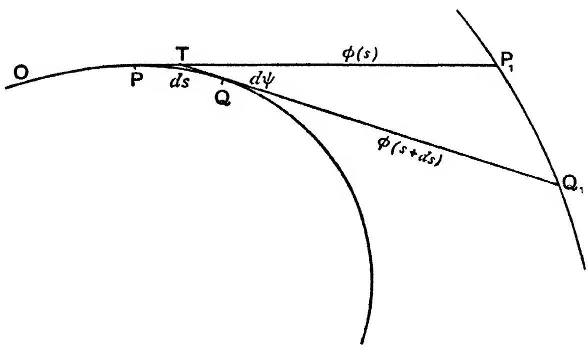


Fig. 72.

to any specified law measured along a tangent from the point of contact.

Let the length of the tail be $\phi(s)$. Let P_1, Q_1 be at the distances $\phi(s)$, $\phi(s+\delta s)$ measured along the tangents at

contiguous points P and Q respectively from the points of contact. Then the area of the triangular element bounded by the two contiguous tails and the arc P_1Q_1 is to the first order

$$\tfrac{1}{2}\{\phi(s)\}^2\,\delta\psi,$$

and the area swept out by the tail is

$$\frac{1}{2}\int\{\phi(s)\}^2\,d\psi.$$

If $\phi(s)=$ a constant $=c$, Area swept $=\dfrac{1}{2}\displaystyle\int c^2\,d\psi$, and for a closed oval of continuous curvature $=\pi c^2$, viz. the area of a circle of radius c.

If the tail be of length equal to the corresponding radius of curvature, the area swept out $=\dfrac{1}{2}\displaystyle\int\rho^2\,d\psi=\dfrac{1}{2}\int\rho\,ds$.

434. If **lengths be taken along the normal drawn outwards,** and specified in the same way, viz. $\phi(s)$, the area between the original curve and the locus traced is

$$\frac{1}{2}\int[\{\rho+\phi(s)\}^2-\rho^2]\,d\psi=\int\phi(s)\,ds+\frac{1}{2}\int\{\phi(s)\}^2\,d\psi,$$

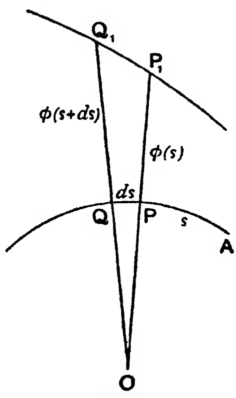


Fig. 73.

or if the distance $\phi(s)$ be on the inward drawn normal

$$\int\phi(s)\,ds-\frac{1}{2}\int\{\phi(s)\}^2\,d\psi.$$

435. **Parallel Curves.**

If, in this case (Art. 434), $\phi(s)$ be constant $=c$, a 'parallel' to the original curve is traced, and the area between a curve and its parallel will be found from

$$\frac{1}{2}\int\{(\rho \pm c)^2 \sim \rho^2\}\,d\psi,$$

i.e.
$$\frac{1}{2}\int(2\rho c \pm c^2)\,d\psi = c\int ds \pm \frac{c^2}{2}\int d\psi,$$

and for a closed oval of one convolution surrounding the pole this becomes $cs \pm \pi c^2$, s being the perimeter of the oval, the positive sign being taken for exterior parallels, the negative sign for interior ones. If the normal makes n revolutions before returning to its original position, the area swept over by PP_1 will be numerically

$$cs \pm n\pi c^2.$$

436. **General Case.**

More generally, let us construct a new curve from a given one by measuring a distance α along the tangent from the point of contact, in the direction of measurement of the arc, and a distance β through the extremity of α, parallel to the outward drawn normal at P, and let the point at which we arrive be called Q; α, β not necessarily being constants.

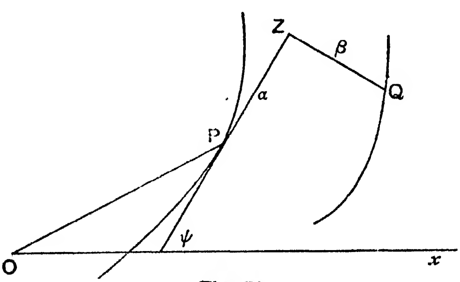


Fig. 74.

Then if x, y be the coordinates of P and ξ, η those of Q, and if ψ be the inclination of the tangent at P to the initial line,

$$\xi = x + \alpha\cos\psi + \beta\sin\psi, \quad \eta = y + \alpha\sin\psi - \beta\cos\psi.$$

Then $d\xi = dx + (d\alpha\cos\psi + d\beta\sin\psi) + (-\alpha\sin\psi + \beta\cos\psi)\,d\psi,$

$$d\eta = dy + (d\alpha\sin\psi - d\beta\cos\psi) + (\alpha\cos\psi + \beta\sin\psi)\,d\psi;$$

$$\begin{aligned}
\therefore\ \xi d\eta-\eta d\xi &= (x+\alpha\cos\psi+\beta\sin\psi)\{dy+(d\alpha\sin\psi-d\beta\cos\psi)\\
&\qquad\qquad+(\alpha\cos\psi+\beta\sin\psi)\,d\psi\}\\
&\quad-(y+\alpha\sin\psi-\beta\cos\psi)\{dx+(d\alpha\cos\psi+d\beta\sin\psi)\\
&\qquad\qquad+(-\alpha\sin\psi+\beta\cos\psi)\,d\psi\}\\
&= x\,dy-y\,dx+\{(\alpha\cos\psi+\beta\sin\psi)\sin\psi\\
&\qquad\qquad-(\alpha\sin\psi-\beta\cos\psi)\cos\psi\}\,ds\\
&\quad+x(d\alpha\sin\psi-d\beta\cos\psi)-y(d\alpha\cos\psi+d\beta\sin\psi)\\
&\qquad+x(\alpha\cos\psi+\beta\sin\psi)\,d\psi\\
&\qquad-y(-\alpha\sin\psi+\beta\cos\psi)\,d\psi\\
&\quad+(\alpha\cos\psi+\beta\sin\psi)(d\alpha\sin\psi-d\beta\cos\psi)\\
&\qquad-(\alpha\sin\psi-\beta\cos\psi)(d\alpha\cos\psi+d\beta\sin\psi)\\
&\qquad+(\alpha^2+\beta^2)\,d\psi\\
&\qquad\qquad(\text{for } dx=ds\cos\psi,\ dy=ds\sin\psi)\\
&= (x\,dy-y\,dx)+2\beta\,ds+(\beta\,d\alpha-\alpha\,d\beta)+(\alpha^2+\beta^2)\,d\psi\\
&\qquad+d\{x(\alpha\sin\psi-\beta\cos\psi)\}\\
&\qquad-d\{y(\alpha\cos\psi+\beta\sin\psi)\},
\end{aligned}$$

a term $\{-dx(\alpha\sin\psi-\beta\cos\psi)+dy(\alpha\cos\psi+\beta\sin\psi)\}$,

i.e. $$\left\{-dx\left(\alpha\frac{dy}{ds}-\beta\frac{dx}{ds}\right)+dy\left(\alpha\frac{dx}{ds}+\beta\frac{dy}{ds}\right)\right\},$$

that is $\beta\,ds$ having been added and subtracted in the arrangement.

Hence, if A and A_1 be the corresponding sectorial areas swept out by the radii vectores OP and OQ,

$$A_1=A+\int\beta\,ds+\frac{1}{2}\int\left(\beta\frac{d\alpha}{ds}-\alpha\frac{d\beta}{ds}\right)ds+\frac{1}{2}\int\frac{\alpha^2+\beta^2}{\rho}\,ds$$
$$+\tfrac{1}{2}[x(\eta-y)-y(\xi-x)],$$

the portion [ ] being between limits corresponding to the beginning and ending of the arc traced by P.

If the curves be closed this term disappears, and

$$A_1=A+\int\beta\,ds+\frac{1}{2}\int\left(\beta\frac{d\alpha}{ds}-\alpha\frac{d\beta}{ds}\right)ds+\frac{1}{2}\int\frac{\alpha^2+\beta^2}{\rho}\,ds.$$

This formula of course includes the foregoing cases.

Thus, for parallels $\alpha=0$, $\beta=c$, and the oval being closed,

$$A_1=A+cs+\frac{1}{2}\int c^2\,d\psi=A+cs+\pi c^2 \text{ as before.}$$

437. Polar Subtangent.

The area bounded by any portion of a given curve, two tangents, and the corresponding portion of the locus of the extremity of the polar subtangent is given by

$$A=\frac{1}{2}\int\frac{r^2}{r_1^{\,2}}(r^2+2r_1^{\,2}-rr_2)\,d\theta,$$

where
$$r_1=\frac{dr}{d\theta},\quad r_2=\frac{d^2r}{d\theta^2}.$$

For if OT be the polar subtangent corresponding to a point

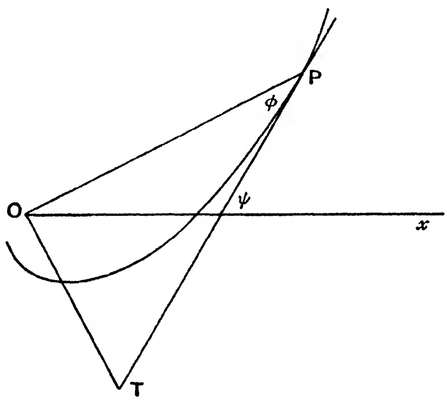


Fig. 75.

P, the point of contact of the tangent, we have with the usual notation
$$PT=r\sec\phi,$$

and
$$\text{Area swept by } PT=\frac{1}{2}\int PT^2\,d\psi$$

$$=\frac{1}{2}\int r^2\sec^2\phi\,d\psi$$

$$=\frac{1}{2}\int r^2\left(\frac{ds}{dr}\right)^2\frac{d\psi}{ds}\cdot\frac{ds}{d\theta}\cdot d\theta$$

$$=\frac{1}{2}\int\frac{r^2}{r_1^{\,2}}\cdot\left(\frac{ds}{d\theta}\right)^3\frac{1}{\rho}\,d\theta$$

$$=\frac{1}{2}\int\frac{r^2}{r_1^{\,2}}(r^2+r_1^{\,2})^{\frac{3}{2}}\frac{r^2+2r_1^{\,2}-rr_2}{(r^2+r_1^{\,2})^{\frac{3}{2}}}\,d\theta$$

$$=\frac{1}{2}\int\frac{r^2}{r_1^{\,2}}(r^2+2r_1^{\,2}-rr_2)\,d\theta,$$

the limits being the initial and final values of θ for the arc specified.

For a closed curve this area therefore exceeds twice the area of the original curve by

$$\frac{1}{2}\int\frac{r^3}{r_1^2}(r-r_2)d\theta.$$

438. **Intrinsic Equation.**

When the intrinsic equation is given, viz.

$$s=f(\psi),$$

the area bounded by the curve, an initial tangent, and an ordinate from any point of the curve to the same, is given by

$$A=\int_0^\psi\int_0^\chi f'(\chi)f'(\omega)\cos\chi\sin\omega\,d\chi\,d\omega,$$

it being assumed that the integrand is finite and continuous and does not change sign within the limits of integration.

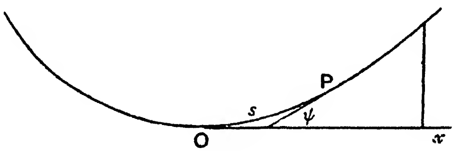


Fig. 76.

This is merely a transformation of

$$A=\int y\,dx.$$

For $\dfrac{dy}{ds}=\sin\psi$ and $y=\displaystyle\int_0^s\sin\psi\,ds=\int_0^\psi f'(\psi)\sin\psi\,d\psi$

$$=\int_0^\psi f'(\omega)\sin\omega\,d\omega.$$

Also $\qquad dx=\cos\psi\,ds=f'(\psi)\cos\psi\,d\psi.$

Hence $\qquad A=\displaystyle\int_0^\psi f'(\psi)\cos\psi\left\{\int_0^\psi f'(\omega)\sin\omega\,d\omega\right\}d\psi.$

$$=\int_0^\psi f'(\chi)\cos\chi\left\{\int_0^\chi f'(\omega)\sin\omega\,d\omega\right\}d\chi.$$

This may clearly be written

$$A=\int_0^\psi\int_0^\chi f'(\chi)\cos\chi f'(\omega)\sin\omega\,d\chi\,d\omega.$$

439. Ex. Taking as a test the case of the circle $s=a\psi$,

$$A=a^2\int_0^\psi\int_0^\chi \cos\chi\sin\omega\,d\chi\,d\omega$$
$$=a^2\int_0^\psi \cos\chi\Big[-\cos\omega\Big]_0^\chi d\chi$$
$$=a^2\int_0^\psi \cos\chi(1-\cos\chi)\,d\chi$$
$$=a^2\int_0^\psi\left(\cos\chi-\frac{1+\cos 2\chi}{2}\right)d\chi$$
$$=a^2\left[\sin\psi-\frac{\psi}{2}-\frac{\sin 2\psi}{4}\right],$$

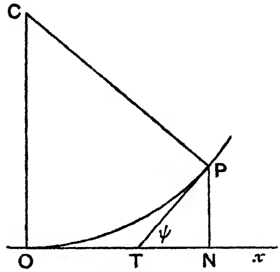


Fig. 77.

which may readily be verified otherwise.

440. **Closed Oval.**

If the area be a closed oval and O a point on the circumference, viz. the starting point for the measurement of s, we may obtain the area of the whole curve by integrating $-\int y\cos\psi\,ds$ round the whole contour, and our formula may be written

$$A=\int_0^{2\pi}\int_\chi^0 f'(\chi)f'(\omega)\cos\chi\sin\omega\,d\chi\,d\omega,$$

the integrand being supposed finite and continuous throughout, and the curve $s=f(\psi)$ having no singularities.

441. **Closed Oval. Another Form.**

Another form may be given for the area of a closed curve whose intrinsic equation is $s=f(\psi)$.

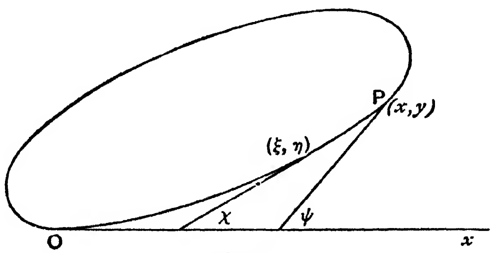


Fig. 78.

Measuring s from the point at which $\psi=0$, we have at any point ξ,η, where the inclination of the tangent to the initial tangent is χ, and the element of arc ds_1,

$$\frac{d\xi}{ds_1}=\cos\chi,\quad \frac{d\eta}{ds_1}=\sin\chi;$$

$$\therefore\ x=\int_0^s \cos\chi\, ds_1=\int_0^\psi \cos\chi f'(\chi)d\chi,$$

$$y=\int_0^s \sin\chi\, ds_1=\int_0^\psi \sin\chi f'(\chi)d\chi;$$

$\therefore\ x\,dy-y\,dx$

$$=\left[\int_0^\psi \cos\chi f'(\chi)d\chi\right]\sin\psi\, ds-\left[\int_0^\psi \sin\chi f'(\chi)d\chi\right]\cos\psi\, ds$$

$$=\left[\int_0^\psi f'(\chi)\sin(\psi-\chi)d\chi\right]ds;$$

$\therefore$ area of curve

$$=\frac{1}{2}\int(x\,dy-y\,dx),\text{ taken round the perimeter,}$$

$$=\frac{1}{2}\int_0^{2\pi} f'(\psi)\left[\int_0^\psi f'(\chi)\sin(\psi-\chi)d\chi\right]d\psi,$$

or, as we may write it.

$$A=\frac{1}{2}\int_0^{2\pi}\int_0^\psi f'(\psi)f'(\chi)\sin(\psi-\chi)\,d\psi\,d\chi,$$

it being understood that the first integration is with regard to χ, considering ψ a constant, from 0 to ψ, and then the result from 0 to 2π with regard to ψ.

Also $$\int_0^\psi f'(\chi)\sin(\psi-\chi)d\chi$$

may be integrated by parts, and becomes

$$=\Big[f(\chi)\sin(\psi-\chi)\Big]_0^\psi+\int_0^\psi f(\chi)\cos(\psi-\chi)d\chi$$

$$=\int_0^\psi f(\chi)\cos(\psi-\chi)d\chi,\quad\text{for } f(0)=0.$$

Hence the result may be exhibited as

$$A=\frac{1}{2}\int_0^{2\pi} f'(\psi)\left\{\int_0^\psi f(\chi)\cos(\psi-\chi)d\chi\right\}d\psi$$

or $$=\frac{1}{2}\int_0^{2\pi}\int_0^\psi f'(\psi)f(\chi)\cos(\psi-\chi)d\psi\,d\chi,$$

it being understood as before that the first integration is with regard to χ from 0 to ψ.

442. If the curve be not closed, and the limits for ψ are from $\psi=\alpha$ to $\psi=\beta$, we find by these formulae, a sectorial

area bounded by the arc and two specified radii vectores, viz. from the origin to the points where $\psi=\alpha$ and $\psi=\beta$, and

$$A=\frac{1}{2}\int_\alpha^\beta\int_0^\psi f'(\psi)f(\chi)\cos(\psi-\chi)\,d\psi\,d\chi.$$

443. Inverse Curve.

If the points P, Q be contiguous points on a curve, and P', Q' their respective inverses, k being the constant of inversion

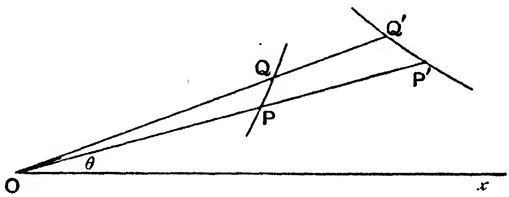


Fig. 79.

and O the pole, we have for any sectorial element $OP'Q'$ of the new curve,

$$\text{area } OP'Q'=\tfrac{1}{2}OP'.OQ'\sin\delta\theta=\tfrac{1}{2}k^4\frac{1}{OP.OQ}\delta\theta \text{ to the first order}$$

$$=\frac{k^4}{2}\frac{1}{r^2}\delta\theta \text{ to the first order,}$$

and the area of any sectorial portion of the inverse is

$$\frac{k^4}{2}\int\frac{1}{r^2}d\theta,$$

r being the radius vector of the original curve.

Ex. Thus the area of the inverse of $Ax^4+By^4=a^2(x^2+y^2)$ with regard to the origin is

$$4.\frac{k^4}{2a^2}\int_0^{\frac{\pi}{2}}(A\cos^4\theta+B\sin^4\theta)\,d\theta$$

$$=\tfrac{3}{8}\pi\frac{k^4}{a^2}(A+B).$$

It will be noted that this amounts to performing the inversion first, and then finding the area as $\frac{1}{2}\int r'^2\,d\theta$, so that our formula $\frac{k^4}{2}\int\frac{1}{r^2}d\theta$ is of but little additional convenience.

444. Locus of Origins of Pedals of given Area.

Let O be a fixed point. Let p, ψ be the polar coordinates of the foot of the perpendicular OY upon any tangent to a

given curve. Let P be any other fixed point, $PY_1\,(=p_1)$, the perpendicular from P upon the same tangent. Then the areas of the pedals, with O and P respectively as origins, are

$$\frac{1}{2}\int p^2\,d\psi \quad \text{and} \quad \frac{1}{2}\int p_1{}^2\,d\psi,$$

taken between the same definite limits. Call these A and A_1 respectively. Let r, θ be the polar coordinates of P with regard to O, and x, y their Cartesian equivalents.

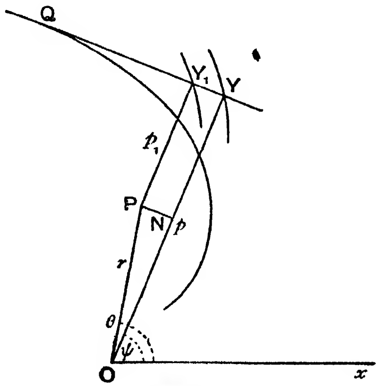


Fig. 80.

Then $\quad p_1 = p - r\cos(\theta-\psi) = p - x\cos\psi - y\sin\psi,$
and p is a known function of ψ.

Hence

$$2A_1 = \int p_1{}^2\,d\psi = \int (p - x\cos\psi - y\sin\psi)^2\,d\psi$$

$$= \int p^2\,d\psi - 2x\int p\cos\psi\,d\psi - 2y\int p\sin\psi\,d\psi$$

$$+x^2\int\cos^2\psi\,d\psi + 2xy\int\cos\psi\sin\psi\,d\psi + y^2\int\sin^2\psi\,d\psi$$

Now $\quad 2\int p\cos\psi\,d\psi,\quad 2\int p\sin\psi\,d\psi,\quad \int\cos^2\psi\,d\psi,$ etc.,

taken between such limits that the whole pedal is described, will be definite constants. Call them respectively

$$-2g,\ -2f,\ a,\ 2h,\ b,$$

and we thus obtain

$$2A_1 = 2A + 2gx + 2fy + ax^2 + 2hxy + by^2.$$

If then P move in such a manner that A_1 is constant, its locus must be a conic section.

By Article 342,

$$\int_0^\psi \cos^2\psi\, d\psi \times \int_0^\psi \sin^2\psi\, d\psi > \left\{\int_0^\psi \cos\psi \sin\psi\, d\psi\right\}^2,$$

i.e. $$ab > h^2.$$

Hence this conic section is **in general an ellipse.**

Moreover, its centre being given by

$$ax + hy + g = 0,$$
$$hx + by + f = 0,$$

its position is *independent of the magnitude* of A_1.

Hence for different values of A_1 these several conic-loci will all be concentric. We shall call this centre Ω.

445. Closed Oval.

Next suppose that the original curve is a closed oval curve, and that the point P is within it. Then the limits of integration are 0 and 2π.

Thus $$a = \int_0^{2\pi} \cos^2\psi\, d\psi = \pi = \int_0^{2\pi} \sin^2\psi\, d\psi = b$$

and $$h = \int_0^{2\pi} \cos\psi \sin\psi\, d\psi = 0.$$

Hence the conic becomes

$$\pi(x^2 + y^2) + 2gx + 2fy + 2(A - A_1) = 0,$$

that is a circle whose centre is at the point

$$\frac{1}{\pi}\int_0^{2\pi} p\cos\psi\, d\psi, \quad \frac{1}{\pi}\int_0^{2\pi} p\sin\psi\, d\psi.$$

Now, if x, y be the point of contact of the tangent, viz. Q,

$$QY = \frac{dp}{d\psi},$$

and $$\left.\begin{aligned} x &= p\cos\psi - \frac{dp}{d\psi}\sin\psi, \\ y &= p\sin\psi + \frac{dp}{d\psi}\cos\psi, \end{aligned}\right\} \text{by projecting } p, \frac{dp}{d\psi} \text{ upon the coordinate axes;}$$

$$\therefore \int x\, d\psi = \int p\cos\psi\, d\psi - [p\sin\psi] + \int p\cos\psi\, d\psi = 2\int p\cos\psi\, d\psi,$$

and

$$\int y\,d\psi=\int p\sin\psi\,d\psi+[p\cos\psi]+\int p\sin\psi\,d\psi=2\int p\sin\psi\,d\psi,$$

for the portions in square brackets disappear in integrating round the whole curve.

Hence the **coordinates of the centre** of the circle may be written

$$\left.\begin{aligned} x_1&=\frac{1}{2\pi}\int x\,d\psi,\ \text{or}\ \frac{\int x\,d\psi}{\int d\psi},\ \text{or}\ \frac{1}{2\pi}\int\frac{x}{\rho}\,ds,\\ y_1&=\frac{1}{2\pi}\int y\,d\psi,\ \text{or}\ \frac{\int y\,d\psi}{\int d\psi},\ \text{or}\ \frac{1}{2\pi}\int\frac{y}{\rho}\,ds,\end{aligned}\right\}\ \text{where}\ \frac{1}{\rho}\ \text{is the curvature at the element}\ ds.$$

446. **Another Determination of the Centre.**

If the original curve be regarded as a material curve of uniform section ω and with a density proportional to the curvature at each point, $=\frac{k}{\rho}$, say, the mass of each element δs is $\frac{k}{\rho}\omega\,\delta s$, and the formulae

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},\quad \bar{y}=\frac{\Sigma my}{\Sigma m}$$

of Statics show that the centroid of any arc of this curve is given by

$$\bar{x}=\frac{\int\frac{k}{\rho}\omega x\,ds}{\int\frac{k}{\rho}\omega\,ds}=\frac{\int\frac{x}{\rho}\,ds}{\int\frac{1}{\rho}\,ds},\quad \text{or}\quad \frac{\int x\,d\psi}{\int d\psi},$$

$$\bar{y}=\frac{\int\frac{k}{\rho}\omega y\,ds}{\int\frac{k}{\rho}\omega\,ds}=\frac{\int\frac{y}{\rho}\,ds}{\int\frac{1}{\rho}\,ds},\quad \text{or}\quad \frac{\int y\,d\psi}{\int d\psi}.$$

Hence the point Ω, which is the centre of these loci, is identical with the centroid of a material wire of fine uniform section, bent into the form of the original curve, and having a density proportional to the curvature at each point; or, which comes to the same thing, having uniform density and cross-section infinitesimally small but proportional at each point to the curvature.

447. Connection of Areas.

The point Ω having been found, let us transfer our origin from O to Ω. The linear terms of the conic will thereby be removed. Thus Ω is a point such that the integrals

$$\int p\cos\psi\, d\psi \quad \text{and} \quad \int p\sin\psi\, d\psi,$$

where p is now measured from Ω, both vanish, and if Π be the area of the pedal whose pole is Ω, we have for any other,

$$2A_1 = 2\Pi + ax^2 + 2hxy + by^2 \text{ in the general case,}$$

and $\quad 2A_1 = 2\Pi + \pi(x^2+y^2)$ in the particular case, when the oval is closed.

The area of the conic is $\dfrac{2\pi(A_1-\Pi)}{\sqrt{ab-h^2}}$. (Smith, *Conic Sections*, Art. 171.)

Thus, in the general case,

$$A_1 = \Pi + \frac{\sqrt{ab-h^2}}{2\pi} \times \text{area of conic.}$$

And in the particular case of the closed oval,

$$A_1 = \Pi + \tfrac{1}{2}\pi r^2,$$

where r is the radius of the circle on which P lies for constant values of A_1, *i.e.* the distance of P from Ω.

448. Position of the Point Ω for a Centric Closed Oval.

In any oval which has a centre the point Ω is plainly at that centre. For when the centre is taken as origin, the integrals

$$\int p\cos\psi\, d\psi \text{ and } \int p\sin\psi\, d\psi, \quad i.e.\ \frac{1}{2}\int x\, d\psi \text{ and } \frac{1}{2}\int y\, d\psi,$$

both vanish when the integration is performed for the complete oval, opposite elements of the integration cancelling; or, which is the same thing, the centroid of a material centric oval curve for a law of density, which varies as the curvature at each point, is obviously at the centre of the oval.

449. Origin for Pedal of Minimum Area.

When Ω is taken as origin, it appears that

$$2A_1 = 2\Pi + \int (x\cos\psi + y\sin\psi)^2\, d\psi.$$

Hence, as the term $\int (x\cos\psi + y\sin\psi)^2\, d\psi$ is necessarily positive, it is clear that A_1 can never be less than Π.

Ω is therefore the origin for which the corresponding pedal curve has a minimum area.

450. **A Statical View of the Case.**

Let O be the origin, QRS the closed oval, OY the perpendicular from O upon a tangent to the curve. Let P be any other point, and Ω the centre of gravity of the curve, QRS having a density at each point proportional to the curvature.

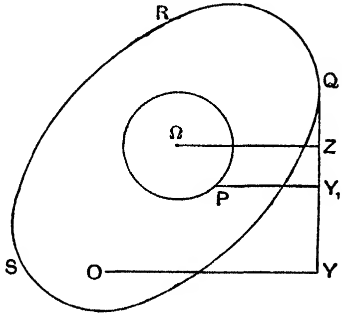


Fig. 81.

A theorem by Lagrange (Routh, *Statics*, vol. i. Art. 436) states that if m_1, m_2, m_3, ... be the masses of a system of heavy particles at Q_1, Q_2, Q_3, ..., and Ω their centre of gravity, and if P be any other point, then

$$m_1PQ_1^2+m_2PQ_2^2+m_3PQ_3^2+\ldots=m_1\Omega Q_1^2+m_2\Omega Q_2^2+m_3\Omega Q_3^2 +\ldots+(m_1+m_2+m_3+\ldots)\Omega P^2.$$

Applying this theorem to our curve of density $\dfrac{k}{\rho}$, uniform small section ω, and total mass $\lambda k\omega$, say,

$$\int\frac{PQ^2}{\rho}ds=\int\frac{\Omega Q^2}{\rho}ds+\lambda.P\Omega^2.$$

Now it has been proved in Art. 426 that the area of the pedal of a closed oval exceeds $\frac{1}{2}$ the area of the oval by $\frac{1}{4}\int\frac{r^2}{\rho}ds$.

$$\therefore\ \text{pedal with regard to } P=\frac{1}{2}\text{oval }+\frac{1}{4}\int\frac{PQ^2}{\rho}ds;$$

and $$\text{pedal with regard to } \Omega=\frac{1}{2}\text{oval }+\frac{1}{4}\int\frac{\Omega Q^2}{\rho}ds;$$

$\therefore$ pedal with regard to P = pedal with regard to $\Omega + \frac{\lambda}{4} P\Omega^2$

and $\lambda k\omega$ = mass of curve $= \int \frac{k}{\rho} \omega\, ds = k\omega \int d\psi = 2\pi k\omega$;

$\therefore\ \lambda = 2\pi$.

$\therefore$ pedal with regard to P = pedal with regard to $\Omega + \frac{\pi}{2} P\Omega^2$.

Hence we are led by statical considerations to the same result as already obtained, viz. that the loci of the origins P, of which the pedal curves of a closed oval are of constant area, are concentric circles, their centre being the origin of the pedal of minimum area and the centroid of a fine wire bent into the form of the original oval, and having uniform cross-section and a density varying as the curvature.

Illustrative Examples.

Ex. 1. Find the area of the pedal of a circle with regard to any point within the circle at a distance c from the centre *i.e.* a limaçon.

Here $$A_1 = \Pi + \frac{\pi c^2}{2}$$

and $$\Pi = \pi a^2.$$

Hence $$A_1 = \pi a^2 + \tfrac{1}{2}\pi c^2.$$

Ex. 2. Find the area of the pedal of an ellipse with regard to any point at a distance c from the centre.

In this case, Π is the area of the pedal with regard to the centre

$$= 2\int_0^{\frac{\pi}{2}} (a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta = (a^2+b^2)\frac{\pi}{2}.$$

Hence $$A_1 = \frac{\pi}{2}(a^2+b^2+c^2).$$

Ex. 3. The area of the pedal of the cardioide $r = a(1-\cos\theta)$ taken with respect to an internal point on the axis at a distance c from the pole is

$$\frac{3\pi}{8}(5a^2 - 2ac + 2c^2).$$

[MATH. TRIPOS, 1876.]

Let O be the pole, P the given internal point; p and p_1 the two perpendiculars OY_2 and PY_1 upon any tangent from O and P respectively; ϕ the angle $Y_2\hat{O}P$ and $OP = c$; then $p_1 = p - c\cos\phi$, and

$$2A_1 = 2A - 2c\int p\cos\phi\, d\phi + \int c^2\cos^2\phi\, d\phi.$$

Now, in order that p may sweep out the whole pedal, we must integrate between limits $\phi=0$ and $\phi=\frac{3\pi}{2}$ and double. Now in the cardioide (Fig. 82).

$$p=OQ\sin Y_2QO=OQ\sin\tfrac{1}{2}xOQ.$$

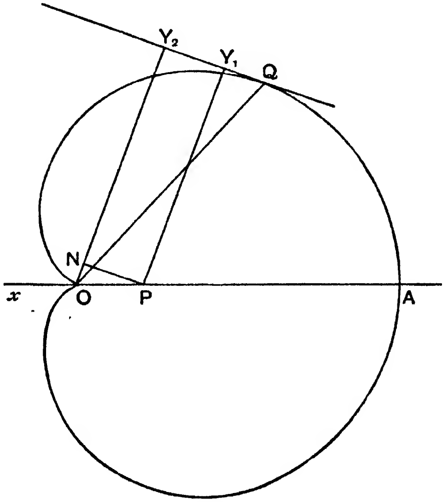


Fig. 82.

For $$Y_2QO=\tfrac{1}{2}xOQ=\frac{\theta}{2}.$$

Hence $$\frac{\pi}{2}-\left\{\phi-(\pi-\theta)\right\}=\frac{\theta}{2},$$

or $$\frac{3\pi}{2}-\phi=\frac{3\theta}{2}\quad\text{and}\quad\frac{\theta}{2}=\frac{\pi}{2}-\frac{\phi}{3}.$$

So $$p=r\sin\frac{\theta}{2}=2a\sin^3\frac{\theta}{2}=2a\cos^3\frac{\phi}{3}.$$

Hence

$$\int p\cos\phi\,d\phi=2\int_0^{\frac{3\pi}{2}}2a\cos^3\frac{\phi}{3}\cos\phi\,d\phi=4a\times3\int_0^{\frac{\pi}{2}}\cos^3z\cos3z\,dz$$

$$=12a\int_0^{\frac{\pi}{2}}\left[4\cos^6z-3\cos^4z\right]dz$$

$$=12a\left[4\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-3\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]=\frac{3\pi a}{4}.$$

Also $\int c^2\cos^2\phi\,d\phi=3\,.\,2c^2\,.\,\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3\pi c^2}{2}.$

Finally, $$2A=2\int_0^{\frac{3\pi}{2}}4a^2\cos^6\frac{\phi}{3}\,d\phi=24a^2\int_0^{\frac{\pi}{2}}\cos^6z\,dz;$$

$$\therefore\ A=12a^2\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{15\pi a^2}{8}.$$

Thus, $$A_1=\frac{15\pi a^2}{8}-\frac{3\pi ac}{4}+\frac{3\pi c^2}{4}$$
$$=\frac{3\pi}{8}(5a^2-2ac+2c^2).$$

Ex. 4. Let A, B, C be any three points and P a fourth point whose areal coordinates are x, y, z when the triangle ABC is regarded as the triangle of reference. To find the relation of the areas of the pedals of any closed curve with respect to A, B, C and P.

Let $[A]$, $[B]$, $[C]$, $[P]$ represent the areas of the pedals. Let X, Y, Z be the areal coordinates of Ω, the centre for the pedal of minimum area.

Then
$$[A]=[\Omega]+\tfrac{1}{2}\pi A\Omega^2,$$
$$[B]=[\Omega]+\tfrac{1}{2}\pi B\Omega^2,$$
$$[C]=[\Omega]+\tfrac{1}{2}\pi C\Omega^2,$$
$$[P]=[\Omega]+\tfrac{1}{2}\pi P\Omega^2;$$
$$\therefore\ [P]-[A]x-[B]y-[C]z=\frac{\pi}{2}(P\Omega^2-xA\Omega^2-yB\Omega^2-zC\Omega^2).$$

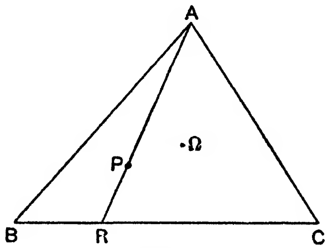


Fig. 83.

Now (Ferrers' *Trilinears*, p. 6) the distance from x, y, z to X, Y, Z is given by
$$P\Omega^2=-a^2(y-Y)(z-Z)-b^2(z-Z)(x-X)-c^2(x-X)(y-Y)$$
and $$A\Omega^2=-a^2(0-Y)(0-Z)-b^2(0-Z)(1-X)-c^2(1-X)(0-Y)$$
$$=-a^2YZ-b^2ZX-c^2XY+b^2Z+c^2Y,$$
$$B\Omega^2=-b^2ZX-c^2XY-a^2YZ+c^2X+a^2Z,$$
$$C\Omega^2=-c^2XY-a^2YZ-b^2ZX+a^2Y+b^2X;$$
$$\therefore\ P\Omega^2-xA\Omega^2-yB\Omega^2-zC\Omega^2=-a^2yz-b^2zx-c^2xy.$$

Now, if $S\equiv a^2yz+b^2zx+c^2xy$, $S=0$ is the equation of the circumcircle, and S is equal to minus the square of the tangent from the point (x, y, z) to the circle $S=0$ if the point lie without the circle, or to the rectangle of the segments of any chord through x, y, z if within. Therefore with this meaning for S,
$$[P]=[A]x+[B]y+[C]z-\tfrac{1}{2}\pi S.$$

PROBLEMS ON QUADRATURE.

1. Interpret geometrically $\int_{p_0}^{p_1} \sqrt{r^2-p^2}\,dp$ in the case of the curve $r=f(p)$

Prove that the value of $\int \sqrt{r^2-p^2}\,dp$, taken all round an ellipse whose semiaxes are a, b, and whose centre is the pole, is $\pi(a-b)^2$.
[OXFORD I. P., 1903.]

2. Use the pedal equation of an ellipse, viz. $\frac{a^2b^2}{p^2}=a^2+b^2-r^2$, to show that the area of the portion of an ellipse included between the curve, the semi-major axis and a central radius vector r, is

$$\frac{ab}{2}\tan^{-1}\sqrt{\frac{a^2-r^2}{r^2-b^2}},$$

a, b being the semiaxes of the ellipse. [COLLEGES, 1882.]

3. Find the area of the part of the ellipse $p^2(2a-r)=b^2r$ included between two focal radii vectores drawn, one to an extremity of the minor axis and the other to the nearer extremity of the major axis.
[OXFORD I. P., 1889.]

4. Find the area included between an ellipse and its evolute and bounding radii of curvature, the one coinciding with the major axis and the other inclined at an angle of $\frac{\pi}{4}$ to it.
[COLLEGES, 1884, AND β, 1888.]

5. Through every point of an ellipse a line is drawn outwards normal to the ellipse and equal to the radius of curvature at the point. Show that the area of the curve thus obtained is

$$\pi\frac{9a^4+14a^2b^2+9b^4}{2ab}.$$

[COLLEGES α, 1891.]

6. Show that the area of that part of the evolute of an ellipse $\left(\text{eccentricity} > \frac{1}{\sqrt{2}}\right)$ which lies outside the ellipse is

$$a^4b^4\int_{b^2}^{\frac{a^2+b^2}{3}} \frac{(a^2+b^2-3\rho)^2}{\rho^3(a^2+b^2-\rho)^2}\frac{d\rho}{\sqrt{(\rho-b^2)(a^2-\rho)}}.$$

[COLLEGES, 1882.]

7. Find the area of the pedal of the curve

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}},$$

the origin being taken at $x=\sqrt{a^2-b^2}$, $y=0$. [OXFORD I. P., 1888.]

8. Show that the area of the space between the epicycloid $p = A \sin B\psi$ and its pedal curve taken from cusp to cusp is $\frac{1}{4}\pi A^2 B$.

[Colleges, 1878.]

9. Show that the area between an epicycloid and the arc of the fixed circle included between two consecutive cusps is

$$\frac{\pi b^2}{a}(3a + 2b),$$

where a and b are the radii of the fixed and rolling circles respectively. [Colleges α, 1884.]

Show also that the area of the corresponding sector of the fixed circle is that of an ellipse with semiaxes the radii of the two circles.

[Oxford I. P., 1913.]

10. Show that the p-ψ equation to a cycloid when one of the cusps is taken as origin is

$$p = 2a(\sin\psi - \psi\cos\psi),$$

where a is the radius of the generating circle; and find the area between the curve from cusp to cusp and the corresponding arc of the pedal with regard to a cusp. [Oxford II. P., 1903.]

11. Show that the area bounded by that portion of the cardioide $r^{\frac{1}{2}} = a^{\frac{1}{2}} \sin \frac{1}{2}\theta$, which lies in the first quadrant, the terminal tangents, and the corresponding portion of the locus of the extremity of the polar subtangent, is

$$3a^2(10 - 3\pi)/16. \qquad \text{[Math. Tripos, 1896.]}$$

12. Show that in the curve in which the area bounded by the curve and the radii vectores from a certain fixed point varies as the square of the length of the bounding arc, the radius of curvature varies as the projection of the radius vector on the tangent.

[Colleges α, 1891.]

13. The pedal of a cycloid with regard to any point on its axis meets the cycloid at the vertex A and cuts the tangent at the cusp in Q; find the area between it and the chord AQ; and prove that this area is least when the origin is the middle point of the axis.

[St. John's, 1883]

14. An elliptic wire is pushed in one plane through a very short straight tube; find the equation to the locus of the centre, and prove that the area of each loop is $\frac{\pi}{2}(a-b)^2$, where a and b are the semiaxes. [Colleges, 1886].

15. A point Q is taken on the normal drawn outward at a point P of a catenary, the parameter of which is c. Prove that if PQ is equal to the length of the arc of the catenary measured from the vertex to P, the area between the locus of Q and the catenary, and bounded by the normal at the vertex and by another normal inclined at an angle ψ to this, is

$$\frac{c^2}{2}(\tan^2\psi + \tan\psi - \psi).$$

[COLLEGES γ, 1882.]

16. Prove that the pedal of the cardioide $r = a\cos^2\frac{\theta}{2}$ with respect to the cusp consists of two closed regions of areas A and B, A consisting of the inner loop and B being external to A and bounded by the outer line of the curve and such that $2A + B = \frac{15\pi a^2}{32}$.

[COLLEGES γ, 1899.]

17. Prove that the area of the pedal of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ with respect to the point $(a, 0)$ is five times as great as the area of its pedal with respect to the origin. [OXFORD II. P., 1899.]

18. The tangent at a point P of a lemniscate cuts the curve again at Q, R. Prove that the middle point of QR is at the same distance from the nodal point as P; and that the equation to its locus is

$$a^{10}(x^2 - y^2) = r^4\{a^8 + 4(a^4 - r^4)(a^4 - 4r^4)\},$$

where $$r^2 \equiv x^2 + y^2.$$

Show that it can be written $$r^2 = a^2\cos\tfrac{2}{3}\theta.$$

Trace the curve completely, and prove that the portion corresponding to the upper half of one branch of the lemniscate divides the other branch into two parts whose areas are in the ratio of

$$6 - 3\sqrt{3} : 3\sqrt{3} - 4.$$

[ST. JOHN'S, 1884.]

19. Show that the area of a loop of the curve

$$(x^2 - a^2)^2 + (y^2 - 3a^2)^2 = a^4$$

is $$a^2\sqrt{2}\left(\frac{\pi}{3} - \log_e\frac{\sqrt{3}+1}{2}\right).$$

[MATH. TRIPOS, 1882.]

20. The tangent at every point P of a closed finite curve is produced to Q so that PQ is constant. Find the area between the locus of Q and the original curve. How is the result to be explained, (i) if the curvature of the first curve is sometimes in one direction, sometimes in the opposite direction; (ii) if the curve cuts itself a given number of times. [ST. JOHN'S COLL., 1881.]

21. A straight line of constant length c is drawn from each point of a closed oval curve making a given angle α with the normal at that point. Prove that the area of the curve traced out by the end of the line is
$$S + \pi c^2 \pm lc \cos \alpha,$$
where S is the area of the given oval curve and l is its length.

[COLL. γ, 1893.]

22. Show that the area of the polar reciprocal of a curve whose equation is given in rectangular coordinates is

$$\tfrac{1}{2}k^4 \int \frac{\dfrac{d^2y}{dx^2}}{\left(y - x\dfrac{dy}{dx}\right)^2} dx,$$

x, y being the coordinates of a point on the original curve.

Apply this to find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[COLLEGES, 1886.]

23. The area of a given closed oval curve is A; the bisectors of the internal and external angles between tangents to it which meet at a given constant angle 2α envelop curves whose areas are A_1 and A_2; show that
$$A_1 \cos^2\alpha + A_2 \sin^2\alpha = A.$$
[COLLEGES γ, 1888.]

24. Prove that for any closed curve which has a centre, the area of the locus of intersection of tangents at right angles, and the area of the locus of intersection of normals at right angles differ by twice the area of the curve. [MATH. TRIPOS, 1888.]

25. O being a fixed point, OP a radius vector of any curve, OP is produced to Q so that $OP \,.\, PQ = a^2$, and A is the area between the locus of Q and the given curve. If A' be the area of the inverse of the curve with respect to O, the constant of inversion being a, show that $A' - A$ is independent of the form of the curve.

If the given curve be a circle, and O a point on its circumference, find the area of any part bounded by the locus of Q, the circle and two radii vectores from O. [ST. JOHN'S, 1891.]

26. A circle rolls on the outside of an oval curve, the pedals of the curve, of the locus of the centre of the circle and of the envelope of the circle are of areas A_0, A_1, A_2, respectively; prove that $A_2 - 2A_1 + A_0$ depends only on the rolling circle.

Show that if the area of the oval curve, of the locus of the centre of the circle and of the envelope of the circle be S_0, S_1, S_2 respectively,
$$A_2 - 2A_1 + A_0 = S_2 - 2S_1 + S_0.$$
[TRINITY, 1878.]

27. One of the curves given by the equation

$$y = a^2 \frac{d}{ds}\left\{\frac{dy}{dx} + \frac{1}{3}\left(\frac{dy}{dx}\right)^3\right\}$$

cuts the axis of x twice at the angle α. Prove that the area between the curve and the axis is

$$a^2\{\tan\alpha\sec\alpha + \log(\sec\alpha + \tan\alpha)\}. \quad \text{[Oxf. I. P., 1912.]}$$

28. A curve concave to the axis of x is such that the product of the ordinate and radius of curvature at any point is constant and equal to c^2 (The Elastica, or Bent Bow). Prove that the maximum value of the ordinate is $2c\sin\frac{\alpha}{2}$, where α is the angle at which the curve crosses the axis of x. [Ox. I. P., 1903.]

Show that the area which lies between the bow and the bow-string is $2c^2\sin\alpha$.

29. Show that the area of a closed curve, which is the envelope of the line $x\cos\psi + y\sin\psi = p$, is the value of the integral

$$-\frac{1}{2}\int\left(\frac{dp}{d\psi} + p\right)\left(\frac{dp}{d\psi} - p\right)d\psi$$

taken completely round the curve. [Math. Trip., 1898.]

30. The integral $-\frac{1}{2}\int\left(\frac{dp}{d\psi} + np\right)^2 d\psi$ is taken round a closed curve, n being taken equal to $\tan\psi$ or to $-\cot\psi$, according as the one or the other is numerically less than unity. Show that the value of the integral differs from the area of the curve by the sum of the squares of the perpendiculars from the origin upon the tangents at the points where the integral changes form. [Math. Trip., 1898.]

31. In the cycloid prove that the conic locus of points with regard to which the area of the pedal is constant, is in general a circle, and find the point for which the area of the pedal is a minimum. [Ox. I. P., 1900.]

32. In a catenary, A is the vertex, P any point on the curve, AO, PN perpendiculars upon the directrix, PY a tangent and NY perpendicular to it. Show that the area of the figure $ONPA$ is double that of the triangle YNP.

33. Show that the area of the first positive pedal of the curve $p = f(r)$ may be obtained by the formula

$$\frac{1}{2}\int\frac{p^2}{\sqrt{r^2 - p^2}}\frac{dp}{dr}dr,$$

where the letters p and r are the pedal coordinates of a point on the original curve.

Apply this method to find the area of the cardioide, which is the first positive pedal of the circle $r^2 = ap$.

34. Employ the formula

$$\frac{1}{2}\int \frac{pr}{\sqrt{r^2 - p^2}}\, dr$$

to find the area of

$$r^2 + a^2 = b^2 + 2ap \quad (a > b).$$

To what curve does this pedal equation belong?

35. In the epicycloid $\qquad p^2 = a^2 \dfrac{r^2 - a^2}{c^2 - a^2}$,

where a and $\dfrac{c-a}{2}$ are the radii of the fixed and rolling circles respectively, obtain a formula for the area of any sectorial portion with centre of the sector at the origin. Hence show that the area between one foil of the curve and the fixed circle is

$$\pi(c-a)^2(c+2a)/4a.$$

36. When $a < b$ the conchoid of Nicomedes, viz.

$$x^2y^2 = (a+y)^2(b^2 - y^2) \quad \text{or} \quad r = a \operatorname{cosec} \theta \pm b$$

has a loop. Find its area.

37. Let S be the focus of a parabola, SP_1, SP_2 two focal radii vectores of lengths r_1, r_2. The latus rectum is $4a$ and $P_1P_2 = c$. Prove Lambert's expression for the sectorial area SP_1P_2, viz.

$$\frac{\sqrt{a}}{3}\left[s^{\frac{3}{2}} - (s-c)^{\frac{3}{2}}\right],$$

where $2s = r_1 + r_2 + c$.

Show that the segment cut off by a focal chord of length c is

$$\frac{1}{3}\, a^{\frac{1}{2}} c^{\frac{3}{2}}.$$

38. In the case of the Cotes's spirals, whose equations are of the form

$$\frac{1}{p^2} = \frac{A}{r^2} + B,$$

show that the area of the sectorial portion bounded by the curve and the radii vectores r_1 and r_2 is

$$\frac{1}{2B}\left\{(Br_1^2 + A - 1)^{\frac{1}{2}} \sim (Br_2^2 + A - 1)^{\frac{1}{2}}\right\}, \quad B \neq 0.$$

Examine in detail the particular cases of

(i) the equiangular spiral;

(ii) the reciprocal spiral;

(iii), (iv) and (v) the cases which reduce to the polar forms,
$u = a \sinh n\theta$, $u = a \cosh n\theta$, $u = a \sin n\theta$, respectively.

39. Riccati's *Syntractory* * is generated as follows. The tractory is an involute of a common catenary of parameter c, starting from the vertex. PT is a tangent at any point P of the tractory, cutting the directrix of the catenary at T. Q is a point on PT or PT produced such that $QT = c'$. The locus of Q is the syntractory.

Show that the areas between the two branches and the directrix are

$$\frac{\pi}{2} c'(2c \pm c').$$

40. If A be the area of the 'Helmet,'

$$(k+1)\{(x^2+ka^2)y^2 - 2ay(a^2-x^2)\} + (a^2-x^2)^2 = 0, \quad (k \neq -1),$$

and V the volume formed by its revolution about the y-axis, prove that

$$A = \frac{\pi a^2}{\sqrt{k(k+1)}}\left[2(k+1)^{\frac{3}{2}} - (2k+3)k^{\frac{1}{2}}\right],$$

$$V = \frac{2\pi a^3}{3\sqrt{k+1}}\left[3(k+1)^{\frac{3}{2}} \log \frac{\sqrt{k+1}+1}{\sqrt{k+1}-1} - 2(3k+4)\right]$$

[For the first part of the example, and for several others of similar character, see Wolstenholme's *Problems*, Nos. 1886 to 1870.]

* *Comment. Bononensia*, Tom. iii., 1755.

CHAPTER XIV.

QUADRATURE, ETC. (III)

SURFACE INTEGRALS, AREALS, CORRESPONDING CURVES.

451. **Use of Second Order Infinitesimals as Elements of Area. "Surface Integrals," Centroids, etc.**

For many purposes it is found desirable, and often necessary, to use for the element of area a second order infinitesimal.

Suppose, for instance, we desire to find the mass of the area bounded by a given curve, the x-axis and a pair of ordinates, where there is a distribution of surface density over the area, not uniform, but represented at any point by $\sigma=\phi(x, y)$, say, where x, y are the coordinates of the point in question.

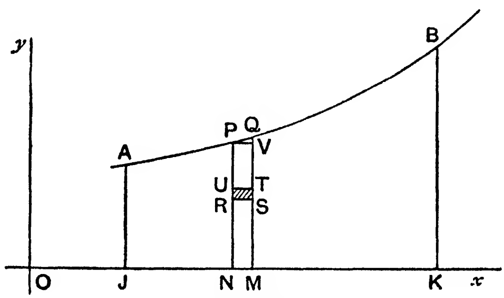


Fig. 84.

Let Ox, Oy be the coordinate axes, AB any arc of the curve whose equation is

$$y=f(x),$$

$\{a, f(a)\}$ and $\{b, f(b)\}$ the coordinates of the points A, B upon it, AJ and BK the ordinates of A and B. Let PN, QM be any contiguous ordinates of the curve, and x, $x+\delta x$ the abscissae of the points P, Q. Let R, U be contiguous points on the

ordinate of P, their ordinates being y, $y+\delta y$; and we shall suppose δx, δy to be small quantities of the first order of smallness.

Draw RS, UT, PV parallel to the x-axis. Thus the area of the rectangle $RSTU$ is $\delta x\,\delta y$, and its mass may be regarded as $\phi(x, y)\,\delta x\,\delta y$ to the second order of smallness.

Then the mass of the strip $PNMV$ may be written

$$Lt_{\delta y=0}[\Sigma\phi(x, y)\,\delta y]\,\delta x,$$

and in conformity with the notation of the Integral Calculus may be expressed as

$$\left[\int \phi(x, y)\,dy\right]\delta x$$

between the limits $y=0$ and $y=f(x)$.

In performing this integration with regard to y, x is to be regarded as constant, for we are finding the limit of the sum of the masses of all elements in the elementary strip PM, parallel to the y-axis, for which x retains the same value, *i.e.* we are finding the mass of the strip PM.

If then we search for the mass of the area $AJKB$, all such strips as the above must now be summed which lie between the ordinates AJ, BK, and the result may be written

$$Lt_{\delta x=0}\Sigma\left[\int_0^{f(x)} \phi(x, y)\,dy\right]\delta x,$$

which may be further written as

$$\int_a^b \left[\int_0^{f(x)} \phi(x, y)\,dy\right]dx,$$

the limits of the integration with regard to x being from $x=a$ to $x=b$.

Thus the mass of the area $AJKB$ for surface density $\phi(x, y)$

$$=\int_a^b \left[\int_0^{f(x)} \phi(x, y)\,dy\right]dx.$$

452. **Notation.**

This will be written

$$\int_a^b \int_0^{f(x)} \phi(x, y)\,dx\,dy,$$

the elements dx, dy being written in the reverse order to that in which they occur in the previous expression, and it

will be remembered that the *right-hand one refers to the first integration*, and the left-hand one to the second. It has already been stated (Art. 363) that we shall throughout the book adopt this order.

If we put $\sigma \equiv \phi(x, y) = 1$, the result of our integration will be to find the area.

Thus,
$$\text{Area} = \int_a^b \int_0^{f(x)} dx\,dy$$
$$= \int_a^b f(x)dx$$
$$= \int_a^b y\,dx, \text{ as before;}$$

or, in the case of the area being bounded by two curves,
$$y = \phi(x), \quad y = \psi(x), \text{ as in Art. 395,}$$
$$\text{Area} = \int_a^b \int_{\psi(x)}^{\phi(x)} dx\,dy$$
$$= \int_a^b [\phi(x) - \psi(x)]dx.$$

Ex. If the surface density of a circular disc bounded by $x^2+y^2=a^2$ be given to vary as the square of the distance from the y-axis, find the mass of the disc.

Here we have μx^2 for the density of the element $\delta x\,\delta y$, and its mass is therefore
$$\mu x^2 \delta x\, \delta y,$$
and the whole mass will be $\displaystyle\iint \mu x^2 dx\,dy.$

The limits for y will be $y=0$ to $y=\sqrt{a^2-x^2}$ for the positive quadrant and for x from $x=0$ to $x=a$. The result must then be multiplied by 4, for the distribution being symmetrical in the four quadrants, the mass is four times the mass of the first quadrant.

Thus,
$$\text{Mass} = 4\int_0^a \int_0^{\sqrt{a^2-x^2}} \mu x^2 dx\,dy$$
$$= 4\mu \int_0^a x^2 \Big[y\Big]_0^{\sqrt{a^2-x^2}} dx$$
$$= 4\mu \int_0^a x^2\sqrt{a^2-x^2}\,dx.$$

Putting $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$, we have
$$\text{Mass} = 4\mu a^4 \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$$
$$= 4\mu a^4 \cdot \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{2\Gamma(3)} = 4\mu a^4 \cdot \frac{\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{2\,.\,2} = \frac{\pi\mu a^4}{4}.$$

453. Other Uses of Double Integration.

The same process may be used for many other purposes, of which we give a few illustrative examples, which will serve to indicate to the student the field of investigation now open to him.

Ex. Find the statical moment of a quadrant of the ellipse

$$x^2/a^2+y^2/b^2=1$$

about the y-axis, the surface density being supposed uniform.

Here each element of area $\delta x\,\delta y$ is to be multiplied by its surface density σ (which is by hypothesis constant in the case supposed), and by its distance from the y-axis; the sum of such elementary quantities is then to be found over the whole quadrant. The limits of integration will be from $y=0$ to $y=\frac{b}{a}\sqrt{a^2-x^2}$ for y; and from $x=0$ to $x=a$ for x. Thus we have

$$\begin{aligned}\text{Moment}&=\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\sigma x\,dx\,dy\\&=\frac{\sigma b}{a}\int_0^a x\sqrt{a^2-x^2}\,dx\\&=\frac{\sigma b}{a}\left[-\frac{(a^2-x^2)^{\frac{3}{2}}}{3}\right]_0^a=\frac{\sigma ba^2}{3}\\&=M\frac{4a}{3\pi},\end{aligned}$$

where M is the mass of the quadrant, *i.e.*

$$\frac{\pi ab}{4}\sigma.$$

454. Centroid of a Plane Area.

The formulae proved in Analytical Statics for the coordinates of the centroid of a number of masses $m_1, m_2, m_3, \ldots$ at points (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., are

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},\quad \bar{y}=\frac{\Sigma my}{\Sigma m}.$$

We may apply these to find the coordinates of the centroid of a given area on which there is any proposed distribution of surface density.

Let σ be the surface density at a given point, which may be either a constant, as for a uniform distribution, or a given function of x and y. Then the mass of the element $\delta x\,\delta y$ is $\sigma\,\delta x\,\delta y$ and

$$\bar{x}=\frac{\iint\sigma x\,dx\,dy}{\iint\sigma\,dx\,dy}.$$

Similarly, $$\bar{y}=\frac{\iint \sigma y\,dx\,dy}{\iint \sigma\,dx\,dy},$$

the limits in each case being determined so that the summation will be effected for the whole area in question.

Ex. Find the centroid of the elliptic quadrant of the example in the last article.

It was proved there that

$$\iint \sigma x\,dx\,dy=\frac{\sigma ba^2}{3}=M\frac{4a}{3\pi}$$

and $$\iint \sigma\,dx\,dy=\text{mass of quadrant}=M;$$

$$\therefore\ \bar{x}=\frac{4a}{3\pi}.$$

Also $$\iint \sigma y\,dx\,dy=\sigma\int_0^a\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx$$

$$=\frac{\sigma}{2}\frac{b^2}{a^2}\int_0^a (a^2-x^2)\,dx$$

$$=\frac{\sigma b^2}{2a^2}\left(a^3-\frac{a^3}{3}\right)=\frac{\sigma ab^2}{3}=M\frac{4b}{3\pi};$$

$$\therefore\ \bar{y}=\frac{4b}{3\pi}.$$

Hence the coordinates of the centroid are $\frac{4a}{3\pi}, \frac{4b}{3\pi}$.

455. Moment of Inertia.

When every element of mass of a given body is multiplied by the square of its distance from a given line, the limit of the sum of such products is called the Moment of Inertia with regard to the line.

Ex. 1. Find the moment of inertia of the quadrant of an ellipse about the y-axis, again taking uniform surface density

Here we have to multiply each element of mass, viz. $\sigma\,\delta x\,\delta y$, by x^2, and then integrate as before.

Moment of Inertia $=\iint \sigma x^2\,dx\,dy$

$$=\int \sigma x^2\Big[y\Big]_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx$$

$$=\sigma\frac{b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

$=\sigma\frac{b}{a}\frac{\pi a^4}{16}$, this integral having been worked out in the example of Art. 452,

$$=\pi\frac{a^3b\sigma}{16}=M\frac{a^2}{4},\quad \text{since } M=\frac{\pi ab\sigma}{4}.$$

Ex. 2. Find the moment of inertia of the portion of the parabola $y^2=4ax$, bounded by the axis and the latus rectum, about the x-axis, supposing the surface density at each point to vary as the n^{th} power of the abscissa.

Here the mass-element is $\mu x^n\,\delta x\,\delta y$, μ being a constant, and the moment of inertia is

$$Lt\sum \mu y^2 x^n\,\delta x\,\delta y \quad \text{or} \quad \mu\iint y^2 x^n\,dx\,dy,$$

where the limits for y are from $y=0$ to $2\sqrt{ax}$, and for x from 0 to a.

We thus get

$$\text{Mom. of In.} = \frac{\mu}{3}\int_0^a \Big[y^3\Big]_0^{2\sqrt{ax}} x^n\,dx = \frac{\mu}{3}\int_0^a 8a^{\frac{3}{2}}x^{n+\frac{3}{2}}\,dx$$

$$= \frac{8\mu}{3}a^{\frac{3}{2}}\left[\frac{x^{n+\frac{5}{2}}}{n+\frac{5}{2}}\right]_0^a = \frac{16\mu}{3(2n+5)}a^{n+4}$$

Again, the *Mass* of this portion of the parabola is given by

$$M=\int_0^a\int_0^{2\sqrt{ax}}\mu x^n\,dx\,dy = \mu\int_0^a\Big[y\Big]_0^{2\sqrt{ax}}x^n\,dx$$

$$=2\mu a^{\frac{1}{2}}\int_0^a x^{n+\frac{1}{2}}\,dx = \frac{4\mu}{2n+3}a^{n+2}.$$

Thus we have

$$\text{Moment of Inertia about } Ox = \frac{4}{3}\,\frac{2n+3}{2n+5}\,Ma^2.$$

Examples.

1. In the first quadrant of the circle $x^2+y^2=a^2$ the surface density varies at each point as xy.

Find (i) the mass of the quadrant
(ii) its centroid,
(iii) its moment of inertia about the y-axis.

2. Work out the corresponding results for the portion of the parabola $y^2=4ax$ bounded by the axis and the latus rectum, the surface density varying as $x^p y^q$.

3. Find the centroid of a fine rod of uniform sectional area and of which the line-density varies as the n^{th} power of the distance from one end. Also its moment of inertia about that end, about the other end, and about the middle point.

4. Find the centroid of the triangle bounded by the lines $y=mx$, $x=a$ and the x axis when the surface density at each point varies as the square of the distance from the origin. Also find the moment of inertia about the y-axis.

5. Find the centroid of

(i) either of the areas bounded by the circle $(x-a)^2+y^2=a^2$ and the parabola $y^2=ax$;

(ii) the centroid of the area bounded by the parabolas

$$y^2=4ax, \quad x^2=4by;$$

(iii) the centroid of the area bounded by

$$y^2=4ax, \quad y=2x,$$

the surface density being uniform in each case.

6. Find the moment of inertia of a triangle of uniform surface density

(i) about one of its sides;

(ii) about an axis perpendicular to its plane through an angular point.

456. Polar Coordinates. Second Order Element.

For polar curves it is desirable to use for our element of area a second order infinitesimal of different form.

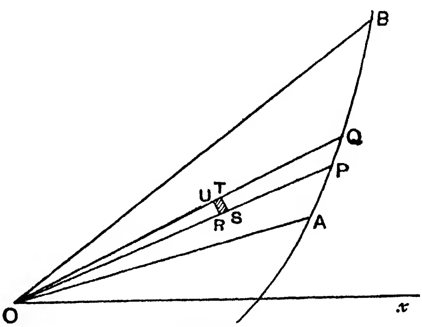


Fig. 85.

Let OP, OQ be two contiguous radii vectores of the curve $r=f(\theta)$; Ox the initial line. Let θ, $\theta+\delta\theta$ be the vectorial angles of the points P, Q on the curve. Draw two circular arcs RU, ST cutting the radii OP, OQ, with centre O and radii r $r+\delta r$ respectively, and let δr, $\delta\theta$ be small quantities of the first order of smallness.

Then area $RSTU$ = sector OST − sector ORU

$$=\tfrac{1}{2}(r+\delta r)^2\,\delta\theta-\tfrac{1}{2}r^2\,\delta\theta$$

$$=r\,\delta\theta\,\delta r \text{ to the second order.}$$

And to this order $RSTU$ may therefore be considered a rectangle of sides δr ($=RS$) and $r\,\delta\theta$ ($=$arc RU).

Thus, if the surface density at each point $R(r, \theta)$ be $\sigma=\phi(r, \theta)$, the mass of the element $RSTU$ is (to second order quantities) $\sigma r\,\delta\theta\,\delta r$, and the mass of the elementary sector OPQ is

$$Lt_{\delta r=0}\,[\Sigma\sigma r\,\delta r]\,\delta\theta$$

the summation being effected for all elements from $r=0$ $r=f(\theta)$,

i.e.
$$\left[\int_0^{f(\theta)} \sigma r\, dr\right] \delta\theta,$$

in which integration θ is to be regarded as constant; and taking the limit of the sum of the elementary sectors for infinitesimal values of $\delta\theta$ between any specified radii vectores $\theta=\alpha$ and $\theta=\beta$, we get the mass of the sectorial area OAB

$$=\int_\alpha^\beta\left[\int_0^{f(\theta)} \sigma r\, dr\right] d\theta,$$

or, as we have agreed to write it (Art. 360),

$$\int_\alpha^\beta \int_0^{f(\theta)} \sigma r\, d\theta\, dr.$$

Obviously when $\sigma=1$ this formula gives the area of the sector.

457. Ex. 1. Find the mass of a circular lamina of radius a in which the surface density at each point varies as the n^{th} power of the distance of that point from a point O on the circumference.

Taking O as origin, and the diameter through O as the initial line, the equation of the curve is

$$r=2a\cos\theta.$$

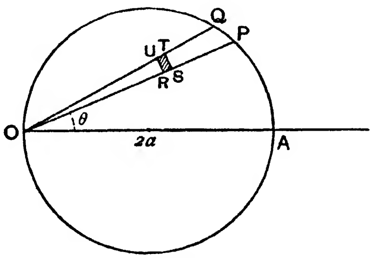


Fig. 86.

Then we have for the density at a point R distant r from O, $\sigma=\mu r^n$ where μ is a constant. The mass of the element $RSTU=\mu r^n\,(r\,\delta\theta\,\delta r)$. Hence the mass of the circular lamina is

$$M=2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} \mu r^n r\, d\theta\, dr$$

$$=\frac{2\mu}{n+2}\int_0^{\frac{\pi}{2}}(2a\cos\theta)^{n+2}\, d\theta$$

$$=\frac{2\mu}{n+2}(2a)^{n+2}\frac{n+1}{n+2}\,\frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{2}{3}$$

or
$$\frac{2\mu}{n+2}(2a)^{n+2}\frac{n+1}{n+2}\,\frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{1}{2}\,\frac{\pi}{2},$$

according as n is odd or even.

Ex. 2. If the moment of inertia were required about a perpendicular to the plane of the lamina through O, each elementary mass $\mu r^n(r\,\delta\theta\,\delta r)$ is to be multiplied by r^2 before integration. The result merely changes n into $n+2$ in the former work, and writing M for the value found for the mass,

$$\text{Moment of Inertia} = M\frac{n+2}{n+4}\frac{n+3}{n+4}(2a)^2.$$

458. Centroids, etc. Polars.

The distance of the centroid of an area whose boundary is defined by a polar equation, from any straight line in the plane of the area and passing through the pole, may be found, as before (Art. 454). Take the line proposed as the x-axis and a perpendicular through the pole as the y-axis. Then the distance of the centroid from the x-axis is obtained by forming the sum of the moments of the masses of the polar elements of area about that line and dividing by the sum of masses; *i.e.* by the use of the formula $\bar{y} = \dfrac{\Sigma my}{\Sigma m}$.

Let σ be the surface density. Then $\sigma r\,\delta\theta\,\delta r$ being the element of mass and $r\cos\theta$, $r\sin\theta$ being its abscissa and ordinate respectively, its moments about the axes of y and x through O are respectively

$$r\cos\theta\,.\,\sigma r\,\delta\theta\,\delta r \quad\text{and}\quad r\sin\theta\,.\,\sigma r\,\delta\theta\,\delta r.$$

Thus

$$\bar{x} = \frac{\iint r\cos\theta\,.\,\sigma r\,d\theta\,dr}{\iint \sigma r\,d\theta\,dr}, \quad \bar{y} = \frac{\iint r\sin\theta\,.\,\sigma r\,d\theta\,dr}{\iint \sigma r\,d\theta\,dr},$$

the limits to be assigned so that the summations for all elements are thereby effected.

459. Ex. 1. Find the centroid of the circular lamina of Art. 457 when the surface density is μr^n.

Obviously the centroid lies on the diameter through O. Hence $\bar{y}=0$.

To find $\bar{x}$ we have to integrate $2\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r\cos\theta\,.\,\mu r^n r\,d\theta\,dr$, and then to divide by M, which has been found before (Art. 457, Ex. 1).

$$\text{This integral} = \frac{2\mu}{n+3}\int_0^{\frac{\pi}{2}}(2a\cos\theta)^{n+3}\cos\theta\,d\theta = \frac{2\mu}{n+3}(2a)^{n+3}\int_0^{\frac{\pi}{2}}\cos^{n+4}\theta\,d\theta$$

$$\left.\begin{aligned} &= \frac{2\mu}{n+3}(2a)^{n+3}\frac{n+3}{n+4}\frac{n+1}{n+2}\cdots\frac{2}{3}, \quad n \text{ odd,} \\ \text{or}\quad &= \frac{2\mu}{n+3}(2a)^{n+3}\frac{n+3}{n+4}\frac{n+1}{n+2}\cdots\frac{1}{2}\frac{\pi}{2}, \quad n \text{ even.}\end{aligned}\right\}$$

$$\left.\begin{aligned} \text{Hence}\qquad \bar{x} &= \frac{n+2}{n+3}\cdot 2a\,\frac{n+3}{n+4} = \frac{n+2}{n+4}\cdot 2a \\ \text{and}\qquad \bar{y} &= 0. \end{aligned}\right\}$$

If the centroid of the upper half only of the lamina had been required, we should have had the same value of $\bar{x}$ but for $\bar{y}$ we shall have to evaluate the additional integral

$$\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r\sin\theta\,.\,\mu r^n r\,d\theta\,dr$$

and divide by $\frac{1}{2}M$, where M is the mass found for the whole lamina.

$$\text{This integral} = \frac{\mu}{n+3}\int_0^{\frac{\pi}{2}}(2a\cos\theta)^{n+3}\sin\theta\,d\theta$$

$$= \frac{\mu}{n+3}(2a)^{n+3}\left[\frac{-\cos^{n+4}\theta}{n+4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\mu}{(n+3)(n+4)}(2a)^{n+3}.$$

$$\left.\begin{aligned} \text{Hence}\qquad \bar{y} &= \frac{n+2}{(n+3)(n+4)}\cdot 2a\,.\,\frac{n+2}{n+1}\cdot\frac{n}{n-1}\cdots\frac{3}{2}, \qquad n \text{ odd}, \\ \text{or}\qquad &\ \frac{n+2}{(n+3)(n+4)}\cdot 2a\,.\,\frac{n+2}{n+1}\cdot\frac{n}{n-1}\cdots\frac{2}{1}\cdot\frac{2}{\pi}, \qquad n \text{ even}. \end{aligned}\right\}$$

Ex. 2. Find the centroid of a lamina in the form of the cardioide

$$r = a(1+\cos\theta)$$

in the case of uniform surface density.

As the initial line is an axis of symmetry, $\bar{y}$ is evidently $=0$ (see Fig. 82).

To find the abscissa we have

$$\bar{x} = \iint r\cos\theta\,.\,r\,d\theta\,dr \Big/ \iint r\,d\theta\,dr,$$

the limits for r being

from $r=0$ to $r=a(1+\cos\theta)$,

and for θ, from $\theta=0$ to $\theta=\pi$

(and double to include the lower half).

$$2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\cos\theta\,.\,r\,d\theta\,dr = 2\int_0^{\pi}\cos\theta\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}d\theta$$

$$= \frac{2}{3}a^3\int_0^{\pi}(\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\,d\theta$$

$$= \frac{4}{3}a^3\int_0^{\frac{\pi}{2}}(3\cos^2\theta + \cos^4\theta)\,d\theta$$

$$= \frac{4}{3}a^3\left[3\frac{1}{2}\frac{\pi}{2} + \frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right]$$

$$= \frac{4}{3}a^3\frac{3\pi}{4}\cdot\frac{5}{4} = \frac{5}{4}\pi a^3.$$

The denominator $=2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)}d\theta$

$$=a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=2a^2\int_0^{\frac{\pi}{2}}(1+\cos^2\theta)d\theta$$

$$=2a^2\left[\frac{\pi}{2}+\frac{1}{2}\frac{\pi}{2}\right]=\frac{3\pi a^2}{2}.$$

Hence $$\bar{x}=\frac{5}{4}\pi a^3\Big/\frac{3\pi a^2}{2}=\frac{5a}{6}.$$

Ex. 3. Calculate the surface integral of μr^{2n} taken over *one loop* of a Bernoulli's Lemniscate.

The curve is $r^2=a^2\cos 2\theta$ (*Diff. Calc.*, Art. 458).

The surface integral is plainly

$$S\equiv 2\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}}\mu r^{2n}\,.\,r\,d\theta\,dr$$

$$=\frac{2\mu}{2n+2}\int_0^{\frac{\pi}{4}}(a^2\cos 2\theta)^{n+1}\,d\theta$$

$$=\frac{1}{2}\frac{\mu}{n+1}a^{2n+2}\int_0^{\frac{\pi}{2}}\cos^{n+1}\phi\,.\,d\phi,\quad \text{where } \phi=2\theta,$$

$$=\frac{\mu}{2}\frac{a^{2n+2}}{n+1}\cdot\frac{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+3}{2}\right)}=\frac{\mu}{4}\frac{a^{2n+2}\sqrt{\pi}}{n+1}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}=\text{etc.}\quad\text{.........(1)}$$

If the moment of inertia be required about an axis perpendicular to the plane through the pole,

$$\text{Mom. In.}=2\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}}r^2\,.\,\mu r^{2n}r\,d\theta\,dr$$

$$=\frac{2\mu}{2n+4}\int_0^{\frac{\pi}{4}}(a^2\cos 2\theta)^{n+2}\,d\theta=\frac{\mu}{4}\frac{a^{2n+4}\sqrt{\pi}}{n+2}\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}$$

$$=Ma^2\frac{n+1}{n+2}\frac{\left[\Gamma\left(\frac{n+3}{2}\right)\right]^2}{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+4}{2}\right)}=2Ma^2\frac{n+1}{(n+2)^2}\left\{\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\right\}^2,\quad\text{.........(2)}$$

where M is the mass.

If we put $n=0$ in (1), we get the mass M of the loop for uniform surface density μ, viz.

$$M=\frac{\mu}{4}a^2\sqrt{\pi}\frac{1}{\Gamma\left(\frac{3}{2}\right)}=\mu\frac{a^2}{2},$$

and $\mu=1$ gives the area, viz. $A=\dfrac{a^2}{2}$.

Putting $n=1$ in (1), we have the moment of inertia for a uniform lamina about a perpendicular through the pole to the plane (or the mass for a superficial distribution μr^2), viz.

$$\text{Mom. In.}=\frac{\mu}{4}\frac{a^4}{2}\sqrt{\pi}\frac{\Gamma(\frac{3}{2})}{\Gamma(2)}=\frac{\mu a^4\pi}{16}=\frac{M\pi a^2}{8}.$$

Similarly $n=2$ in (1) gives the moment of inertia for a superficial distribution μr^2 or the mass for a superficial distribution μr^4, etc.

EXAMPLES.

1. Find the centroid of a sector of a circle

   (α) when the surface density is uniform ;

   (β) when the surface density varies as the n^{th} power of the direct distance from the centre.

2. Find the centroid of a circular lamina whose surface density varies as the n^{th} power of the distance from a point O on the circumference.

   Find also its moment of inertia

   (1) about the tangent at O;

   (2) about the diameter through O;

   (3) about a perpendicular to the plane through O.

3. (a) Show that the moment of inertia of the triangle of uniform surface density, bounded by the y-axis and the lines

$$y=m_1x+c_1, \quad y=m_2x+c_2$$

about the y-axis, is

$$\frac{M}{6}\left(\frac{c_1-c_2}{m_1-m_2}\right)^2,$$

where M is the mass of the triangle.

(b) Find the moments of inertia of the triangle of uniform surface density, bounded by the lines

$$y=m_1x+c_1, \quad y=m_2x+c_2, \quad y=m_3x+c_3,$$

about the coordinate axes; and show that if M be the mass of the triangle, they are the same as those of equal masses $\frac{M}{3}$ placed at the mid-points of the sides.

4. Find the centre of gravity and the moments of inertia about the coordinate axes of the rectangle $x=a_1$, $x=a_2$, $y=b_1$, $y=b_2$, the surface density being $\sigma=\mu x^p y^q$.

5. If A, B be the moments of inertia of any plane area about a pair of perpendicular axes Ox, Oy in its plane, and C the moment of inertia about an axis through O at right angles to the plane, prove that

$$C=A+B$$

for any law of surface density.

6. Show that the moments of inertia of a uniform ellipse bounded by $x^2/a^2+y^2/b^2=1$ about the major and minor axes are respectively $\frac{Mb^2}{4}$ and $\frac{Ma^2}{4}$, and about a line through the centre and perpendicular to its plane, $M\frac{a^2+b^2}{4}$, M being the mass of the ellipse.

7. Find the area remote from the pole between the circles

$$r=a, \quad r=2a\cos\theta;$$

and assuming a surface density varying inversely as the distance from the pole, find

(1) the centroid;

(2) the moment of inertia about a line through the pole perpendicular to the plane.

8. Find for the area included between the curves

$$y^2=4ax, \quad x^2=4ay,$$

(i) the moment of inertia about the x-axis;

(ii) the moment of inertia about an axis through the origin and at right angles to the plane of the area.

9. Find the coordinates of the centroid of the area bounded by the catenary $y=c\cosh\frac{x}{c}$, an ordinate, and the coordinate axes.

10. If the density at any point of a circular disc whose radius is a vary directly as the distance from the centre and a circle described on a radius as diameter be cut out, prove that the centroid of the remainder will be at a distance $\frac{6a}{5(3\pi-2)}$ from the centre. [MATH. TRIP., 1875.]

460. Trilinears and Areals.

These coordinates are not well adapted for metrical purposes. Their special rôle is the discussion of descriptive properties of curves.

With the usual notation of the trilinear system [Smith's *Conics*, Chapter XIII.], we have

$$a\alpha+b\beta+c\gamma=2\Delta,$$

as an identical relation between the three coordinates α, β, γ of a point, and in the areal system this is replaced by

$$x+y+z=1.$$

The transformation formulae from the one system to the other are

$$x=\frac{a\alpha}{2\Delta}, \quad y=\frac{b\beta}{2\Delta}, \quad z=\frac{c\gamma}{2\Delta}.$$

Variations $d\alpha$, $d\beta$, $d\gamma$ or dx, dy, dz of the coordinates are therefore connected by the equations

$$\left.\begin{array}{r} a\,d\alpha+b\,d\beta+c\,d\gamma=0, \\ dx+\ \ dy+\ \ dz=0, \end{array}\right\} \text{respectively.}$$

The evaluation of an area for such coordinates is best done by throwing back the homogeneous equation given into a Cartesian form, taking two sides of the triangle of reference as

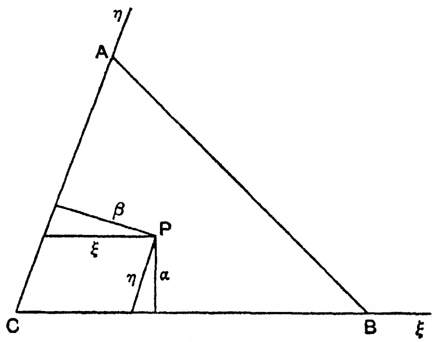


Fig. 87.

coordinate axes. Thus taking CB and CA, sides of the reference triangle, as axes of ξ and η, if ξ, η be the Cartesian coordinates of the point α, β, γ, we obviously have

$$\alpha=\eta\sin C, \quad \beta=\xi\sin C$$

and
$$\gamma=(2\Delta-a\eta\sin C-b\xi\sin C)/c$$
$$=\frac{2\Delta}{c}\left(1-\frac{\xi}{a}-\frac{\eta}{b}\right),$$

and then the evaluation of the area will be obtained by

$$A=-\sin C\int\eta\,d\xi \quad\text{or}\quad \sin C\int\xi\,d\eta \quad\text{or}\quad \sin C\iint d\xi\,d\eta$$

or any of the methods customary for Cartesians.

461. Formulae can, however, be exhibited expressing the area directly in terms of areal or trilinear coordinates for use if necessary.

In the Case of Areals, since x, y, z, the areal coordinates of a point, are linear functions of ξ, η, the Cartesian coordinates

with reference to any chosen rectangular axes and $x+y+z=1$, we have

$$\iint d\xi\, d\eta = \lambda \iint dx\, dy \quad \text{or} \quad \mu \iint dy\, dz \quad \text{or} \quad \nu \iint dz\, dx,$$

where λ, μ, ν are determinate constants depending upon the triangle of reference alone. To determine λ we shall apply the first of these formulae to the triangle of reference itself.

If Δ be the area of the triangle of reference,

$$\iint d\xi\, d\eta = \Delta,$$

where the integration is conducted over the triangle.

Now let us evaluate $\iint dx\, dy$ for the triangle.

The limits of y, keeping x constant, are from $y=0$ to $z=0$, *i.e.* to $y=1-x$, and for x from $x=0$ to $x=1$.

Thus $\iint dx\, dy$ for the triangle $= \int_0^1 \int_0^{1-x} dx\, dy$

$$= \int_0^1 (1-x)\, dx = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2};$$

$$\therefore \lambda = 2\Delta.$$

Hence if $f(x, y, z) = 0$ be the equation of a closed curve in areals, its area is

$$2\Delta \iint dx\, dy,$$

the limits of integration being obtained from

$$f(x, y, 1-x-y) = 0.$$

The corresponding **result for trilinears** will be

$$\frac{1}{\sin C} \iint d\alpha\, d\beta,$$

where the limits are to be found from

$$f\left(\alpha, \beta, \frac{2\Delta - a\alpha - b\beta}{c}\right) = 0,$$

$f(\alpha, \beta, \gamma) = 0$ being the curve to be considered.

462. **Illustrative Cases.**

Ex. 1. As a test let us apply this method to find the area of the circum-circle of the triangle of reference, viz. $a^2yz + b^2zx + c^2xy = 0$ (in areals).

The result, from elementary considerations, should be

$$\pi R^2 = \pi\left(\frac{abc}{4\Delta}\right)^2, \quad R \text{ being the radius of the circle.}$$

Substituting $1-x-y$ for z, we have

$$(a^2y+b^2x)(1-x-y)+c^2xy=0,$$

$$a^2y+b^2x-a^2y^2-b^2x^2-2ab\cos C\,xy=0,$$

$$a^2y^2+(2ab\cos C\,x-a^2)y=b^2x-b^2x^2,$$

$$y^2+\left(2\frac{b}{a}\cos C\,x-1\right)y+\frac{1}{4}\left(2\frac{b}{a}\cos C\,x-1\right)^2$$

$$=\frac{1}{4}-\frac{b}{a}\cos C\,x+\frac{b^2}{a^2}\cos^2 C\,x^2+\frac{b^2}{a^2}x-\frac{b^2}{a^2}x^2$$

$$=\frac{1}{4}+\frac{bc}{a^2}\cos A\,x-\frac{b^2}{a^2}\sin^2 C\,x^2$$

$$=\frac{1}{4}+\frac{1}{4}\frac{c^2}{a^2}\frac{\cos^2 A}{\sin^2 C}-\frac{b^2}{a^2}\sin^2 C\left(x-\frac{1}{2}\frac{c}{b}\frac{\cos A}{\sin^2 C}\right)^2$$

$$=\frac{1}{4}\operatorname{cosec}^2 A\left[1-4\sin^2 B\sin^2 C\left(x-\frac{1}{2}\frac{\cos A}{\sin B\sin C}\right)^2\right]$$

$$=p^2-q^2(x-r)^2, \text{ say.}$$

The limits for y are therefore

$$-\frac{1}{2}\left(\frac{2b\cos C}{a}x-1\right)\pm\sqrt{p^2-q^2(x-r)^2},$$

and for x,

$$r\pm\frac{p}{q}.$$

$$\text{The area}=2\Delta\iint dx\,dy=4\Delta\int\sqrt{p^2-q^2(x-r)^2}\,dx$$

$$=4\Delta\,.\,\frac{1}{2q}\left[q(x-r)\sqrt{p^2-q^2(x-r)^2}+p^2\sin^{-1}\frac{q(x-r)}{p}\right]_{r-\frac{p}{q}}^{r+\frac{p}{q}}$$

$$=\frac{2\Delta}{q}p^2\left[\sin^{-1}1-\sin^{-1}(-1)\right]=2\pi\Delta\frac{p^2}{q}$$

$$=2\pi\Delta\,.\,\frac{1}{4}\frac{\operatorname{cosec}^2 A}{\dfrac{\sin B}{\sin A}\sin C}=\frac{\pi\Delta}{2}\frac{1}{\sin A\sin B\sin C}$$

$$=\frac{\pi\Delta}{2}\cdot\frac{a^2b^2c^2}{(2\Delta)^3}=\pi\left(\frac{abc}{4\Delta}\right)^2,$$

the result to be expected.

Ex. 2. More generally consider the areal equation of an ellipse

$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0.$$

To obtain the integration limits put $z=1-x-y$.

We obtain $$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

where

$$a=w+u-2v', \qquad g=-w+v',$$
$$h=w+w'-u'-v', \qquad f=-w+u',$$
$$b=w+v-2u', \qquad c=w.$$

Solving for y,

$$by = -(hx+f) \pm \sqrt{-Cx^2+2Gx-A} = -(hx+f) \pm \sqrt{\frac{G^2-AC}{C} - C\left(x-\frac{G}{C}\right)^2},$$

where $$A = \frac{\partial H}{\partial a}, \quad 2G = \frac{\partial H}{\partial g}, \quad C = \frac{\partial H}{\partial c},$$

and H = the Hessian, viz. $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = \begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}$.

The limits for y are

$$\left\{-(hx+f) \pm \sqrt{\frac{G^2-AC}{C} - C\left(x-\frac{G}{C}\right)^2}\right\}\Big/ b,$$

and for x, $$\frac{G}{C} \pm \frac{\sqrt{G^2-AC}}{C}.$$

Writing the radical

$$\sqrt{\frac{G^2-AC}{C} - C\left(x-\frac{G}{C}\right)^2} \text{ as } \sqrt{p^2-q^2(x-r)^2},$$

$$\text{area} = 2\Delta \iint dx\, dy = \pm \frac{4\Delta}{b} \int \sqrt{p^2-q^2(x-r)^2}\, dx$$

$$= \pm \frac{2\Delta}{bq}\left[q(x-r)\sqrt{p^2-q^2(x-r)^2} + p^2 \sin^{-1}\frac{q(x-r)}{p}\right]_{r-\frac{p}{q}}^{r+\frac{p}{q}}$$

$$= \pm \frac{2\Delta}{bq} p^2 [\sin^{-1} 1 - \sin^{-1}(-1)] = \pm 2\pi\Delta \frac{p^2}{bq}.$$

Now $q = \sqrt{C} = \sqrt{ab-h^2} = \sqrt{\sum(vw-u'^2) + 2\sum(v'w'-uu')} = \sqrt{-K}$,

where $K = \begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$, the "bordered Hessian," and $G^2 - AC = -bH$.

Hence $$\frac{p^2}{bq} = \frac{G^2-AC}{bC^{\frac{3}{2}}} = \frac{-H}{(-K)^{\frac{3}{2}}}.$$

Therefore the area sought is $\pm 2\pi\Delta \dfrac{H}{(-K)^{\frac{3}{2}}}$, the positive value to be taken, where

$\Delta \equiv$ area of triangle of reference,

$H \equiv$ the Hessian, viz. $\begin{vmatrix} u, & w', & v' \\ w', & v, & u' \\ v', & u', & w \end{vmatrix}$,

$K \equiv$ the bordered Hessian, viz. $\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$.

463. Corresponding Points and Areas.

Let $f(x, y)$ be any closed curve.

Its area (A_1) is expressed by taking the line-integral $-\int y\,dx$ or the line-integral $\int x\,dy$ round the complete contour.

If the coordinates of the current point x, y be connected with those of a second point (ξ, η) by the relations

$$x = m\xi, \quad y = n\eta,$$

this second point will trace out the curve

$$f(m\xi, n\eta) = 0,$$

whose area (A_2) is expressed by the line-integral $-\int \eta\,d\xi$ or the line-integral $\int \xi\,d\eta$ taken round the contour.

And we have

$$A_1 = -\int y\,dx = -\int n\eta\,m\,d\xi = -mn\int \eta\,d\xi = mnA_2,$$

or $$A_1 = \int x\,dy = \int m\xi\,n\,d\eta = mn\int \xi\,d\eta = mnA_2,$$

or, if we use surface integrals,

$$A_1 = \iint dx\,dy = \iint mn\;d\xi\,d\eta = mn\iint d\xi\,d\eta \equiv mnA_2,$$

whence it appears that the area of any closed curve $f(x, y) = 0$ is mn times that of the closed curve $f(mx, ny) = 0$.

464. Ex. 1. Thus, in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ put } \frac{x}{a} = \frac{\xi}{r}, \; \frac{y}{b} = \frac{\eta}{r}.$$

The corresponding point traces out the circle $\xi^2 + \eta^2 = r^2$, and area of the ellipse $= \frac{ab}{r^2} \times$ area of circle $= \frac{ab}{r^2}\pi r^2 = \pi ab$.

Ex. 2. Find the area of the curve $(m^2x^2 + n^2y^2)^2 = a^2x^2 + b^2y^2$.

Put $mx = \xi$, $ny = \eta$. Then the corresponding curve is

$$(\xi^2 + \eta^2)^2 = \frac{a^2}{m^2}\xi^2 + \frac{b^2}{n^2}\eta^2,$$

or in polars $$r^2 = \frac{a^2}{m^2}\cos^2\theta + \frac{b^2}{n^2}\sin^2\theta,$$

the central pedal of an ellipse, symmetrical about both coordinate axes.

Hence the area of the given curve

$$=\frac{1}{mn}\times \text{area of derived curve}$$

$$=\frac{1}{mn}\,2\int_0^{\frac{\pi}{2}}\left(\frac{a^2}{m^2}\cos^2\theta+\frac{b^2}{n^2}\sin^2\theta\right)d\theta$$

$$=\frac{\pi}{2mn}\left(\frac{a^2}{m^2}+\frac{b^2}{n^2}\right).$$

It will be noted that it is often possible by a selection of such a change of the variables to arrange that the derived curve is of a much more convenient form, and its area readily obtainable when expressed in polars.

Ex. 3. Find the area of the curve

$$\left(c^2+\frac{a^2y^2}{b^2}+\frac{b^2x^2}{a^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\left(\frac{a^2y^2}{b^2}+\frac{b^2x^2}{a^2}\right),$$

where c is less than both a and b.

Let $\dfrac{ay}{b}=\eta$, $\dfrac{bx}{a}=\xi$.

Then the derived curve is

$$(c^2+\xi^2+\eta^2)\left(\frac{\xi^2}{b^2}+\frac{\eta^2}{a^2}\right)=\xi^2+\eta^2,$$

or in polars, $$(c^2+r^2)\left(\frac{\cos^2\theta}{b^2}+\frac{\sin^2\theta}{a^2}\right)=1,$$

i.e. $$r^2=\frac{a^2b^2}{a^2\cos^2\theta+b^2\sin^2\theta}-c^2.$$

There is obviously symmetry about both axes, and though there is a conjugate point in the original curve at the origin, the curve does not pass through the origin, and the derived curve is one which could be obtained from an ellipse by writing r^2+c^2 for r^2.

Let $r^2+c^2=r'^2$. Then $r'^2=\dfrac{a^2b^2}{a^2\cos^2\theta+b^2\sin^2\theta}$, and the area of this ellipse is πab. The area of our first derived curve is therefore

$$4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}r^2\,d\theta=4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}(r'^2-c^2)\,d\theta=4\cdot\frac{1}{2}\left(\frac{\pi ab}{2}-\frac{\pi}{2}c^2\right)=\pi(ab-c^2);$$

$\therefore$ the area of the original curve is

$$\frac{b}{a}\cdot\frac{a}{b}\pi(ab-c^2),$$

which also $=\pi(ab-c^2)$.

465. In connection with the last example, it is worth noting that in any curve $r=f(\theta)$ if the area if any portion from $\theta=\alpha$ to $\theta=\beta$ be found as

$$\frac{1}{2}\int_\alpha^\beta[f(\theta)]^2\,d\theta \quad\text{and}\quad =A,$$

then the sectorial area of the curve $r^2=[f(\theta)]^2 \pm c^2$ between the same limits is

$$\frac{1}{2}\int_\alpha^\beta \{[f(\theta)]^2 \pm c^2\}\, d\theta = A \pm \frac{c^2}{2}(\beta-\alpha);$$

and if both be closed and the origin within both, then the area of the new curve differs from the area of the original curve by the area of a circle of radius c, supposing c to be such that r is real throughout the range of integration in each case.

EXAMPLES.

1. Find the whole area of a loop of each of the curves

$$\text{(i)} \qquad x(x^2+y^2)=a(x^2-y^2).$$

$$\text{(ii)} \quad (m^2x^2+n^2y^2)^2=a^2x^2-b^2y^2.$$

[ST. JOHN'S, 1887.]

2. Trace the shape of the following curves, and find their areas:

$$\text{(i)} \quad (x^2+\ \ y^2)^3=axy^4.$$

$$\text{(ii)} \quad (x^2+2y^2)^3=axy^4.$$

[BARNES SCHOLARSHIPS, 1887.]

3. Prove that the area of

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 \quad \text{is} \quad \frac{\pi c^2}{2ab}(a^2+b^2).$$

4. Prove that the area in the positive quadrant of the curve

$$(a^2x^2+b^2y^2)^{\frac{5}{2}}=mx^3+ny^3 \quad \text{is} \quad \frac{1}{3ab}\left(\frac{m}{a^3}+\frac{n}{b^3}\right).$$

[α, 1890.]

5. Prove that the area of the curve

$$(a^2x^2+b^2y^2)^2=c^6(x^2-y^2) \quad \text{is} \quad \frac{c^6}{a^3b^3}\left\{ab+(b^2-a^2)\tan^{-1}\frac{b}{a}\right\}.$$

[ST. JOHN'S, 1883.]

6. Show that the area of the loop of the curve

$$\frac{x^5}{a^5}+\frac{y^5}{b^5}=5\frac{x^2y^2}{a^2b^2} \quad \text{is} \quad \frac{5}{2}ab.$$

7. Find the area of the curve

$$\frac{l}{\sqrt{r^2-c^2}}=1+e\cos\theta \quad (e<1).$$

8. Show that the area bounded by

$$(x^2+y^2-c^2)(x^2+y^2)=4a^2x^2 \quad \text{is} \quad (2a^2+c^2)\pi.$$

9. Find the area included within the curve whose equation is

$$\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1.$$

[COLLEGES, 1885.]

10. Trace the curve

$$\left(\frac{x}{a}+\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{x}{a}-\frac{y}{b}\right)^{\frac{2}{3}}=2,$$

and show that its area is half as great again as that of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

[MATH. TRIPOS, 1884.]

11. Prove that the area of the curve

$$(x^2+y^2)^5=a^2y(x^7+y^7) \quad \text{is} \quad \tfrac{105}{512}\pi a^2+\tfrac{125}{384}a^2.$$

[ST. JOHN'S, 1889.]

12. Prove that the area of the curve

$$(a^2x^2+b^2y^2)^5=8a^4b^4xy(a^4x^6+b^4y^6) \quad \text{is} \quad a^2+b^2.$$

[ST. JOHN'S, 1889.]

13. Show that the area in the first quadrant of the curve

$$c^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^5=\left(\frac{x^3}{a}+\frac{y^3}{b}\right)^2 \quad \text{is} \quad \frac{ab(a^2+b^2)}{3c^2}.$$

14. Trace the curve $4(x^2+2y^2-2ay)^2=x^2(x^2+2y^2)$, proving that the area of a loop is $4\pi(2-\sqrt{3})a^2/\sqrt{3}$, and that the area included between the loops is

$$8a^2(2\pi-3\sqrt{3})/3\sqrt{3}.$$

[TRINITY, 1896.]

15. Find the whole area of the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{2xy}{ab},$$

[OXFORD I. P., 1890.]

and of a loop of the curve

$$\frac{x^4}{a^4}+\frac{y^4}{b^4}=\frac{2xy}{ab}.$$

[OXFORD II. P., 1900.]

16. Show that the area of either oval of

$$x^2\{x^2/a^2+y^2/b^2-1\}+c^2=0 \quad \text{is} \quad \tfrac{1}{2}\pi b(a-2c).$$

[ST. JOHN'S, 1890.]

17. If $f(x, y)=0$ be a closed curve, show that its area is mn times the area of the closed curve $f(mx, ny)=0$. Trace the curve $(4x^2+9y^2)^4=axy^6$, and find its area. [OXFORD II. P., 1890.]

18. Trace the curve $\dfrac{x^3}{a^3}+\dfrac{y^3}{b^3}=\dfrac{3xy}{ab}$, and show that the area of its loop is $\dfrac{3}{2}ab$.

19. A curve is defined by the equations

$$x = 6a\sin^2\phi, \quad y = 6a\sin^2\phi\tan\phi,$$

where ϕ is a variable parameter. Show that the centroid of the portion enclosed between the infinite branches and the asymptote is situated on the x-axis at a distance $5a$ from the origin.

[OXFORD II. P., 1889.]

20. (i) In an involute of a circle, show that the area swept out by the radius vector drawn from the centre of the circle to a point on the curve varies as the cube of the central perpendicular upon the tangent, the initial line being the radius to the point where the involute meets the circle.

(ii) In the Conchoid of Nicomedes $r = a\sec\theta - b$ in the case when $a < b$, show that the area of the loop is

$$a^2(\alpha\sec^2\alpha - 2\sec\alpha\cosh^{-1}\sec\alpha + \tan\alpha),$$

and that the distance of the centroid of the loop from the node is

$$\frac{2}{3}a\,\frac{3\alpha\sec\alpha - 3\cosh^{-1}\sec\alpha - \sin\alpha\tan^2\alpha}{\alpha\sec\alpha - 2\cosh^{-1}\sec\alpha + \sin\alpha},$$

where $$\alpha = \cos^{-1}a/b.$$

21. Prove that the area contained by the curve

$$x^4 + 2x^2y^2 + 4ax^2y + 2a^2(y^2 - x^2 - 2ay) + a^4 = 0 \quad \text{is} \quad \pi a^2(4 - 5/\sqrt{2}).$$

Find also the distance from the axis of y of the centre of gravity of that portion of the area which lies in the first quadrant.

[COLLEGES β, 1890.]

22. Show that the area included between the curve

$$s = a\tan\psi, \quad \text{its tangent at } \psi = 0 \quad \text{and its tangent at } \psi = \phi$$

is $$\tfrac{1}{2}a^2\tan\phi + a^2\tan\tfrac{1}{2}\phi - a^2\log(\sec\phi + \tan\phi).$$

[TRINITY, 1892.]

23. Show that an expression for the element of area in trilinear coordinates is

$$\operatorname{cosec} C\,d\alpha\,d\beta.$$

Show that the area of the conic whose trilinear equation is

$$a^{-1}\beta\gamma + b^{-1}\gamma\alpha + c^{-1}\alpha\beta = 0$$

is to that of the triangle of reference as

$$4\pi : 3\sqrt{3}.$$

[OXFORD II. P., 1890.]

24. Show that the coordinates of the centroid of the area bounded by half the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, the line of cusps and the y-axis are given by

$$\frac{9\pi\bar{x}}{(3\pi - 4)(3\pi + 4)} = \frac{3\bar{y}}{7} = \frac{a}{2}.$$

[WALLIS.]

25. OB and OC are any two semi-diameters of an ellipse conjugate to each other; find the locus of the intersection of the normals at B and C, and show that the area of the curve is

$$\frac{\pi(a^2-b^2)^2}{4ab}$$ [R. P.]

26. Tangents to a system of similar and similarly situated concentric ellipses are drawn such that the distance of each from the centre is the same. Find the area of the curve formed by the points of contact. [TRINITY, 1885.]

27. Show that the moment of inertia of the portion of a uniform parabolic lamina cut off by the latus rectum about the tangent at an extremity of the latus rectum, is equal to $\frac{12Ma^2}{7}$, $4a$ being the latus rectum and M the mass of the lamina. [OXF. I. P., 1914.]

28. Prove by integration that the moment of inertia of a uniform triangular lamina ABC of mass M about a perpendicular axis at A is $\frac{1}{12}M(3b^2+3c^2-a^2)$. [Ox. I. P., 1915.]

CHAPTER XV.

QUADRATURE (IV).

MISCELLANEOUS THEOREMS, CONNEXION OF A LINE-INTEGRAL AND A SURFACE-INTEGRAL, MECHANICAL INTEGRATION, ETC.

466. A Theorem due to Stokes.

Let u and v be two functions of x and y, finite, single-valued and continuous at every point within and along the boundary of a given region bounded by any given contour line in the plane of x, y having no multiple points, and let the differential coefficients $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$ be also functions which are finite, single-valued, and continuous at all points of the region; then the line-integral

$$\int\left(u\frac{dx}{ds}+v\frac{dy}{ds}\right)ds$$

taken round the perimeter of the contour is equal to the surface-integral

$$\iint\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy$$

taken over the region bounded by the contour. We shall first consider u and v to be *real* functions of x and y.

Let the region referred to be indicated, as shown in the accompanying figure, with an inner boundary and an outer boundary, the inner boundary enclosing a region *within* which the integration is *not* to be performed.

Divide the whole contour into two systems of strips of infinitesimal breadth parallel to the coordinate axes. Two typical strips are shown in the figure, the one parallel to the x-axis being bounded by lines with ordinates y and $y+\delta y$,

and that parallel to the y-axis bounded by lines with abscissae x and $x+\delta x$. The first intercepts elementary arcs

$P_1Q_1=\delta s_1$, $P_2Q_2=\delta s_2$, $P_3Q_3=\delta s_3$, etc., an even number,

and the second intercepts

$P_1'Q_1'=\delta s_1'$, $P_2'Q_2'=\delta s_2'$, $P_3'Q_3'=\delta s_3'$, etc., an even number.

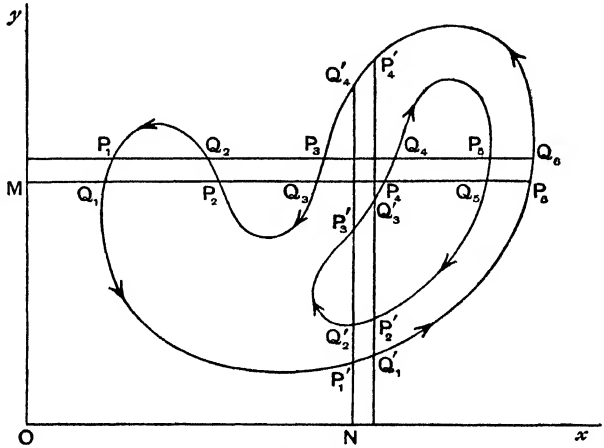


Fig. 88.

The direction of integration is indicated in the figure; the region to be integrated over being on the left hand as a person travels along either boundary, following the direction of increase of s. The signs of δy at the several points P_1, P_2, P_3, P_4, ... are respectively $-\delta y$, $+\delta y$, $-\delta y$, $+\delta y$, ..., and the signs of δx at the points P_1', P_2', P_3', P_4', ... are respectively $+\delta x$, $-\delta x$, $+\delta x$, $-\delta x$, etc.

Let u_r, v_r be the respective values of u, v at P_r, and u_r', v_r' those at P_r'.

And let the abscissae and ordinates of the points P_r, Q_s, P_r', Q_s' be x, y with the corresponding accents and suffixes.

If we integrate $\dfrac{\partial v}{\partial x}\delta y$ with regard to x along the strip $P_1Q_2P_3Q_4$, ..., we have $[v\,\delta y]$, taken between proper limits, viz.

$$\begin{aligned}(v_2\,\delta y_2-v_1\,\delta y_1)+(v_4\,\delta y_4-v_3\,\delta y_3)+\dots+(v_{2n}\,\delta y_{2n}-v_{2n-1}\,\delta y_{2n-1})\\ =v_1\,\delta y+v_2\,\delta y+v_3\,\delta y+\dots+v_{2n}\,\delta y\\ =\Sigma v\,\delta y,\text{ say, for the strip.}\end{aligned}$$

If then we sum the result for the whole set of strips parallel to the x-axis by integration, we have $\int v\,dy$, where the integration is taken for the whole perimeter of the contour. Similarly for the strips parallel to the y-axis, if we integrate $\frac{\partial u}{\partial y}\delta x$ with regard to y along the strip $P_1'Q_2'P_3'Q_4'$, ..., we obtain $[u\,\delta x]$, taken between proper limits, viz.

$$\begin{aligned}(u_2'\delta x_2' - u_1'\delta x_1') &+ (u_4'\delta x_4' - u_3'\delta x_3') + \text{etc.}\\ &= -(u_1'\delta x + u_2'\delta x + u_3'\delta x \ldots)\\ &= -\Sigma u\,\delta x, \text{ say};\end{aligned}$$

and, summing for the strips, we obtain $-\int u\,dx$, where the integration is taken for the whole perimeter of the contour.

Hence $$\iint\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy = \int(u\,dx + v\,dy).$$

467. *A line-integral taken round a closed plane contour may therefore be represented by a surface-integral taken over the surface bounded by the contour, and vice versa.*

Or, we may say that if u, v be the components parallel to the axes of x and y of any vector quantity, then $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ may be regarded as another vector quantity at right angles to the plane of xy, and such that the line-integral of u, v round a contour in the plane of x, y is equal to the surface-integral of the vector quantity $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ taken over the surface. This theorem is part of a more general three-dimension theorem due to Professor Stokes.*

468. **Extension to Complex Functions.**

If the functions u and v be not entirely real, let them be separated into their real and imaginary parts, viz.

$$u = u_1 + \iota u_2, \quad v = v_1 + \iota v_2,$$

where u_1, u_2, v_1, v_2 are single-valued finite and continuous functions of x and y for all points within and upon the contour, as also their first differential coefficients.

* Smith's Prize, 1854; Maxwell, *Elect. and Mag.*, vol. i., p. 25.

Then we have

$$\iint\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)dx\,dy=\int(u_1\,dx+v_1\,dy),$$

$$\iint\left(\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}\right)dx\,dy=\int(u_2\,dx+v_2\,dy).$$

Therefore, multiplying the second line by ι and adding to the first,

$$\iint\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy=\int(u\,dx+v\,dy),$$

the integrations to be taken as before. Hence the theorem is true whether the functions u, v be real or complex.

In any case in which $\dfrac{\partial v}{\partial x}=\dfrac{\partial u}{\partial y}$ it will follow that

$$\int(u\,dx+v\,dy)=0,$$

the integration being taken round the perimeter of the contour.

The theorem has many very important applications.

469. An Interpretation.

We may interpret the theorem thus:

Let $$\sigma=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},\quad \rho=u\frac{dx}{ds}+v\frac{dy}{ds}.$$

Then $$\iint\sigma\,dx\,dy=\int\rho\,ds;$$

that is the mass of a plane lamina bounded by any closed contour for surface density $\sigma=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$ is equal to the mass of the perimeter with a line density

$$\rho=u\frac{dx}{ds}+v\frac{dx}{ds}.$$

470. Illustrations.

Ex. 1. Taking $$u=-y\quad v=x,$$

we have at once $\displaystyle\iint dx\,dy=\frac{1}{2}\int(x\,dy-y\,dx)$, which expressions have been established (Arts. 409 and 452) as measures of the area.

Ex. 2. Let $$u=e^x\sin y-ay,\quad v=e^x\cos y-a.$$

Then $$\int\left[(e^x\sin y-ay)\frac{dx}{ds}+(e^x\cos y-a)\frac{dy}{ds}\right]ds$$

taken round the perimeter of the contour

$$=\iint[e^x\cos y-(e^x\cos y-a)]\,dx\,dy=\iint a\,dx\,dy$$

$$=a\times\text{area of the figure enclosed by the contour.}$$

Ex. 3. Consider the effect of integrating

$$I=\int[(\cos x\cosh y-Ay)\,dx+(\sin x\sinh y-Bx)\,dy]$$

round any closed contour.

Here $\qquad u=\cos x\cosh y-Ay \quad$ and $\quad v=\sin x\sinh y-Bx.$

Therefore $\quad \dfrac{\partial v}{\partial x}=\cos x\sinh y-B \quad$ and $\quad \dfrac{\partial u}{\partial y}=\cos x\sinh y-A.$

Hence

$$I=\iint(A-B)\,dx\,dy=(A-B)\times\text{area enclosed by the contour.}$$

Ex. 4. If U, V be any single-valued conjugate functions of x and y *i.e.* real functions of x and y, such that $U+\iota V=f(x+\iota y)$, and if

$$u=V-Ay, \quad v=U-Bx,$$

then $\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=\dfrac{\partial U}{\partial x}-\dfrac{\partial V}{\partial y}-B+A=A-B$ [see *Diff. Cal.*, Art. 190],

and $\int[(V-Ay)\,dx+(U-Bx)\,dy]$ round a closed contour

$$=\iint(A-B)\,dx\,dy=(A-B)\times\text{area bounded by the contour.}$$

That many different forms of U and V may lead to the same result is obvious from the consideration that the mass of the area bounded by the contour for a given distribution of surface density may be equal to the mass of the perimeter for many distributions of line density.

471. Two Resulting Theorems.

If P, Q, U be any three functions of x and y, finite and continuous throughout and along the boundary of a given contour, as also their first differential coefficients, we have

$$\iint\left\{\frac{\partial}{\partial x}(PU)+\frac{\partial}{\partial y}(QU)\right\}dx\,dy$$

$$=-\int U(Q\,dx-P\,dy)=\int U\left(P\frac{dy}{ds}-Q\frac{dx}{ds}\right)ds,$$

i.e. $$\iint\left(P\frac{\partial U}{\partial x}+Q\frac{\partial U}{\partial y}\right)dx\,dy$$

$$=-\iint U\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy+\int U\left(P\frac{dy}{ds}-Q\frac{dx}{ds}\right)ds,$$

the double integrals being understood to be taken over the whole area bounded by the contour, and the single integral being taken round the perimeter in the positive direction, *i.e.* leaving the area bounded to the left in travelling in the direction in which s is measured.

472. If R, S, T, U be any four functions of x, y which, with their first and second differential coefficients, are continuous and finite throughout and along the boundary of a given contour, we have, supposing suffixes to denote partial differential coefficients,

$$\frac{\partial}{\partial x}\left\{\begin{vmatrix} R, & R_x \\ U, & U_x \end{vmatrix} + SU_y\right\} - \frac{\partial}{\partial y}\left\{\begin{vmatrix} U, & U_y \\ T, & T_y \end{vmatrix} + S_x U\right\}$$

$$= (R_x U_x + R U_{xx} - R_{xx} U - R_x U_x + S_x U_y + S U_{xy})$$

$$-(U_y T_y + U T_{yy} - U_{yy} T - U_y T_y + S_{xy} U + S_x U_y)$$

$$= (RU_{xx} + SU_{xy} + TU_{yy}) - U(R_{xx} + S_{xy} + T_{yy}).$$

Hence

$$\iint[(RU_{xx} + SU_{xy} + TU_{yy}) - U(R_{xx} + S_{xy} + T_{yy})]\,dx\,dy$$

$$= \iint\left[\frac{\partial}{\partial x}\left\{\begin{vmatrix} R, & R_x \\ U, & U_x \end{vmatrix} + SU_y\right\} - \frac{\partial}{\partial y}\left\{\begin{vmatrix} U, & U_y \\ T, & T_y \end{vmatrix} + S_x U\right\}\right]dx\,dy$$

$$= \int\left[\left\{\begin{vmatrix} U, & U_y \\ T, & T_y \end{vmatrix} + S_x U\right\}\frac{dx}{ds} + \left\{\begin{vmatrix} R, & R_x \\ U, & U_x \end{vmatrix} + SU_y\right\}\frac{dy}{ds}\right]ds,$$

the double integral being taken over the area bounded by the contour and the single integral round the perimeter.

Thus

$$\iint\left(RU_{xx} + SU_{xy} + TU_{yy}\right)dx\,dy = \iint U\left(R_{xx} + S_{xy} + T_{yy}\right)dx\,dy$$

$$+ \int\left[\left\{\begin{vmatrix} U, & U_y \\ T, & T_y \end{vmatrix} + S_x U\right\}\frac{dx}{ds} + \left\{\begin{vmatrix} R, & R_x \\ U, & U_x \end{vmatrix} + SU_y\right\}\frac{dy}{ds}\right]ds.$$

These results will be useful later (Chapter XXXIV.).

473. Motion of a Rod in a Plane.

Let O be the origin and Ox, Oy any fixed rectangular axes in the plane.

Let a rod move in any manner in the plane.

Let P_1, P_2, P_3 be points attached to it, their coordinates being

$$(x_1, y_1); \quad (x_2, y_2); \quad (x_3, y_3).$$

Let $\qquad P_2P_3 = a_1, \quad P_3P_1 = a_2, \quad P_1P_2 = a_3,$

so that $\qquad a_1 + a_2 + a_3 = 0.$

Let θ be the angle the rod makes at any instant with the x-axis.

Then $\quad x_1 = x_2 - a_3 \cos\theta, \qquad x_3 = x_2 + a_1 \cos\theta,$

$$y_1 = y_2 - a_3 \sin\theta, \qquad y_3 = y_2 + a_1 \sin\theta;$$

$$\therefore \quad dx_1 = dx_2 + a_3 \sin\theta\, d\theta, \quad dx_3 = dx_2 - a_1 \sin\theta\, d\theta,$$

$$dy_1 = dy_2 - a_3 \cos\theta\, d\theta, \quad dy_3 = dy_2 + a_1 \cos\theta\, d\theta;$$

$$\therefore \; x_1 dy_1 - y_1 dx_1 = (x_2 - a_3 \cos\theta)(dy_2 - a_3 \cos\theta\, d\theta) - (y_2 - a_3 \sin\theta)(dx_2 + a_3 \sin\theta\, d\theta)$$

$$= x_2 dy_2 - y_2 dx_2 + a_3^2 d\theta - a_3(R\cos\theta - S\sin\theta),$$

where
$$R = dy_2 + x_2 d\theta$$
$$S = dx_2 - y_2 d\theta,$$

and $\quad x_3 dy_3 - y_3 dx_3 = x_2 dy_2 - y_2 dx_2 + a_1^2 d\theta + a_1(R\cos\theta - S\sin\theta).$

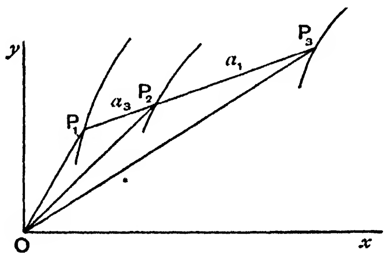


Fig. 89.

Hence, eliminating $R\cos\theta - S\sin\theta$,

$$a_1(x_1 dy_1 - y_1 dx_1) + a_3(x_3 dy_3 - y_3 dx_3) = (a_1 + a_3)(x_2 dy_2 - y_2 dx_2) + a_1 a_3 (a_1 + a_3) d\theta,$$

i.e. $\quad a_1(x_1 dy_1 - y_1 dx_1) + a_2(x_2 dy_2 - y_2 dx_2) + a_3(x_3 dy_3 - y_3 dx_3) + a_1 a_2 a_3 d\theta = 0.$

If, then, O be the origin and dA_1, dA_2, dA_3 the elementary sectorial areas described by OP_1, OP_2, OP_3, respectively,

$$a_1 dA_1 + a_2 dA_2 + a_3 dA_3 + \tfrac{1}{2} a_1 a_2 a_3 d\theta = 0.$$

Hence, if the points P_1, P_2, P_3 describe closed curves, and A_1, A_2, A_3 be the areas of these curves, and if the rod returns to its original position after making one complete revolution, then

$$a_1 A_1 + a_2 A_2 + a_3 A_3 + \pi a_1 a_2 a_3 = 0.$$

474. **Various Cases.**

If the rod returns to its original position without completing a revolution, rotating in one direction during part

of its motion and in the opposite direction during another part, then $\int d\theta=0$; and

$$a_1A_1+a_2A_2+a_3A_3=0.$$

475. If then the contours of A_1 and A_3 be such that the rod cannot complete a rotation, but must oscillate as in the case of the connecting rod in a steam engine, we have

$$A_2=\frac{a_1A_1+a_3A_3}{a_1+a_3}.$$

476. If it makes *several complete rotations forwards, say m times, and backwards n times*, whilst the several points P_1, P_2, P_3 describe closed curves once, then $\int d\theta=(m-n)2\pi$; and

$$a_1A_1+a_2A_2+a_3A_3+(m-n)\pi a_1a_2a_3=0.$$

477. *If two of the points, say P_1 and P_3, are constrained to move on fixed curves* and the rod rotates *once* round, as, for

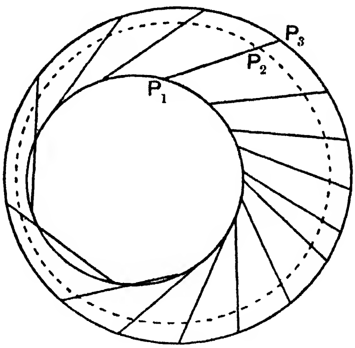


Fig. 90.

instance, if the ends were one on each of a pair of confocal ellipses, or on a pair of circles, as in Fig. 90,

$$A_2=\frac{a_1A_1+a_3A_3}{a_1+a_3}-\pi a_1a_3.$$

478. *If P_1 and P_3 move on the same curve* $A_1=A_3$, and the theorem reduces to $A_2=A_1-\pi a_1a_3$.

This last result is known as HOLDITCH'S THEOREM.

479. It should be noticed that in the above results, if any of the contours are described in a sense opposite to others, such areas are to be reckoned of opposite sign to the others.

480. Leudesdorf's Theorem.

As an application of this theorem, consider the motion of a lamina on which A, B, C, P are fixed points, the lamina being constrained to move so that A, B, C and P describe closed

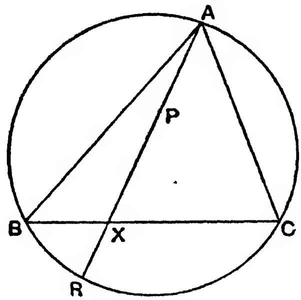


Fig. 91.

curves of areas $[A]$, $[B]$, $[C]$, $[P]$. Let x, y, z be the areal coordinates of P referred to ABC as triangle of reference. Let AP cut BC at X and the circumcircle at R. Let X describe a curve of area $[X]$.

Then $$[P]=\frac{PX[A]+AP[X]}{AX}-n\pi AP.PX,$$

$$[X]=\frac{XC[B]+BX[C]}{BC}-n\pi BX.XC.$$

Hence, eliminating the area $[X]$,

$$[P]=\frac{PX}{AX}[A]+\frac{AP}{AX}\cdot\frac{XC}{BC}[B]+\frac{AP}{AX}\cdot\frac{BX}{BC}[C]$$
$$-n\pi\cdot\frac{AP}{AX}\cdot BX.XC-n\pi AP.PX.$$

Now $$\frac{PX}{AX}=x,\quad \frac{AP}{AX}\cdot\frac{XC}{BC}=y,\quad \frac{AP}{AX}\cdot\frac{BX}{BC}=z,$$

and $$\frac{AP}{AX}\cdot BX.XC+AP.PX=AP\left(\frac{AX.XR}{AX}+PX\right)$$
$$=AP.PR$$

= rectangle of segments of any chord of the circumcircle through P;

$\therefore$ $[P]=x[A]+y[B]+z[C]-n\pi\times$ rectangle of segments of chord.

If P lies outside the circle, instead of the rectangle of segments, we may put $-(\text{tangent})^2$, and the theorem may be written

$$[P]=x[A]+y[B]+z[C]+n\pi t^2,$$

t being the tangent from P to the circumcircle.

This theorem is due to Leudesdorf.*

481. **Motion of a Plane Lamina sliding in any Manner upon a Fixed Plane. Two Theorems.**

When a plane lamina moves in any manner upon a fixed plane, so that in the end it again takes up its original position, it is clear that every point in the lamina will take up its original position, that is that the several points in their motion have travelled along paths back to the same points from which they started, and may therefore be regarded as having travelled along closed curves. This will be supposed to include paths which are retraced, which may be regarded as closed curves of infinitesimal distance between the outgoing and returning paths. For instance, a finite straight line of length $2a$ might be regarded as a closed oval—say an ellipse of semimajor axis a and infinitesimal minor axis.

Suppose two points on the lamina P_1 and P_3 to trace out known closed curves on the fixed plane. This will define the motion of the lamina, and P_1P_3 may be regarded as a straight rod whose ends are describing the given closed curves. Let P be any other carried point on the lamina and PP_2 a perpendicular from P to P_1P_3.

Let a fixed point O in the plane be taken as origin, and let

$$P_2P_3=a_1,\quad P_3P_1=a_2,\quad P_1P_2=a_3 \quad\text{and}\quad PP_2=p,$$

so that
$$a_1+a_2+a_3=0.$$

We shall continue to adopt the convenient notation $[P]$ for the area swept out by the radius vector OP to any moving point P.

Let E be the point of contact of P_1P_3 with its envelope.

Through P draw a parallel PE' to P_3P_1, and let the outward normal to the E locus meet PE' at E'. Then $EE'=p$, and the

* See Williamson, *Int. Calc.*, p. 220; Leudesdorf, *Messenger of Mathematics*, 1878.

E' locus is a parallel to the E locus, the area between them being in the case of n complete revolutions $n\pi p^2+pS$, where S is the perimeter of the envelope of the line P_1P_3 (Art. 435), *i.e.* $[E']-[E]=n\pi p^2+pS$ or πp^2+pS if there be but one revolution of the lamina.

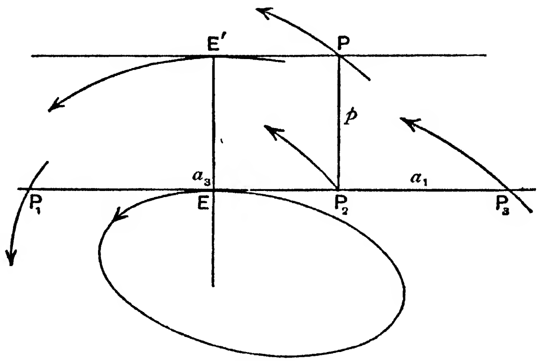


Fig. 92.

Let $E'P=EP_2=r$. Then $P_1E=a_3-r$, $EP_3=a_1+r$, and let P_1P_3 make an angle ψ with any fixed line.

Now
$$[P_1]-[E]=\frac{1}{2}\int(a_3-r)^2d\psi,$$
$$[P_2]-[E]=\frac{1}{2}\int r^2d\psi=[P]-[E'],$$
$$[P_3]-[E]=\frac{1}{2}\int(a_1+r)^2d\psi.$$

$\therefore$ multiplying by a_1, a_2, a_3 and adding,

$$a_1[P_1]+a_2[P_2]+a_3[P_3]=-\frac{1}{2}a_1a_2a_3\int d\psi \quad \text{(cf. Art. 473)};$$

and if the lamina reoccupies its original position after n positive revolutions, or if n be the excess of the number of positive revolutions over the number of negative ones, the right-hand side is

$$-\frac{1}{2}a_1a_2a_3\,2n\pi\,;$$

$$\therefore\ a_1[P_1]+a_2[P_2]+a_3[P_3]+n\pi a_1a_2a_3=0. \quad \text{.........(A)}$$

Also it has been shown that

$$[P]=[P_2]+[E']-[E]=[P_2]+n\pi p^2+pS\,;$$

$\therefore$ eliminating $[P_2]$,

$$[P]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}-n\pi a_1a_3+n\pi p^2+pS, \quad \text{......(B)}$$

which may be written as

$$a_1[P_1]+a_2[P]+a_3[P_3]+n\pi a_1a_2a_3=a_2p(n\pi p+S).$$

482. **Remarks.**

It is assumed that all the areas are described *in the same "sense."* If in any case one of them be described by its tracing point in the clockwise direction, then in this equation the corresponding quantity [ ] is *to be interpreted as the area counted negatively*; and if one of the paths cuts itself so as to form several loops, the interpretation of [ ] is the same as that in Art. 399, viz. *the difference of the odd and even portions.*

The sign of p is positive when in the same sense measured from P_2 as the outward drawn normal of the envelope of P_1P_3.

483. **Deductions.**

Corollary I. When $p=0$ the tracing point P is at P_2, and supposing there to be *one complete revolution* of the lamina we get the case already considered in Art. 477, viz.

$$[P_2]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}-\pi a_1a_3$$

which is Woolhouse's Extension of Holditch's Theorem.*

484. **Cor. II.** If in addition P_1 and P_3 are *tracing the same curve,* then $[P_1]=[P_3]$ and $[P_2]=[P_1]-\pi a_1a_3$ (Art 478),

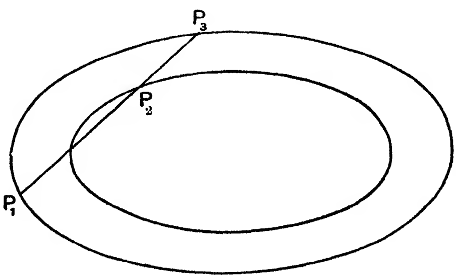


Fig. 93.

and therefore a point upon any chord of constant length inscribed in an oval curve, and which divides the chord into two portions a_1, a_3, traces out another curve whose area is less

* See Williamson's *Integral Calculus*, p. 206.

than that of the original oval by the area of an ellipse whose semiaxes are a_1, a_3. This is **Holditch's** original theorem.*

If a_1, a_3 were interchanged the result would not be affected in this case. If the tracing point be on the chord produced, one of the letters a_1, a_3 is negative and the traced oval is greater than the original oval by the same amount.

485. **Cor. III.** If the line P_1P_3 oscillates back to its original position *without performing* a *complete revolution*, or if *the number of forward revolutions is equal to the number of backward revolutions*, $n=0$, and

$$[P]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}+pS.$$

This is the case when the contours are two ovals each lying entirely outside the other and the line P_1P_3 cannot revolve completely, but oscillates. It is moreover assumed that the line a_1+a_3 is sufficiently long to allow of the full description of both ovals. If not, the particular oval which is not fully described contributes nothing.

For instance, if P_3 travel along an arc of a circle ACB from A to B via C and back along the same arc, it has described what we may regard as a contour of zero area.

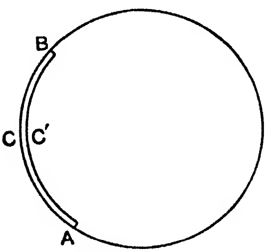


Fig. 94.

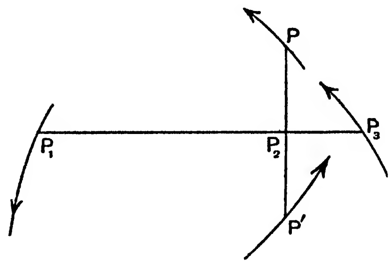


Fig. 95.

486. **Cor. IV.** If P' be the image of P in the line P_1P_3 (*i.e.* $PP_2=P_2P'$),

$$[P]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}-n\pi a_1a_3+n\pi p^2+pS,$$

$$[P']=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}-n\pi a_1a_3+n\pi p^2-pS,$$

and $[P]-[P']=2pS$, which is independent of the position of P_2.

* See Bertrand, *Calc. Intég.*, p. 365; Williamson, *Integ. Calc.*, p. 206; *Lady's and Gentleman's Diary*, 1858.

487. **Cor. V.** If P_1 and P_3 lie upon the same curve,

$$[P]=[P_1]-n\pi a_1 a_3+n\pi p^2+pS.$$

In case $a_1=0$, we have

$$[P_2]=[P_3] \quad \text{and} \quad [P]=[P_1]+n\pi p^2+pS.$$

488. **Cor. VI.** Let O, the mid-point of P_1P_3, be taken as origin, OP_3 as x-axis, and let $OP_2=x$, $P_2P=p=y$. Let the length of the rod be $2a$.

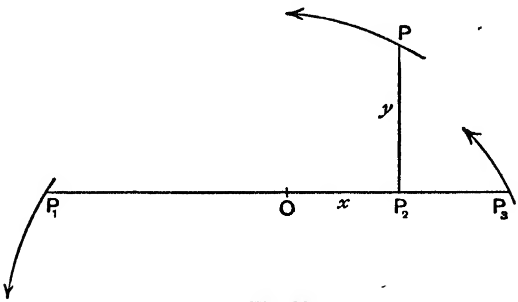


Fig. 96.

Then $$a_1=a-x, \quad a_3=a+x,$$

and $$[P]=\frac{(a-x)[P_1]+(a+x)[P_3]}{2a}-n\pi(a^2-x^2)+n\pi y^2+Sy,$$

i.e. $$x^2+y^2-\frac{[P_1]-[P_3]}{2an\pi}x+\frac{S}{n\pi}y$$
$$+\frac{1}{2n\pi}\{[P_1]+[P_3]-2[P]\}-a^2=0.$$

Hence the locus of point P on the lamina for which the contours $[P]$ are all equal is a circle whose centre is at

$$\frac{1}{4}\frac{[P_1]-[P_3]}{an\pi}, \quad -\frac{S}{2n\pi}.$$

These coordinates are independent of $[P]$. Hence, for specific values of $[P]$, the loci of the P-points are concentric circles on the lamina.

This theorem is due to Mr. A. B. Kempe.*

489. We note that if $[P_1]$ and $[P_3]$ be the same contour, the centre of this circle lies on the perpendicular bisector of the line P_1P_3.

* *Messenger of Mathematics*, 1878, cited by Williamson, *Integ. Calc.*, p. 210, where it is deduced from Holditch's form of the theorem geometrically.

490. If the closed "contours" are merely portions of two straight lines $[P_1]=[P_3]=0$, and taking $n=1$,

$$[P]=-\pi a_1 a_3+\pi p^2+pS,$$

or when $p=0$ also, $\qquad [P]=-\pi a_1 a_3,$

which is the case of a rod of given length sliding with its ends on the coordinate axes, which are drawn in Fig. 97 as long closed ovals to indicate the direction of rotation.

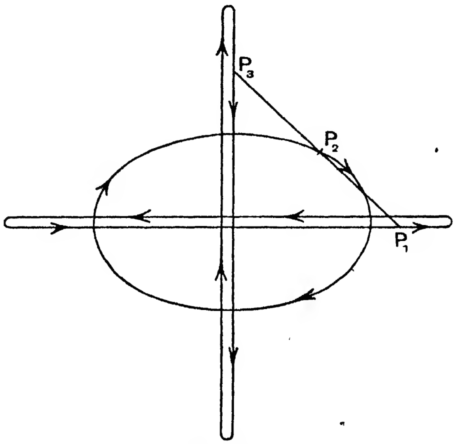


Fig. 97.

Note that in the case shown in Fig. 97 the elliptic area is traced clockwise, the ovals, which are in the limit the axes, are traced one counter-clockwise, one clockwise, and that the areas of the two ovals traced by P_1 and P_3 are both ultimately zero.

It is a well-known theorem that in this case the locus of P_2 is an ellipse of which the product of the semiaxes is the product of the segments of the moving line, whether the axes be rectangular or oblique.

491. **Cor. VII.** If P lie anywhere on the circle on P_1P_3 as diameter, we have $p^2=a_1a_3$, and the theorem reduces to

$$[P]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}+pS,$$

or if $[P_1]$ and $[P_3]$ be the same contour,

$$[P]=[P_1]+pS.$$

492. A General Theorem on the Motion of the Centroid of a System of Moving Particles, connected or otherwise.

If

$$\left.\begin{array}{lllll} m_1, & m_2, & m_3, & \dots & m_n, \\ x_1, & x_2, & x_3, & \dots & x_n, \\ y_1, & y_2, & y_3, & \dots & y_n, \\ \dot{x}_1, & \dot{x}_2, & \dot{x}_3, & \dots & \dot{x}_n, \\ \dot{y}_1, & \dot{y}_2, & \dot{y}_3, & \dots & \dot{y}_n, \end{array}\right\} \text{be five groups of } n \text{ quantities each}$$

it may readily be proved by induction that

$$\Sigma mx\,\Sigma m\dot{y} = \Sigma m\,\Sigma mx\dot{y} - \Sigma m_r m_s(x_r - x_s)(\dot{y}_r - \dot{y}_s)$$

and $$\Sigma my\,\Sigma m\dot{x} = \Sigma m\,\Sigma my\dot{x} - \Sigma m_r m_s(y_r - y_s)(\dot{x}_r - \dot{x}_s),$$

and therefore that

$$\Sigma mx\,\Sigma m\dot{y} - \Sigma my\,\Sigma m\dot{x} = \Sigma m\,\Sigma m(x\dot{y} - y\dot{x})$$
$$- \Sigma m_r m_s[(x_r - x_s)(\dot{y}_r - \dot{y}_s) - (y_r - y_s)(\dot{x}_r - \dot{x}_s)].$$

Let there be n particles of masses in the ratios

$$m_1 : m_2 : m_3 : \dots : m_n$$

and (x_1, y_1), (x_2, y_2), etc., their coordinates; and let $\dot{x}$, $\dot{y}$ be the differentials of x and y, viz. dx, dy.

The centroid of the system is given by

$$\Sigma m\,.\,\bar{x} = \Sigma mx, \quad \Sigma m\,.\,\bar{y} = \Sigma my;$$

whence $$\Sigma m\,.\,d\bar{x} = \Sigma m\,dx,$$
$$\Sigma m\,.\,d\bar{y} = \Sigma m\,dy.$$

Let each particle describe continuously a closed contour in the plane, m_1 describing a contour of area A_1, m_2 describing a contour of area A_2, and so on, and let $\bar{x}$, $\bar{y}$ in consequence describe a closed contour of area $\bar{A}$. Also let the area of the contour which m_2 describes relatively to m_1 be called S_{12}, and so on for other pairs. Then the above equation may be written

$$[\Sigma m]^2[\bar{x}\,d\bar{y} - \bar{y}\,d\bar{x}] = \Sigma m\,\Sigma m(x\,dy - y\,dx)$$
$$- \Sigma m_r m_s[(x_r - x_s)(dy_r - dy_s) - (y_r - y_s)(dx_r - dx_s)],$$

and therefore integrating round the contours

$$[\Sigma m]^2\bar{A} = \Sigma m\,\Sigma mA - \Sigma m_r m_s S_{rs},$$

an equation which expresses the area of the contour described by the centroid of the system in terms of the areas of the n

contours described by the several particles and of the $\frac{n(n-1)}{2}$ relative contours.

It will be noticed that the particles are in no wise rigidly connected, but are capable of independent motion; also that the result obtained is necessarily homogeneous as regards the masses.

493. If the revolutions of any particles of the system be *not complete*, the various integrals

$$\frac{1}{2}\int(\bar{x}\,d\bar{y}-\bar{y}\,d\bar{x}),\quad \frac{1}{2}\int(x\,dy-y\,dx),$$

$$\frac{1}{2}\int[(x_r-x_s)(dy_r-dy_s)-(y_r-y_s)(dx_r-dx_s)],$$

refer to the sectorial portions of the several contours which have been actually described during the several displacements of the particles, and represent *sectorial areas swept out by the several radii vectores from the origin to the centroid*, or from the origin to x, y in the first two cases, or the relative area by a radius vector from x_r, y_r to x_s, y_s in the third class of integral.

494. *When the several particles are rigidly connected*, the several *relative contours are circles*, with radii the distances between the several pairs, and traced as many times over as the whole system revolves before re-attaining its original position; and in case of no rigid connection, if one or more of the mutual distances returns to its original position without making a complete relative revolution, in such case the corresponding relative area S vanishes.

495. In the case where there are *two particles only*, we have

$$\bar{A}=\frac{m_1A_1+m_2A_2}{m_1+m_2}-\frac{m_1m_2}{(m_1+m_2)^2}S_{12},$$

a result established by Mr. Elliott, and reproduced in Dr. Williamson's *Integral Calculus*, p. 209, with Mr. Elliott's Enunciation of this Theorem.

496. If in this case there be a *rigid connection between the points* A_1 and A_2, say a connecting rod, we may take a_1, a_2 as the distances of A_2, A_1 from the centroid, and $\frac{a_1}{m_1}=\frac{a_2}{m_2}$.

Also the relative contour has area $\pi(a_1+a_2)^2$.

Hence $$\bar{A}=\frac{m_1A_1+m_2A_2}{m_1+m_2}-\frac{m_1m_2}{(m_1+m_2)^2}S_{12}$$

becomes $$\bar{A}=\frac{a_1A_1+a_2A_2}{a_1+a_2}-\frac{a_1a_2}{(a_1+a_2)^2}\pi(a_1+a_2)^2$$
$$=\frac{a_1A_1+a_2A_2}{a_1+a_2}-\pi a_1a_2.$$

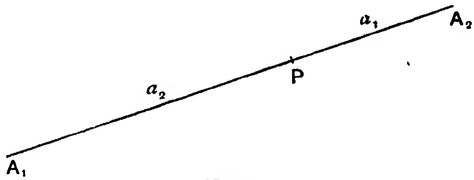


Fig. 98.

Holditch's theorem is therefore deduced as a particular case of the two particle motion, there being a rigid connection.

497. If there be *three particles* the theorem takes the form
$$\bar{A}=\frac{m_1A_1+m_2A_2+m_3A_3}{m_1+m_2+m_3}-\frac{m_2m_3S_{23}+m_3m_1S_{31}+m_1m_2S_{12}}{(m_1+m_2+m_3)^2}.$$

498. Let us apply this result to find the area described by *any* point P *attached to a triangle* ABC which moves in its own plane and after one revolution re-occupies its original position. If x, y, z, be the areal coordinates of P with reference to the triangle ABC, P is the centroid of masses proportional to m_1, m_2, m_3, at A, B, C respectively, where $\frac{x}{m_1}=\frac{y}{m_2}=\frac{z}{m_3}$, and the several "relative areas" are πa^2, πb^2, πc^2;
$$\therefore\ [P]=\frac{m_1[A]+m_2[B]+m_3[C]}{m_1+m_2+m_3}-\frac{m_2m_3\pi a^2+m_3m_1\pi b^2+m_1m_2\pi c^2}{(m_1+m_2+m_3)^2};$$
whence $$[P]=x[A]+y[B]+z[C]-\pi(a^2yz+b^2zx+c^2xy)$$
$$=x[A]+y[B]+z[C]+\pi t^2,$$
where t^2 is the square of the tangent from x, y, z to the circumcircle if the point be without, zero if upon, or $-$the rectangle of the segments of a chord through x, y, z if the point be within the circumcircle; which gives Mr. Leudesdorf's result of Art. 480 already established in a different manner.

499. It is worth observing that the locus of points P which give equal areas $[P]$

is $a^2yz+b^2zx+c^2xy+$linear terms $=0$, *i.e.* a circle,

or making it homogeneous,

$$a^2yz+b^2zx+c^2xy-\left(\frac{[A]}{\pi}x+\frac{[B]}{\pi}y+\frac{[C]}{\pi}z\right)(x+y+z)$$
$$-\frac{[P]}{\pi}(x+y+z)^2=0,$$

and the centre of this circle is given by

$$b^2z+c^2y-\frac{[A]}{\pi}(x+y+z)-\left(\frac{[A]}{\pi}x+\frac{[B]}{\pi}y+\frac{[C]}{\pi}z\right)$$
$$-\frac{2[P]}{\pi}(x+y+z)=\text{two similar expressions},$$

i.e. $$b^2z+c^2y-\frac{[A]}{\pi}=c^2x+a^2z-\frac{[B]}{\pi}=a^2y+b^2x-\frac{[C]}{\pi},$$

which is independent of $[P]$, and therefore indicates that such loci for different values of $[P]$ form a set of concentric circles, which is Mr. Kempe's Theorem of Art. 488 (Cor. VI).

500. It is also worth notice that the area described by *the centroid of the triangle* is given for the case of one complete revolution by

$$[G]=\frac{[A]+[B]+[C]}{3}-\frac{1}{9}\pi(a^2+b^2+c^2);$$

and for *the orthocentre O*,

$$[O]=\frac{[A]\tan A+[B]\tan B+[C]\tan C}{\tan A\tan B\tan C}-8\pi R^2\cos A\cos B\cos C,$$

where R is the radius of the circumcircle.

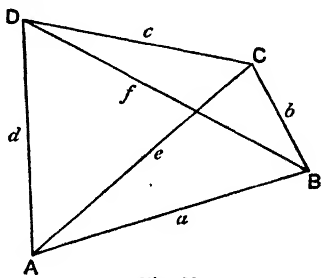


Fig. 99.

501. In the case of *four particles in rigid connection* if a, b, c, d be the sides and e, f the internal diagonals of

the quadrilateral formed, we have, in the one-revolution case,

$$[P]=\frac{m_1[A]+m_2[B]+m_3[C]+m_4[D]}{m_1+m_2+m_3+m_4}$$
$$-\pi\frac{m_1m_2a^2+m_2m_3b^2+m_3m_4c^2+m_4m_1d^2+m_1m_3e^2+m_2m_4f^2}{(m_1+m_2+m_3+m_4)^2},$$

and similarly if there be a greater number of points.

502. In a case where there is *no rotation, i.e.* where the line joining each pair of particles remains parallel to its original position, or if there be rotation of any of these joins and an opposite equal rotation of the same join, it is clear that all the "relative contours" will disappear and

$$[P]=\frac{\Sigma m[A]}{\Sigma m}.$$

503. The same result will also hold in the case when the "relative contours," though not individually vanishing, are such as in *the aggregate to destroy each other*, some being positive and others negative, for in such case $\Sigma m_r m_s S_{rs}=0$.

504. If the several particles be *in rigid connection* and the figure describe n revolutions before re-occupying its original position,

$$\Sigma m_r m_s S_{rs}=n\pi\Sigma m_r m_s A_r A_s{}^2=n\pi M\Sigma mGA^2,$$

by Lagrange's "Second Theorem." (Routh, *Anal. Statics*, vol. i., Art. 437); and in that case

$$[G]=\frac{\Sigma m[A]}{M}-n\pi\frac{\Sigma mGA^2}{M}=\frac{\Sigma m[A]}{M}-n\pi\kappa^2,$$

where $M=\Sigma m$ and κ the radius of gyration about the centroid G.

505. Mechanical Integrators or Planimeters.

Consider the case of two rods OP, PQ of lengths a_1 and a_2, freely hinged together at P and the first one OP hinged to a fixed point O in a plane in which both rods can otherwise move freely.

Let x, y be the coordinates of Q relative to a pair of rectangular axes through O, let the rods make angles θ_1, θ_2 respectively with the x-axis, and let $\theta_2-\theta_1=\psi$.

Then

$$x = a_1 \cos\theta_1 + a_2 \cos\theta_2, \qquad y = a_1 \sin\theta_1 + a_2 \sin\theta_2,$$

$$dx = -a_1 \sin\theta_1 d\theta_1 - a_2 \sin\theta_2 d\theta_2, \quad dy = a_1 \cos\theta_1 d\theta_1 + a_2 \cos\theta_2 d\theta_2;$$

$$\therefore\ x\,dy - y\,dx = a_1^2 d\theta_1 + a_2^2 d\theta_2 + a_1 a_2 \cos(\theta_2 - \theta_1)(d\theta_1 + d\theta_2)$$

$$= a_1^2 d\theta_1 + a_2^2 d\theta_2 + a_1 a_2 \cos\psi\, d\psi + 2a_1 a_2 \cos\psi\, d\theta_1.$$

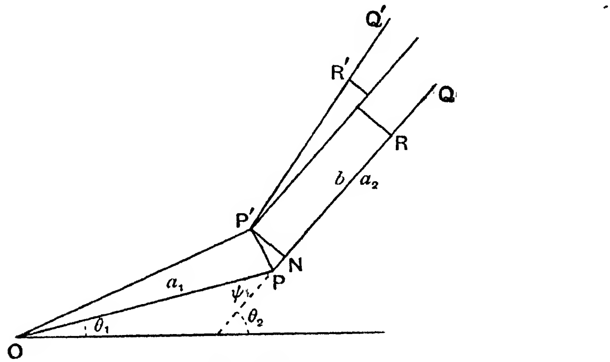


Fig. 100.

Let R be a point on PQ at distance b from P, and let P', Q', R' be the positions taken up by P, Q, R after displacements $d\theta_1$, $d\theta_2$ of the rods.

Then R has advanced perpendicularly to PQ a distance

$$a_1 d\theta_1 \cos\psi + b\, d\theta_2 = ds, \text{ say, to the first order.}$$

Then $x\,dy - y\,dx = a_1^2 d\theta_1 + a_2^2 d\theta_2 + a_1 a_2 \cos\psi\, d\psi + 2a_2(ds - b\, d\theta_2)$.

If Q be made to travel round the contour of any closed curve whose area is to be found, in the positive direction, on completion of the circuit, supposing the point O to be outside the contour and OP and OQ to have *oscillated back to their original positions,*

$$\int d\theta_1 = 0, \quad \int d\theta_2 = 0, \quad \int \cos\psi\, d\psi = [\sin\psi] = 0,$$

and we have

$$\text{Area bounded by the contour} = a_2 S,$$

where S is the total distance travelled over by a point R on the rod PQ, in a direction at right angles to the rod. And it is further to be noticed that this result does not

depend upon b, the term involving b disappearing upon integration round the contour. Hence the particular position of the attachment of the point R to the rod is immaterial.

506. But if the point O be within the contour considered, and both rods make a complete revolution before regaining their original position,

$$\int d\theta_1=2\pi,\quad \int d\theta_2=2\pi,\quad \int \cos\psi\, d\psi=[\sin\psi]=0;$$

and therefore $$A=\pi(a_1^2+a_2^2-2a_2b)+a_2S.$$

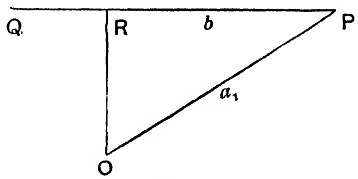


Fig. 101.

Now $a_1^2+a_2^2-2a_2b$ is the value of OQ^2 when the rods are clamped at the joint P in such a position that OR is perpendicular to PQ. Call this value of OQ^2, r_0^2.

$$\therefore\quad A=\pi r_0^2+a_2S.$$

A circle with centre O and radius r_0 is called the zero circle. When the system is clamped in this position the motion of R is at right angles to OR, *i.e.* in the direction of PQ, and R has no motion at all at right angles to the rod PQ on which it lies. Hence when O lies *within* the contour the area of the zero circle, viz. πr_0^2, must be added to a_2S to give the area of the contour.

Again, if one rod, say OP_1, oscillates back to its original position whilst the other PQ makes a complete turn, then

$$\int d\theta_1=0,\quad \int d\theta_2=2\pi,\quad \int \cos\psi\, d\psi=0;$$

and $$A=\pi(a_2^2-2a_2b)+a_2S.$$

Similarly, if PQ oscillates but OP revolves,

$$\int d\theta_1=2\pi,\quad \int d\theta_2=0,\quad \int \cos\psi\, d\psi=0,$$

and $$A=\pi a_1^2+a_2S.$$

507. The general result is therefore that the area traced by the pointer is

	(1) a_2S
or	(2) $\pi(a_1^2+a_2^2-2a_2b)+a_2S$
or	(3) $\pi(a_2^2-2a_2b)+a_2S$
or	(4) $\pi a_1^2+a_2S$,
according as	(1) neither a_1 nor a_2 complete a revolution,
	(2) both complete a revolution,
	(3) a_2 completes a revolution but a_1 does not,
	(4) a_1 completes a revolution but a_2 does not,

in each case the arms of the instrument occupying the same position as they did at the beginning of the tracing.

508. This principle is made use of in the construction of a Mechanical Integrator known as AMSLER'S PLANIMETER, which is used for the practical measurement of an area. The rod PQ is provided at R with a small graduated wheel with axis parallel to the rod, which is allowed to rest on the paper and to turn by friction with the paper. It can then only register the amount of travel of R at right angles to the rod, the amount of travel in the direction of the rod being necessarily unregistered as it is due to slide along the surface of the paper and not to the rolling of the wheel. A reading of the wheel gives the value of S. Then

area of contour $=a_2S$ or $a_2S+\pi r_0^2$,

according as the point O is outside or within the contour.

509. Several forms of Mechanical Integrators are in use, but for the most part they are modifications of Professor Amsler's form and based upon the general principle described above.

Description of the Instrument.

The figure shown (Fig. 102) is an illustration of a form of the instrument made by Messrs. John J. Griffin & Sons, Scientific Instrument Makers, Kingsway, London. The lettering corresponds to the preceding general explanation of the principle. O is the fixed point, ABC the contour of the area required, Q the tracing point which is being made

to traverse the contour, P is the joint connecting the two beams of the instrument, R the graduated wheel or roller whose axis is parallel to PQ and which rolls upon the paper when there is any motion at right angles to PQ. Its position upon the beam PQ being immaterial, it is placed in this form of the instrument on QP produced. D is a dial whose axis is perpendicular to the axis of the wheel and turned by a worm on the axis of the roller. There is a pointer attached to the beam PQ, serving to mark the amount of rotation of the dial plate. V is a vernier assisting to read small amounts of rotation of the wheel. There is

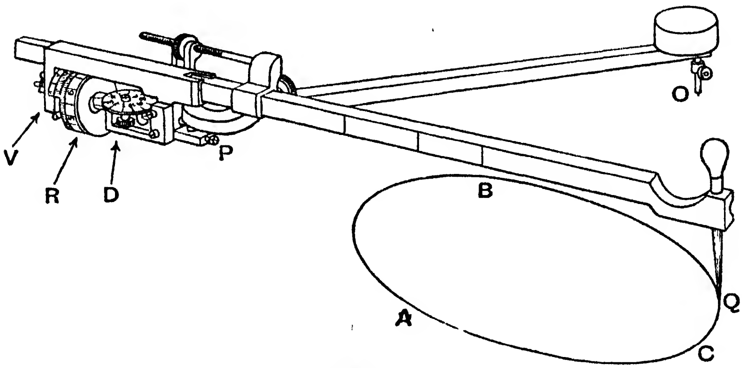


Fig. 102.

a pointer at Q by means of which the contour can be carefully followed.

The graduations on the rim of the wheel are such that the circumference is divided into 10 equal segments indicated by 1, 2, 3, 4, ... 0, and each segment into 10 further subdivisions. The dial D is such as to rotate once for 10 revolutions of the roller, and is itself divided into 10 segments, which are again subdivided, an advance of a segment of the dial indicating one complete revolution of the wheel. The readings of the dial therefore indicate the number of complete revolutions of the wheel. In the vernier a length equal to 9 subdivisions of the wheel is divided into 10 equal portions on the vernier.

If the figures on the dial be taken as units, the figured graduations on the wheel will represent 10ths and the subdivisions

100$^{\text{ths}}$, the difference between the distance of two consecutive divisions of the vernier and two consecutive subdivisions of the wheel, being $(\frac{1}{100}-\frac{1}{10}\times\frac{9}{100})$ of the circumference of the wheel, is $\frac{1}{1000}$ of the circumference of the wheel. Hence, by means of the vernier, readings may be made to three places of decimals. The area to be found has been shown to be

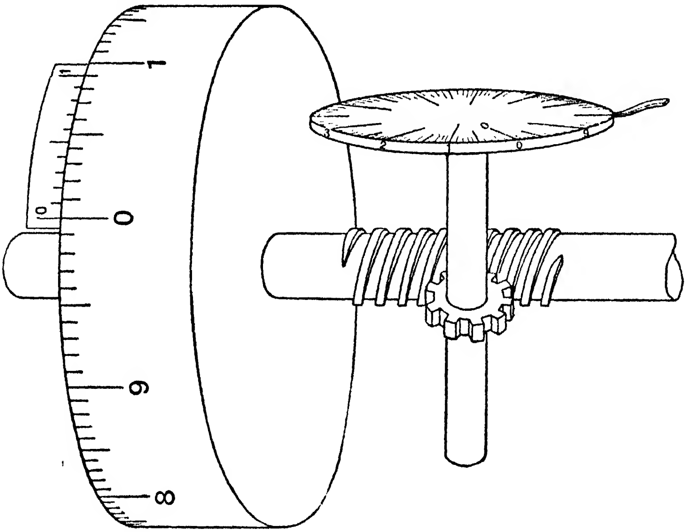


Fig. 103.

proportional to the number registered by the roll of the wheel, the component of motion parallel to the axis, *i.e.* slide, being unregistered. Let S be the number registered by the wheel, then

$$A = CS,$$

where C is some constant called the constant of the instrument. Apply the instrument first to any figure of known area A_0, say a square or a circle, as may be most convenient; let the difference of initial and final readings of the instrument be S_0, then $A_0 = CS_0$, which determines C. If now we apply it to the contour whose quadrature is required and S be the difference of the initial and final readings of the instrument,

$$A = A_0 \frac{S}{S_0}.$$

It has been assumed that the fixed point O has been taken outside the perimeter of the contour. If inside, we have still to add the area of the "zero" circle, and

$$A = \pi r_0^2 + A_0 \frac{S}{S_0}.$$

The area of the zero circle is usually marked on the instrument.

Mode of Procedure.

The procedure is then as follows:

(1) Fix the point O to the drawing board on which the area to be found has been previously pinned.

(2) Bring the pointer Q to some point of the perimeter of the contour and mark the starting point.

(3) Read the instrument by means of the dial, the wheel and the vernier, and note the initial reading.

(4) Trace carefully the whole perimeter of the contour with the pointer Q.

(5) Read the instrument again.

(6) Subtract the two readings. The difference is S.

Then the constant of the instrument being known, or having been found previously in like manner,

$$A = A_0 \frac{S}{S_0} \quad \text{or} \quad A = A_0 \frac{S}{S_0} + \pi r_0^2,$$

according as it has been convenient to take O outside or within the contour.

EXAMPLES.

1. Ox, Oy being perpendicular axes, A, B fixed points on Oy and $AMBA$ any closed region of area S lying in the positive quadrant, show that the integral

$$\int [\{\phi(y)e^x - my\}\, dx + \{\phi'(y)e^x - m\}\, dy],$$

taken round the curve from A to B, is equal to

$$m(S + a - b) + \phi(b) - \phi(a),$$

$\phi(y)$, $\phi'(y)$ being finite and continuous, m a constant and $OA = a$, $OB = b$. [J. MATH. SCHOL. OXFORD, 1904.]

2. P_1, P_2 are points on a closed oval of area A, such that P_1, P_2 subtends a right angle at a fixed point O. Show that the area of the curve traced out by the middle point of P_1P_2 is equal to

$$\frac{1}{2}A+\frac{1}{8}\int_0^{2\pi}\left(r_1\frac{dr_2}{d\theta_1}-r_2\frac{dr_1}{d\theta_2}\right)d\theta_1,$$

where $$\theta_2=\theta_1+\frac{\pi}{2} \quad \text{and} \quad OP_1=r_1, \quad OP_2=r_2.$$

[COLLEGES β, 1889.]

3. A fixed point O is taken on a central oval which is such that through any point inside it other than the centre one and only one chord can be drawn which is bisected at that point; prove that the locus of the middle point of the chord PQ for a constant sum 2σ of the arcs OP, OQ cuts at right angles the same locus for a constant difference $2\sigma'$ of these arcs; and deduce that the area of the oval is

$$\frac{1}{2}\int_0^l d\sigma\int_0^{\frac{1}{2}l} d\sigma'\sin\theta,$$

where l is the length of the oval, and θ is the angle between the tangents at P and Q. [MATH. TRIPOS, 1889.]

4. A bar AB carries at a point of its length a small wheel having AB for axis and which turns about AB: the end A is constrained to move in a given straight line; show that if the end B is carried round any closed curve without singular points and which does not cut the straight line on which A moves, the area of the curve is measured by the product of AB into the whole length registered by the revolving wheel. [COLLEGES, 1892.]

[This is the principle of construction of Coffin's Planimeter. A full description will be found on p. 159, *Practical Electrical Engineering*, by Briggs and others. It is the case when the rod OP of Fig. 102 is of infinite length, so that P describes a straight line instead of a circle.]

5. A straight line of given length moves with its extremities on the arcs of two closed curves of given areas, and a point is attached to the moving line.

Prove that when the area traced by this attached point has a minimum value for different positions of the point on the line, the difference of the areas of the circles whose radii are the segments into which the point divides the line is equal to the difference of the areas of the given curves. [ST. JOHN'S, 1882.]

6. Show that the path of the mid-point of a rod of constant length $2c$, whose ends lie upon an ellipse, is an oval of area $\pi(ab-c^2)$.

If, instead of both ends being on the ellipse, one end lies on the ellipse and the other on the major axis, or if one end lies on the ellipse and the other on the auxiliary circle, find the areas of the paths described by the centre of the rod in both cases.

7. A rigid cyclic quadrilateral $ABCD$ moves in its plane so as to return to its original position after turning through four right angles. Show that if (A), etc., denote the areas of the curves described by A, etc., and if S_1, S_2, etc., denote the areas of the triangles BCD, CDA, etc., then

$$S_1(A) + S_3(C) = S_2(B) + S_4(D).$$

Find also the equation connecting the areas described by any three vertices with that described by the centre of the circumcircle of the triangle. [I. C. S., 1909.]

8. Two bars OP, RPQ, of lengths $OP = c$, $RPQ = b + a$, respectively turn round a fixed pin at O and a joint at P. dS_1, dS_2 denote the polar elements of area about O of the curves traced by P and Q respectively; prove that

$$dS_2 - dS_1 = a\,d\zeta + a\left(\tfrac{1}{2}a + b\right)d\theta - \tfrac{1}{2}a\,dp,$$

where $PQ = a$, $RP = b$, p is the perpendicular from O on RPQ, $d\zeta$ is the displacement of R perpendicular to RPQ and θ is the inclination of RPQ to a fixed line OA. [MATH. TRIP., PT. I., 1914.]

CHAPTER XVI.

RECTIFICATION (I.). ELEMENTARY.

510. In the following five chapters we propose to illustrate further the methods and processes of integration by showing their application to finding the length of a curved line whose equation is given by one of the ordinary modes of description, Cartesian, Polar, Pedal Equation, Tangential Polar, etc.; and further to discuss some subsidiary matters which arise in connection with such problems.

The process of finding the length of an arc of a curve, *i.e.* of finding a straight line whose length is the same as that of a specified arc, is called Rectification. Curves, the lengths of whose arcs can be found, are said to be Rectifiable.

Any formula which may have been established in the Differential Calculus expressing the differential coefficient of the arc "s" with regard to any independent variable, in terms of that variable, gives rise at once by integration to a formula in the Integral Calculus for the finding of s.

In each case the limits of integration to be assigned are the values of the independent variable corresponding to the two points which terminate the arc whose length is sought.

511. THE WORKING FORMULAE.

Below are added a list of the most common of these formulae. The references are to the articles in the author's *Treatise on the Differential Calculus* where they are established.

Formula in the Differential Calculus.	Formula in the Integral Calculus.	Reference.	Observations.
$\dfrac{ds}{dx}=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$	$s=\displaystyle\int\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx$	Art. 200	For Cartesian Equations of form $y=f(x)$
$\dfrac{ds}{dy}=\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$	$s=\displaystyle\int\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\,dy$	Art. 200	For Cartesian Equations of form $x=f(y)$
$\dfrac{ds}{dt}=\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}$	$s=\displaystyle\int\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}\,dt$	Art. 200	For the case where the curve is defined as $x=f(t)$, $y=F(t)$
$\dfrac{ds}{d\theta}=\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}$	$s=\displaystyle\int\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\,d\theta$	Art. 201	For Polar Equations of form $r=f(\theta)$
$\dfrac{ds}{dr}=\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}$	$s=\displaystyle\int\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}\,dr$	Art. 201	For Polar Equations of form $\theta=f(r)$
$\dfrac{ds}{dr}=\sec\phi=\dfrac{r}{\sqrt{r^2-p^2}}$	$s=\displaystyle\int\dfrac{r\,dr}{\sqrt{r^2-p^2}}$	Arts. 202 and 203	For Pedal Equations of form $p=f(r)$
$\dfrac{ds}{d\psi}=p+\dfrac{d^2p}{d\psi^2}$	$s=\dfrac{dp}{d\psi}+\displaystyle\int p\,d\psi$	Art. 221	For Tangential-Polars of form $p=f(\psi)$

The formulae $s=\displaystyle\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or $s=\displaystyle\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

are applicable to cases where the Cartesian Equation is given, or can readily be expressed, in the forms $y=f(x)$ or $x=f(y)$ respectively, x being regarded as the independent variable in the first case, y in the second, and the axes being supposed to be rectangular.

As explained in the *Differential Calculus*, Art. 200, these formulae arise from the consideration of the infinitesimal right-angled triangle formed by the increments of abscissa, ordinate, and to the first order, the arc.

512. The amended form of these results for oblique axes would be, with the same description of the figure (Fig. 104) as in the article cited,

$$\delta s^2=(\text{chord } PQ)^2=\delta x^2+\delta y^2+2\delta x\,\delta y\cos\omega,$$

to the second order, and after rejecting infinitesimals of higher

order than the second and proceeding to the limit

$$\left(\frac{dx}{ds}\right)^2+2\frac{dx}{ds}\frac{dy}{ds}\cos\omega+\left(\frac{dy}{ds}\right)^2=1,$$

and accordingly we should write

$$s=\int\sqrt{1+2\frac{dy}{dx}\cos\omega+\left(\frac{dy}{dx}\right)^2}\,dx$$

or

$$s=\int\sqrt{1+2\frac{dx}{dy}\cos\omega+\left(\frac{dx}{dy}\right)^2}\,dy,$$

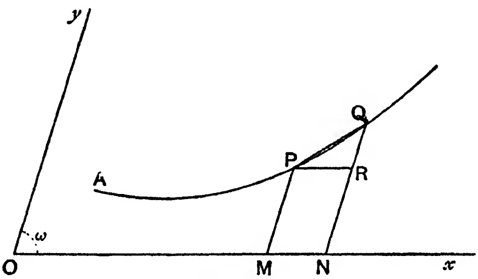


Fig. 104.

according as we take x or y for the independent variable.

513. The formulae may be remembered in a less formal manner as

$$s=\int\sqrt{dx^2+dy^2}$$

or

$$s=\int\sqrt{dx^2+2dx\,dy\cos\omega+dy^2},$$

where the dx or the dy may be brought outside the radical as circumstances demand.

514. Further, when the curve is given by expressing x and y separately in terms of a single variable t, as

$$x=f(t),\quad y=F(t),$$

we have

$$s=\int\sqrt{[\{f'(t)\}^2+\{F'(t)\}^2]}\,dt$$

or

$$s=\int\sqrt{[\{f'(t)\}^2+2f'(t)\,F'(t)\cos\omega+\{F'(t)\}^2}\,dt,$$

according as the coordinate axes are rectangular or oblique.

The coordinate axes will be always assumed to be rectangular unless the contrary is expressly stated, or to be inferred from the context.

515. The Rectification, therefore, of a curve depends upon the possibility of integration of the radical which occurs in these formulae.

ILLUSTRATIVE EXAMPLES.

516. **The Earliest Rectification. William Neil's Problem (1637-1670).***

Ex. 1. Rectification of the Semicubical Parabola.

The equation of this curve is $ay^2=x^3$.

Here
$$\frac{dy}{dx}=\frac{3}{2}\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}},$$
$$s=\int\sqrt{1+\frac{9}{4}\frac{x}{a}}\,dx=\frac{2}{3}\cdot\frac{4a}{9}\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}}.$$

Taken between $x=0$ (the cusp) and $x=x_1$, for the branch in the first quadrant,
$$s=\frac{1}{27a^{\frac{1}{2}}}[(4a+9x)^{\frac{3}{2}}-(4a)^{\frac{3}{2}}].$$

This is stated by Gregory and Walton to have been the first curve to be rectified. The priority is ascribed to Neil by Wallis, but the rectification of the curve was also independently accomplished by Van Huraet.†

517. **The Parabola.**

Ex. 2. Consider the arc of the ordinary parabola $y^2=4ax$.

Here
$$y=2\sqrt{ax},\quad \frac{dy}{dx}=\sqrt{\frac{a}{x}},$$
$$s=\int\sqrt{1+\frac{a}{x}}\,dx.$$

To effect this integration, let $x=a\tan^2\psi$.

Then
$$dx=2a\tan\psi\sec^2\psi\,d\psi,$$
$$s=2a\int\sqrt{1+\cot^2\psi}\tan\psi\sec^2\psi\,d\psi$$
$$=2a\int\sec^3\psi\,d\psi$$
$$=a[\sec\psi\tan\psi+\log(\sec\psi+\tan\psi)]$$
$$=a\left[\sqrt{\frac{x}{a}}\sqrt{1+\frac{x}{a}}+\log\left(\sqrt{1+\frac{x}{a}}+\sqrt{\frac{x}{a}}\right)\right].$$

If taken between any two limits, x_1 and x_2, corresponding to any two points P, Q on the arc, which lie on the same side of the axis,
$$\text{arc } PQ=(\sqrt{x_2}\sqrt{a+x_2}-\sqrt{x_1}\sqrt{a+x_1})+a\log\frac{\sqrt{a+x_2}+\sqrt{x_2}}{\sqrt{a+x_1}+\sqrt{x_1}}.$$

* *Wallisii Opera*, T. 1, 551; Gregory and Walton, p. 420.

† Cajori's *History of Mathematics*, p. 190.

For example, if we require the length from the vertex to the upper end of the latus rectum, $x_1=0,\quad x_2=a,$

and $$\text{arc required}=\sqrt{a}\sqrt{a+a}+a\log\frac{\sqrt{a+a}+\sqrt{a}}{\sqrt{a}}$$
$$=a[\sqrt{2}+\log_e(\sqrt{2}+1)]$$
$$=a(1{\cdot}4142\ldots+{\cdot}8814\ldots)$$
$$=2{\cdot}2956\ldots\times a.$$

Thus the length of an arc of a parabola from one end of the latus rectum to the other is $\quad 1{\cdot}1478\ldots$ times the latus rectum.

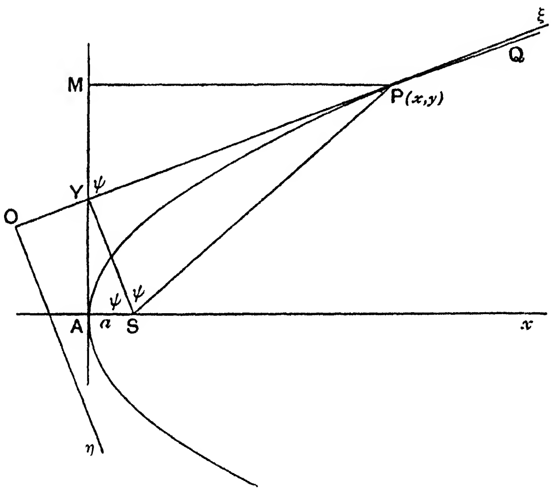


Fig. 105.

It is worth considering the angle ψ which has been used as a subsidiary variable to facilitate integration.

It is the angle which the tangent at the current point P makes with the y-axis, viz. the tangent at the vertex. For if PM be the perpendicular upon the y-axis, PY the tangent, S the focus, SY the perpendicular upon the tangent, and if we call MYP, ψ, we have

$$x=MP=YP\sin\psi=SY\frac{\sin^2\psi}{\cos\psi}=a\tan^2\psi.$$

The intrinsic equation of this curve is therefore

$$s=a\sec\psi\tan\psi+a\log(\sec\psi+\tan\psi)$$

or $$s=a\sec\psi\tan\psi+a\,\mathrm{gd}^{-1}\psi,$$

the tangent at the vertex being the initial tangent.

Let us call PY, t. Then $t=a\sec\psi\tan\psi$.

Hence $$s-t=a\log(\sec\psi+\tan\psi).$$

Hence the logarithmic portion of s, viz. $a \log(\sec\psi + \tan\psi)$ denotes the excess of the arcual distance of P from A over the "tail," *i.e.* the portion of the tangent measured from P to the foot of the perpendicular upon the tangent from the focus.

It will be seen later that in many cases this excess "arc − tail" plays an important part.

In the case under consideration—viz. the parabola—let a length $PO = s$ be measured along the tangent. Then $OY = s - t$. The point O is the point on the tangent at which the vertex A would arrive if we regard the tangent as a fixed line, and the parabola to roll upon it without sliding. Consider it in this way. O is then a fixed point. Take the tangent OP as the ξ-axis, and a perpendicular through O as the η-axis. Then, if ξ, η be the coordinates of the focus,

$$\xi = OY = s - t = a\log(\sec\psi + \tan\psi),$$
$$\eta = YS \qquad = a\sec\psi.$$

To find the path of S as the parabola rolls upon its fixed tangent, we have to eliminate ψ.

$$\left.\begin{aligned} \sec\psi + \tan\psi &= e^{\frac{\xi}{a}}. \\ \text{Hence} \qquad \sec\psi - \tan\psi &= e^{-\frac{\xi}{a}}. \end{aligned}\right\} \text{Therefore } \sec\psi = \cosh\frac{\xi}{a}.$$

Therefore the path of the focus of the rolling parabola is

$$\eta = a\cosh\frac{\xi}{a},$$

i.e. the ordinary catenary or chainette.

We also have, putting $\frac{\xi}{a} = u$,

$$\tan\psi = \sinh u, \quad \sin\psi = \tanh u,$$
$$SP = a\sec^2\psi = a\cosh^2 u,$$
$$t = SP\sin\psi = a\sinh u\cosh u = \frac{a}{2}\sinh 2u,$$
$$s = a\sinh u\cosh u + a\log(\sinh u + \cosh u)$$
$$= \frac{a}{2}\sinh 2u + au,$$
$$s - t = au,$$
$$SY = a\sec\psi = a\cosh u, \text{ etc.}$$

Incidentally, we may note that the equation

$$\xi = a\log(\sec\psi + \tan\psi) = a\,\mathrm{gd}^{-1}\psi$$

may be used to indicate the "march" of the function, $\mathrm{gd}^{-1}\psi$ for $a\,\mathrm{gd}^{-1}\psi$ is the abscissa of a point on a catenary curve, and since $\frac{d\eta}{d\xi} = \tan\psi$, ψ is the slope of the tangent to the catenary. Hence a good idea of the graph of $y = a\,\mathrm{gd}^{-1}x$ can be formed by first plotting the catenary itself and then

plotting a new curve, taking as abscissae the circular measures of the angles which the tangent to the catenary makes with its directrix, and for ordinates the corresponding abscissae of the catenary.

If PP' be a focal chord of the parabola, the arc AP has been shown to be

$$AP = a\sec\psi\tan\psi + a\log(\sec\psi + \tan\psi),$$

and the arc $P'A$ can be obtained from it by writing $90-\psi$ for ψ,

i.e. $$P'A = a\operatorname{cosec}\psi\cot\psi + a\log(\operatorname{cosec}\psi + \cot\psi).$$

Hence, by addition, the whole arc $P'AP$ cut off by a focal chord which makes an angle 2ψ with the axis is

$$a\left[\frac{\sin^3\psi + \cos^3\psi}{\sin^2\psi\cos^2\psi} + \log(1+\sec\psi)(1+\operatorname{cosec}\psi)\right].$$

The evaluation of the arc might have been conducted by taking y as the independent variable.

Then $$x = \frac{y^2}{4a}, \quad \frac{dx}{dy} = \frac{y}{2a},$$

$$s = \frac{1}{2a}\int\sqrt{4a^2+y^2}\,dy$$

$$= \frac{1}{4a}\left[y\sqrt{4a^2+y^2} + 4a^2\log\frac{y+\sqrt{y^2+4a^2}}{2a}\right]_{y_1}^{y_2},$$

which reduces to the same form as already obtained.

518. Sir Christopher Wren's Problem (1632–1723). Rectification of the Cycloid.

Ex. 3. The equations of the curve are

$$\left.\begin{aligned} x &= a\theta + a\sin\theta, \\ y &= a(1-\cos\theta). \end{aligned}\right\} \text{(See } \textit{Diff. Calc.}\text{, pp. 337–339.)}$$

Here $$dx = a(1+\cos\theta)\,d\theta,$$

$$dy = a\sin\theta\,d\theta.$$

Hence $$ds^2 = 2a^2(1+\cos\theta)\,d\theta^2 = 4a^2\cos^2\frac{\theta}{2}\,d\theta^2,$$

$$ds = 2a\cos\frac{\theta}{2}\,d\theta,$$

$$s = 4a\sin\frac{\theta}{2}, \quad \text{......(1)}$$

s being measured from the point at which $\theta=0$, *i.e.* the vertex.

Again, with the same description of the figure as in *Diff. Calc.*, Art. 394, chord $CQ = 2a\sin\frac{\theta}{2}$.

Therefore $$\text{arc } CP = 2 \text{ chord } CQ. \quad \text{......(2)}$$

Substituting for θ from $y = 2a\sin^2\frac{\theta}{2}$,

$$s = \sqrt{8ay}. \quad \text{......(3)}$$

If the tangent at P is inclined at an angle ψ to the tangent at the vertex,

$$\tan\psi = \frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2};$$

$$\therefore\ \theta = 2\psi,$$

and

$$s = 4a\sin\psi. \qquad \text{...(4)}$$

This is the intrinsic equation of the curve.

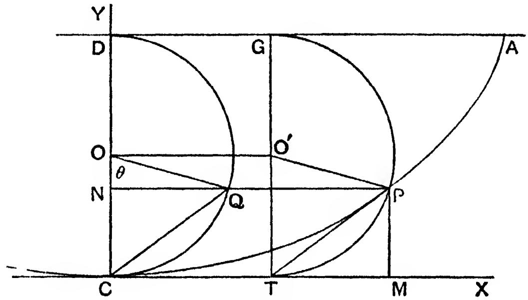


Fig. 106.

The whole length of the curve from cusp to cusp is

$$2\Big[4a\sin\psi\Big]_0^{\frac{\pi}{2}} = 8a. \qquad \text{...............................(5)}$$

The point at which $\psi = 30$ gives $s = 2a$, and therefore bisects the arcual distance from vertex to cusp.

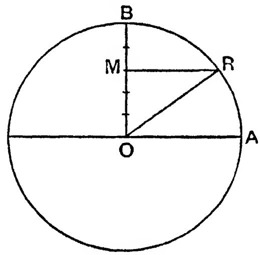


Fig. 107.

If a circle be drawn with any radius, and OA, OB be a pair of radii at right angles, and OB divided into n equal parts so that M being, say, the r^{th} point of division, and MR be then drawn parallel to OA to meet the circle at R, then $\sin AOR = \frac{r}{n}$.

If then in the cycloid a chord CQ of the circle CQD be drawn (Fig 106) so that the angle XCQ = angle AOR, in Fig. 107, the line QP parallel to CX, and cutting the cycloid at P, will cut off an arc $CP = \frac{r}{n}$ of the arc CA, for

$$\text{arc } CP = 4a\sin\psi = 4a \cdot \frac{r}{n} = \frac{r}{n}\,\text{arc } CA.$$

Hence an arc of any proposed ratio to the whole arc can be cut off.

Many of the geometers of the seventeenth century devoted considerable attention to the cycloid.* Wren, the architect of St. Paul's Cathedral, discovered the rectification of the curve and determined the centroid; Fermat, the area bounded by an arc; Huygens invented the cycloidal pendulum; Pascal and Wallis also greatly advanced a knowledge of the curve.†

519. Centroid of an Arc of any Line Density.

If ρ be the line density, the mass of any element δs is $\rho\,\delta s$,

and $$\bar{x}=\frac{\Sigma(\rho\,\delta s)x}{\Sigma(\rho\,\delta s)},\quad \bar{y}=\frac{\Sigma(\rho\,\delta s)y}{\Sigma(\rho\,\delta s)},$$

give the position of the centroid. Hence, taking the limit when δs is infinitesimally small,

$$\bar{x}=\frac{\int\rho x\,ds}{\int\rho\,ds},\quad \bar{y}=\frac{\int\rho y\,ds}{\int\rho\,ds}.$$

If ρ be constant,

$$\bar{x}=\frac{\int x\,ds}{\int ds},\quad \bar{y}=\frac{\int y\,ds}{\int ds};$$

that is, $s\bar{x}=\int x\,ds$, $s\bar{y}=\int \rho\,dy$, s being the length of the arc whose centroid is required, and the integration being taken from one extremity of the arc to the other. (See Art. 446.)

And if x be the independent variable,

$$\bar{x}=\frac{\int\rho x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int\rho\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx},\quad \bar{y}=\frac{\int\rho y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int\rho\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx},$$

with corresponding formulae if it be desirable to express the integral with other independent variables as shown in the table of Art. 511.

*See *Diff. Calc.*, Art. 390.

†Cajori's *Hist. of Math.*, pp. 177, etc.

EXAMPLES.

1. Find the length of the arc of the curve $y^2(2a-x)=x^3$, the cissoid of Diocles. [HUYGENS, 1625–1695.]

2. Find the curve for which the length of the arc measured from the origin varies as the square root of the ordinate.

3. The major axis of an ellipse is 1 foot in length, and its eccentricity is $\frac{1}{10}$. Prove that its circumference is 3·1337 feet nearly. [TRINITY, 1883.]

4. Find the length of any arc of the curve

$$x^{\frac{2}{3}}-y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

5. Show that in the "catenary of equal strength," $y=a\log\sec\frac{x}{a}$,

$$s=a\log\tan\left(\frac{x}{2a}+\frac{\pi}{4}\right),$$

and that the intrinsic equation of the curve is $s=a\,\mathrm{gd}^{-1}\psi$.

6. Show that in the common catenary, or chainette, $y=c\cosh\frac{x}{c}$,

$$s=\sqrt{y^2-c^2},\quad s=c\tan\psi,\quad s^2=c(\rho-c),\quad s=c\sinh\frac{x}{c}.$$

The area bounded by the curve, the directrix, the y-axis and an ordinate is $A=cs$.

The centroid of the arc has coordinates

$$\begin{aligned}\bar{x}&=x-c\tan\frac{\psi}{2} & \bar{y}&=\tfrac{1}{2}(y+x\cot\psi)\\ &=x-\left(y-c\right)\frac{c}{s}, & &=\frac{1}{2}\left(y+\frac{cx}{s}\right).\end{aligned}$$

The centroid of the area bounded by the curve, the directrix, the y-axis and an ordinate is given by

$$\bar{x}=x-(y-c)\frac{c}{s},\quad \bar{y}=\frac{1}{4}\left(y+\frac{cx}{s}\right),$$

and that both centroids lie on the ordinate through the intersection of the terminal tangents.

7. Show that the length of the curve $y=\log\coth\frac{x}{2}$ from the point (x_1, y_1) to the point (x_2, y_2) is $\log\frac{\sinh x_2}{\sinh x_1}$.

8. Show that in the epi- or hypo-cycloid

$$\left.\begin{aligned}x&=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta,\\ y&=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta.\end{aligned}\right\}$$

(i) $s=\frac{4b}{a}(a+b)\cos\frac{a\theta}{2b}$, (ii) $s=\frac{4b}{a}(a+b)\cos\frac{a}{a+2b}\psi$,

s being measured from the point where $\theta=\frac{\pi b}{a}$, *i.e.* a vertex.

9. For the four-cusped hypocycloid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

show (i) that $s=\frac{3a}{4}\cos 2\psi$, s being measured from a vertex;

(ii) the whole length of the curve is $6a$;

(iii) $s^3 \propto x^2$, s being measured from the cusp which lies on the y-axis.

10. In the tractrix

$$x=\sqrt{c^2-y^2}+\frac{c}{2}\log\frac{c-\sqrt{c^2-y^2}}{c+\sqrt{c^2-y^2}},$$

show that $$s=c\log\frac{c}{y}.$$

11. Show that the distance from the vertex of the centroid of a wire in the form of portion of a cycloid, of which the vertex is the middle point, is $\frac{1}{3}$ of the greatest ordinate of the arc.

12. Show that the arc of a parabola of latus rectum $4a$ measured from the vertex, and the radius vector from the focus, are expressible in terms of a parameter t in the respective forms

$$\frac{s}{a}=\frac{t}{1-t^2}+\frac{1}{2}\log\frac{1+t}{1-t}, \quad \frac{r}{a}=\frac{1}{1-t^2}.$$

[MATH. TRIP. PT. II., 1915.]

Prove also that $$s=\sqrt{r(r-a)}+a\tanh^{-1}\sqrt{\frac{r-a}{r}}.$$

520. **Polar Formula.**

In the *Differential Calculus* (Art 201) it is shown from consideration of the small infinitesimal right-angled triangle formed by the increments of arc, radius vector and perpendicular on the radius vector from one extremity of the infinitesimal arc, that to the second order

$$\delta s^2=\delta r^2+r^2\delta\theta^2.$$

This gives rise at once, on proceeding to the limit, to the formulae,

$$s=\int\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta$$

or

$$s=\int\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr,$$

according as we wish to use θ or r as the independent variable, and, as in Art. 513, we may remember it in the less formal manner as

$$s=\int\sqrt{dr^2+r^2\,d\theta^2}.$$

Further, as in the case of Cartesians, if r and θ be given in terms of some third variable t (though this is very unusual) by $r=f(t)$, $\theta=F(t)$, we may say

$$s=\int\sqrt{\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2}\,dt$$

$$=\int\sqrt{[f'(t)]^2+[f(t)]^2[F'(t)]^2}\,dt.$$

521. Illustrative Examples.

Ex. 1. In the case of the **Archimedean Spiral** $r=a\theta$,

$$s=a\int\sqrt{\theta^2+1}\,d\theta=\frac{a}{2}\left[\theta\sqrt{\theta^2+1}+\log(\theta+\sqrt{\theta^2+1})\right],$$

s being measured from the vertex, where $\theta=0$.

As this may be written

$$s=\frac{1}{2a}\left(r\sqrt{r^2+a^2}+a^2\log\frac{r+\sqrt{r^2+a^2}}{a}\right),$$

we see, on comparison with the result of Art. 517, that this is the same as the arc of the parabola $y^2=2ax$, measured from the vertex of the parabola and expressed in terms of the ordinate.

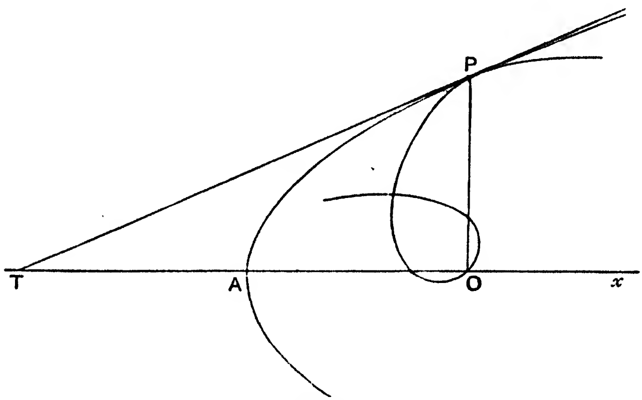


Fig. 108.

Hence it will follow that when an Archimedean spiral $r=a\theta$ rolls without sliding on the concave side of a parabola $y^2=2ax$ so that their vertices come into contact, the roulette of the pole of the spiral is the axis of the parabola. In this case the r of the spiral is the y of the parabola, and the motion of the pole O is always at right angles to the line PO, and arcs AP, OP are equal.

For many examples of this class, see Chapter XIX.

522. Ex. 2. **The Cardioide** $r=a(1-\cos\theta)$. (See Art. 424, *Diff. Calc.*) The curve is symmetrical about the initial line, and θ varies from 0 to π for the upper half.

$$\frac{dr}{d\theta}=a\sin\theta.$$

Hence $s=\int_0^\theta \sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}\,d\theta$

$$=a\int_0^\theta 2\sin\frac{\theta}{2}\,d\theta=\left[-4a\cos\frac{\theta}{2}\right]_0^\theta=4a\left(1-\cos\frac{\theta}{2}\right)=8a\sin^2\frac{\theta}{4}.$$

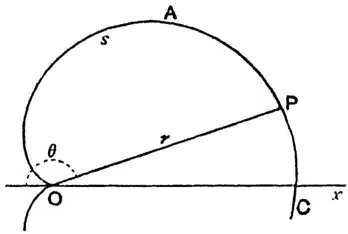


Fig. 109.

This gives the length of any arc OAP.

For the upper half the length is $4a\left(1-\cos\frac{\pi}{2}\right)=4a$.

The whole length of arc $=8a$.

523. **The u, θ Formula.**

The equation of a curve is sometimes given in the form

$$u=f(\theta),\quad \text{where } u=\frac{1}{r}.$$

The appropriate formula for rectification in this case is

$$ds^2=\frac{1}{u^4}du^2+\frac{1}{u^2}d\theta^2\ \left(\text{since } dr=-\frac{1}{u^2}du\right),$$

giving rise to $\qquad s=\int\sqrt{\frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2+\frac{1}{u^2}}\,d\theta$

$$=\int\frac{1}{u^2}\sqrt{\left(\frac{du}{d\theta}\right)^2+u^2}\,d\theta,$$

or $\qquad s=\int\sqrt{\frac{1}{u^4}+\frac{1}{u^2}\left(\frac{d\theta}{du}\right)^2}\,du$

$$=\int\frac{1}{u^2}\sqrt{1+u^2\left(\frac{d\theta}{du}\right)^2}\,du,$$

according as θ or u be taken as the independent variable.

524. Centroid of an Arc of any Line Density; Polars.

Again, exactly as in the case of the curve whose equation is given in Cartesian coordinates, if ρ be the line density, the centroid of the arc of a curve is given by

$$\bar{x}=\frac{\int \rho x\,ds}{\int \rho\,ds}=\frac{\int \rho r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}{\int \rho\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta},$$

$$\bar{y}=\frac{\int \rho y\,ds}{\int \rho\,ds}=\frac{\int \rho r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}{\int \rho\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}.$$

525. Centroid of Arc of a Circle.

Ex. In the case of a uniform circular arc of radius a and terminated

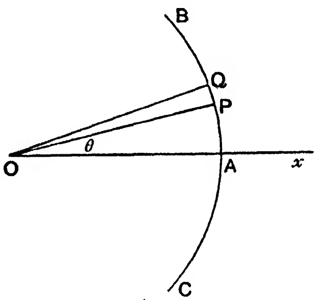


Fig. 110.

by the radii vectores $\theta=\pm\alpha$, the line density being uniform, taking the medial line as x-axis,

$$\bar{x}=\frac{\int_{-\alpha}^{\alpha} a\cos\theta\,.\,a\,d\theta}{\int_{-\alpha}^{\alpha} a\,d\theta}=a\frac{\Big[\sin\theta\Big]_{-\alpha}^{\alpha}}{\Big[\theta\Big]_{-\alpha}^{\alpha}}=a\frac{\sin\alpha}{\alpha}$$

and $\qquad \bar{y}=0$ because the x-axis is an axis of symmetry.

526. Moment of Inertia of a Fine Wire.

The moment of inertia of a fine wire of line density ρ about any straight line in the plane of the wire is $\Sigma\rho\,\delta s\times p^2$, where p is the perpendicular from the element δs upon the straight line.

Thus, Moment of inertia about x-axis $=\int \rho y^2\, ds$,

Moment of inertia about y-axis $=\int \rho x^2\, ds$,

and Moment of inertia about a perpendicular to the plane through the pole $=\int \rho r^2\, ds$,

and for ds is to be substituted from the table of Art. 511, the appropriate expression according to the system of coordinates used in any particular case.

The **Product of Inertia** for such a wire with regard to the axes is defined as

$$\int \rho xy\, ds.$$

EXAMPLES.

1. Find the length of any arc of the curve from the formula

$$s=\int \sqrt{r^2+r'^2}\, d\theta$$

for the following cases:

(i) $r=a\cos\theta$ (circle). (ii) $r=ae^{m\theta}$ (equiang. spiral).

(iii) $r=a\sin^2\dfrac{\theta}{2}$ (cardioide). (iv) $\dfrac{2a}{r}=1+\cos\theta$ (parabola).

(v) $r=a\dfrac{\sin^2\theta}{\cos\theta}$ (cissoid). (vi) $r=\dfrac{3a}{2}\dfrac{\sin^2\theta}{\cos^3\theta}$ (semicub. parab.).

2. Show that the length of the arc of that part of the cardioide

$$r=a(1+\cos\theta),$$

which lies on the side of the line $4r=3a\sec\theta$ remote from the pole, is equal to $4a$. [OXFORD.]

3. Show that the whole length of the limaçon $r=a\cos\theta+b$ is equal to that of an ellipse whose semiaxes are equal in length to the maximum and minimum radii vectores of the limaçon. Hence show how to divide the arc of the limaçon into four equal parts. [COLLEGES α, 1888.]

4. Prove that the length of the n^{th} pedal of a loop of the curve

$$r^m=a^m\sin m\theta$$

is
$$a(mn+1)\int_0^{\frac{\pi}{m}} \sin^k m\theta\, d\theta, \quad \text{where } m(k-n+1)=1.$$

5. Show that the length of a loop of the curve

$$3x^2y-y^3=(x^2+y^2)^3$$

$$=2\int_0^1 \frac{d\xi}{\sqrt{1-\xi^6}}.$$ [ST. JOHN'S, 1881.]

6. Show that the rectification of the curve $r^n=a^n\sin n\theta$ is given by the integral

$$s=a\int_0^{\xi}\frac{d\xi}{\sqrt{1-\xi^{2n}}}.$$ [MATH. TRIP., 1896.]

7. Two radii vectores OP, OQ of the curve

$$r=2a\cos^3\left(\frac{\pi}{4}+\frac{\theta}{3}\right)$$

are drawn equally inclined to the initial line; prove that the length of the intercepted arc is $a\alpha$, where α is the circular measure of the angle POQ. [ASPARAGUS, *Educ. Times.*]

8. Show that the centroid of a wire bent into the form of a cardioide $r=a(1+\cos\theta)$, and with a line density $k\sec\frac{\theta}{2}$, k being a constant, is on the axis of the cardioide at distance $\frac{a}{2}$ from the cusp.

527. The Converse Problem. Given s, find the Curve.

The converse problem, viz. given s in terms of one of the quantities x, y, r or θ, to find the equation of the curve, leads in the first three cases shown below to an application of the same formulae, but in the fourth case there is more difficulty (Art. 529).

(1) If $s=f(x)$, we have

$$\left(\frac{dy}{dx}\right)^2=\left(\frac{ds}{dx}\right)^2-1,\quad y=\int\sqrt{\{f'(x)\}^2-1}\,dx.$$

(2) If $s=f(y)$,

$$\left(\frac{dx}{dy}\right)^2=\left(\frac{ds}{dy}\right)^2-1,\quad x=\int\sqrt{\{f'(y)\}^2-1}\,dy.$$

(3) If $s=f(r)$,

$$r^2\left(\frac{d\theta}{dr}\right)^2=\left(\frac{ds}{dr}\right)^2-1,\quad \theta=\int\frac{\sqrt{\{f'(r)\}^2-1}}{r}\,dr.$$

528. For example.

1. Find the curve for which $s=\dfrac{x^2}{2a}$.

Here $$\left(\frac{dy}{dx}\right)^2=\frac{x^2}{a^2}-1$$

$$\pm a\,dy=\sqrt{x^2-a^2}\,dx$$

$$\pm 2ay=x\sqrt{x^2-a^2}-a^2\cosh^{-1}\frac{x}{a}+\text{constant}.$$

2. Find the curve in which $s=r\sec\alpha$.

Here
$$r^2\left(\frac{d\theta}{dr}\right)^2=\sec^2\alpha-1=\tan^2\alpha,$$
$$\frac{dr}{r}=\pm d\theta\cot\alpha,$$
$$\log r=\pm\theta\cot\alpha+\text{const.},$$
$$r=ae^{\pm\theta\cot\alpha}. \quad \text{(Equiangular spirals.)}$$

3. Find the curve in which $s=\sqrt{8ay}$.

Here
$$\frac{ds}{dy}=\sqrt{\frac{2a}{y}},$$
$$\left(\frac{dx}{dy}\right)^2=\frac{2a}{y}-1, \quad dx=\sqrt{\frac{2a-y}{y}}\,dy.$$

Let
$$y=a(1-\cos\theta),$$
$$dx=\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\,a\sin\theta\,d\theta=a(1+\cos\theta)\,d\theta,$$
$$\left.\begin{aligned}x+\text{const.}&=a(\theta+\sin\theta),\\ y&=a(1-\cos\theta),\end{aligned}\right\}\text{ a cycloid.}$$

529. (4) But the case when $s=f(\theta)$ leads at once to
$$\left(\frac{dr}{d\theta}\right)^2+r^2=\{f'(\theta)\}^2,$$

and the variables r and θ are not now in general "separable" as in the former cases (see *Integral Calculus for Beginners*, Art. 175); nor does this differential equation fall under any of the standard forms. Nevertheless, in some cases useful information may be derived from its consideration.

For example,

1. Is the circle $r=a$ the only curve for which $s=a\theta$?

Here we have $\left(\frac{dr}{d\theta}\right)^2+r^2=a^2$, which is of course satisfied by $r=a$. But if r is not equal to a, we have
$$\int\frac{dr}{\sqrt{a^2-r^2}}=\pm d\theta,$$
$$\sin^{-1}\frac{r}{a}=\alpha\pm\theta, \quad \text{where } \alpha \text{ is a constant.}$$

Hence
$$r=a\sin(\alpha\pm\theta),$$

i.e. a circle of radius $\frac{a}{2}$ and passing through the pole will also give the same result, viz. $s=a\theta$, as is geometrically obvious. But no curve other than $r=a$ or $r=a\sin(\alpha\pm\theta)$ will do so.

2. Is the equiangular spiral $r=ae^{\theta\cot\alpha}$ the only curve for which

$$s=\frac{ae^{\theta\cot\alpha}}{\cos\alpha}\ ?$$

Here
$$\left(\frac{dr}{d\theta}\right)^2+r^2=\frac{a^2}{\sin^2\alpha}e^{2\theta\cot\alpha}.$$

Let $r=ave^{\theta\cot\alpha}$, where v is some function of θ to be determined.

Thus
$$\left(\frac{dv}{d\theta}+v\cot\alpha\right)^2+v^2=\operatorname{cosec}^2\alpha,$$

which is of course obviously satisfied if $v=1$, which leads back to $r=ae^{\theta\cot\alpha}$.

But we have in addition to this the general solution of

$$\frac{dv}{d\theta}+v\cot\alpha=\pm\sqrt{(\operatorname{cosec}^2\alpha-v^2)},$$

i.e. of
$$\int\frac{dv}{v\cot\alpha\mp\sqrt{\operatorname{cosec}^2\alpha-v^2}}=-\theta+\beta,$$

where β is some constant.

To integrate this, let $v=\operatorname{cosec}\alpha\sin\phi$.

Then
$$\int\frac{\cos\phi\,d\phi}{\sin(\phi\mp\alpha)}=\frac{\beta-\theta}{\sin\alpha},$$

i.e.
$$\int\{\cos\alpha\cot(\phi\mp\alpha)\mp\sin\alpha\}\,d\phi=\frac{\beta-\theta}{\sin\alpha},$$

or
$$\left.\begin{aligned}&\cos\alpha\log\sin(\phi\mp\alpha)\mp\phi\sin\alpha=\frac{\beta-\theta}{\sin\alpha},\\ \text{where}\quad&\frac{\sin\phi}{\sin\alpha}=\frac{r}{ae^{\theta\cot\alpha}},\end{aligned}\right\}$$

which upon elimination of ϕ furnishes a set of curves whose arcs are of the same length as the corresponding arcs of the equiangular spiral $r=ae^{\theta\cot\alpha}$

Examples.

1. Find the curves in which

(i) $s=a\sin^{-1}\dfrac{y}{a}$. (ii) $s=\sqrt{y^2-c^2}$.

(iii) $s=\dfrac{r^2}{2a}$. (iv) $y=ce^{-\frac{s}{c}}$.

(v) $s\propto r$. (vi) $s\propto\sqrt{x}$.

(vii) $s=2\sqrt{2ar}$.

2. Show that the equation

$$nx\frac{d^2s}{d^2x}+\frac{ds}{dx}=0$$

leads to a cycloid or a four-cusped hypocycloid according as $n=2$ or $n=3$.

530. **Tangential Polar Equations. Legendre's Formulae.**

Formulae $$t=\frac{dp}{d\psi},\quad \frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2}.$$

These results were proved in Article 221 of the *Differential Calculus*, but are now established in a different manner.

Let PY, $P'Y'$ be the tangents at two contiguous points P, P' of the curve, OY, OY' the perpendiculars upon them from the pole O.

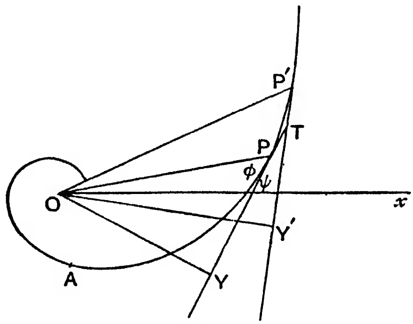


Fig. 111.

Let t be the projection of the radius vector upon the tangent
$$OY=p,\quad OY'=p+\delta p,\quad \text{arc } PP'=\delta s,$$
and $\delta\psi$ the angle YOY'.

Then, projecting the broken line $OYPP'$ upon OY' and upon $Y'P'$,

(1) $p+\delta p=p\cos\delta\psi+t\sin\delta\psi+$second order quantities,

(2) $t+\delta t=\delta s+t\cos\delta\psi-p\sin\delta\psi$,

i.e. $$\left.\begin{aligned}\delta p&=t\,\delta\psi,\\ \delta t&=\delta s-p\,\delta\psi,\end{aligned}\right\}\text{ to the first order.}$$

And ultimately $t=\dfrac{dp}{d\psi},\quad \dfrac{ds}{d\psi}=p+\dfrac{dt}{d\psi}=p+\dfrac{d^2p}{d\psi^2}.$

531. It is to be noted that since $t\equiv\dfrac{dp}{d\psi}=r\cos\phi$, *i.e.* the projection of the radius vector upon the tangent, t is positive or negative according as ϕ is acute or obtuse.

The above figure (Fig. 111) exhibits the standard case. In this case $t\equiv\dfrac{dp}{d\psi}$ is $+PY$, and is in a direction from P opposite to that

of the direction of increase of s; p is increasing with ψ and $\frac{dp}{d\psi}$ is therefore positive. In cases where p increases or decreases as ψ decreases or increases, $\frac{dp}{d\psi}$ (*i.e.* t) is negative, and $=-PY$.

The student should examine the formulae carefully in all four cases:

(1) Curve concave to O, ϕ acute.
(2) Curve convex to O, ϕ acute.
(3) Curve concave to O, ϕ obtuse.
(4) Curve convex to O, ϕ obtuse.

It will be seen that $\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2}$ in all cases and that $t=\pm PY$ according as ϕ is acute or obtuse.

The arc s is measured from a point on the arc on the same side of the radius vector as that on which ϕ is measured; ψ may increase or decrease with the increase of s.

The value of the radius of curvature is, of course, essentially positive; and $\rho=\pm\frac{ds}{d\psi}$ according as s and ψ increase together, or the one increases as the other decreases.

Accordingly we have $\rho=\pm\left(p+\frac{d^2p}{d\psi^2}\right)$ respectively in these cases. The formulae established are due to LEGENDRE.

532. By integration of

$$\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2},$$

we have
$$s=\frac{dp}{d\psi}+\int p\,d\psi,$$

i.e.
$$s-t=\int p\,d\psi;$$

where t is the "tail" referred to in Art. 517.

In the case of *a closed oval of continuous curvature*, the "tail" t returns to its original value when the integration is conducted round the whole contour.

If the origin be *within the curve* and is only enclosed once by it, the length of the contour is given by

$$\int_0^{2\pi} p\,d\psi.$$

If the *origin is enclosed n times* (Fig. 112), so that the tangent makes n complete revolutions as its point of contact travels continuously round the curve, the length will be

$$\int_0^{2n\pi} p\, d\psi.$$

Further modifications may have to be made, for instance, in integrating round *a loop of a curve* (Fig. 113); it may happen that the initial and final values of $\frac{ds}{d\psi}$ are not the same, and that the tangent does not make a complete

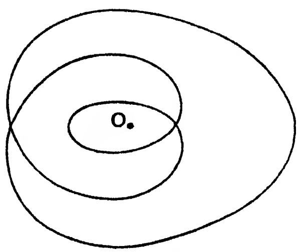


Fig. 112.

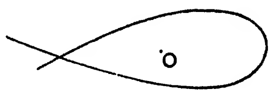


Fig. 113.

revolution, but the student should have no difficulty in such cases in assigning the proper limits.

533. Ex. Show that the perimeter of an ellipse of small eccentricity e exceeds by $\frac{3e^4}{64}$ of its length that of a circle having the same axis.

[γ, 1889.]

Here $$p^2 = a^2\cos^2\psi + b^2\sin^2\psi = a^2(1 - e^2\sin^2\psi),$$

where ψ is the angle p makes with the major axis.

Therefore $$p = a\left(1 - \frac{1}{2}e^2\sin^2\psi - \frac{1}{8}e^4\sin^4\psi - \dots\right).$$

Hence $$s = 4a\left(\frac{\pi}{2} - \frac{1}{2}e^2\frac{1}{2}\frac{\pi}{2} - \frac{1}{8}e^4\frac{3}{4}\frac{1}{2}\frac{\pi}{2} - \dots\right)$$

$$= 2\pi a - \frac{\pi}{2}ae^2 - \frac{3}{32}\pi a e^4 - \dots .$$

The radius r of a circle of the same area is given by

$$r^2 = ab = a^2(1 - e^2)^{\frac{1}{2}},$$

and its circumference is

$$2\pi a\left(1 - \frac{1}{4}e^2 - \frac{3}{32}e^4 - \dots\right);$$

$$\therefore \text{ circumf. ellipse} - \text{circumf. circle} = \left(\frac{3}{16} - \frac{3}{32}\right)\pi a e^4$$
$$= \frac{3}{64} 2\pi a e^4$$
$$= \frac{3e^4}{64}\{\text{circ. of circle}\}$$

as far as terms involving e^4,

$$i.e. \quad \frac{\text{circ. ell.} - \text{circ. circle}}{\text{circ. circle}} = \frac{3e^4}{64};$$

$$\therefore \quad \frac{\text{circ. ell.} - \text{circ. circle}}{\text{circ. ellipse}} = \frac{3e^4}{64}\Big/\left(1 + \frac{3e^4}{64}\right) = \frac{3e^4}{64}$$

to terms involving e^4.

534. Length of the Arc of an Evolute.

It was shown in the *Differential Calculus* (Art. 343) that the difference between the radii of curvature at two points of a curve of continuous curvature is equal to the length of

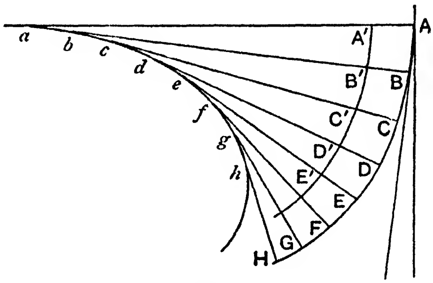


Fig. 114.

the corresponding arc of the evolute; *i.e.* if ah be the arc of the evolute of the portion AH of the original figure, then (Fig. 114)

$$\text{arc } ah = Aa - Hh, \quad i.e. \ \rho \text{ (at } A) - \rho \text{ (at } H).$$

And if the evolute be regarded as a rigid curve, and an inelastic string be unwound from it, being kept tight, then the points of the unwinding string describe a system of parallel curves, each of the parallels being an involute of the curve ha, one of these being the original curve HA itself.

535. Ex. Find the length of the evolute of an ellipse. If α, α', β, β' be the centres of curvature corresponding to the extremities of the axes, viz. A, A', B, B' respectively, the arc $\alpha\beta$ of the evolute corresponds to the arc AB of the ellipse, and we have

$$\text{arc } \alpha\beta = \rho \text{ (at } B) - \rho \text{ (at } A) = \frac{a^2}{b} - \frac{b^2}{a},$$

for the radius of curvature at any point P of the ellipse is $\frac{a^2b^2}{p^3}$ $\left(\text{the pedal equation being } \frac{a^2b^2}{p^2}=a^2+b^2-r^2 \text{ and } \rho=\frac{r\,dr}{dp}\right)$. Thus the length of the entire perimeter of the evolute, which is obviously symmetrical about the axes, is

$$4\left(\frac{a^2}{b}-\frac{b^2}{a}\right).$$

In the application of this rule care is needed, not to pass a point of maximum or minimum curvature on the original curve, for on travelling

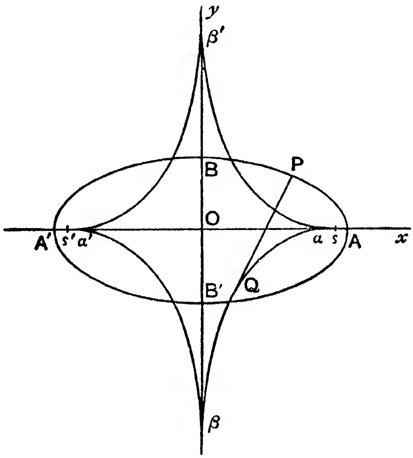


Fig. 115.

in a continuous direction round the original curve the difference of successive radii of curvature changes sign at such points and the evolute has a cusp as in the figure for the ellipse (Fig. 115). In that case, as P travels from A to B and through B to A', the string PQ is wound *off* the arc $a\beta$ and *upon* the arc $\beta a'$. And therefore the arcs $a\beta$ and $\beta a'$ would appear with opposite signs, viz. $\frac{a^2}{b}-\frac{b^2}{a}$ and $\frac{b^2}{a}-\frac{a^2}{b}$, if P travels continuously in one direction. The intervals between the points of maximum and of minimum curvature must therefore be treated separately and the positive results added together.

EXAMPLES.

1. Show that in the parabola $y^2=4ax$, the length of the arc of the evolute intercepted within the parabola is

$$4a(3\sqrt{3}-1).$$

2. Find the whole length of the evolute of the cardioide

$$r=a(1+\cos\theta).$$

3. Show that the length of the evolute of the portion of the Folium of Descartes $x^3+y^3=3axy$, which corresponds to the loop, is $\frac{3a}{4}(4-\sqrt{2})$.

536. Intrinsic Equation of a Curve.

Let s be the length of the arc of a curve measured from a fixed point O to the current tracing point P;

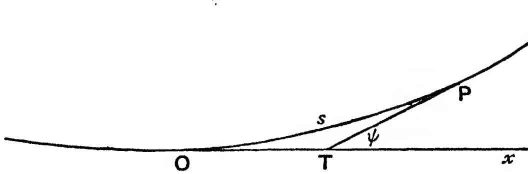


Fig. 116.

ψ the angle of contingence at P, *i.e.* the angle between the tangent at P and any fixed line in the plane, say the tangent at O;

ρ the radius of curvature at P, or κ its reciprocal, viz. the curvature.

Then any given relation between two of these three quantities s, ψ, ρ (or κ) will suffice to determine the shape of the curve, and may in many cases very conveniently replace an extraneous specification of the curve by means of coordinates, Cartesian or Polar. These quantities s, ψ, ρ depend upon no external system of coordinates and leave the position of the curve undefined. The nature of the curve itself is specified by the relation existing between two of the three s, ψ, ρ, which has been very aptly styled by Dr. Whewell the Intrinsic Equation of the curve. Some notice has been already taken of Intrinsic Equations in Arts. 346–349 of the *Differential Calculus*. But the subject is more closely allied to Integral Calculus, and it is convenient to develop the matter more fully here, though at the risk of some repetition.

We shall adopt the notation used in the *Differential Calculus* as to the meanings of the letters involved for the following work.

When the relation is between s and ψ,

$$\text{say} \quad s=f(\psi),$$

that between ρ and ψ is

$$\rho = \pm \frac{ds}{d\psi}, \quad i.e. \quad \rho = \pm f'(\psi).$$

The sign to be taken + when s is increasing with ψ,
− when s increases or decreases as ψ decreases or increases,

and if κ be used $\left(\text{viz. the curvature, } = \frac{1}{\rho}\right)$, instead of the radius of curvature,

$$\kappa f'(\psi) = \pm 1$$

is the relation between κ and ψ, with, of course, the same rule as to choice of sign.

Conversely, if the connection given be between ρ and ψ,

$$\text{say} \quad \rho = f(\psi),$$

then
$$\frac{ds}{d\psi} = \pm f(\psi),$$

and
$$s = \pm \int f(\psi)\, d\psi + C,$$

C being a constant which may be chosen to correspond to the measurement of s from any arbitrarily chosen point of the curve, and the sign selected as before.

When the relation is given between κ and ψ, it is, of course, the same thing, except that we have

$$\kappa = f(\psi), \quad \text{say},$$

i.e.
$$\frac{d\psi}{ds} = \pm f(\psi),$$

and
$$s = \pm \int \frac{d\psi}{f(\psi)} + \text{const.}$$

Finally, when the relation is between ρ and s,

$$\text{say} \quad \rho = f(s),$$

we have
$$\pm \frac{ds}{d\psi} = f(s),$$

and
$$\psi + C = \pm \int \frac{ds}{f(s)}.$$

Hence, these three systems of description of a curve, by means of a specified relation,

(*a*) between s and ψ,
(*b*) between ρ (or κ) and ψ,
(*c*) between ρ (or κ) and s,

are equivalent, and either forms a mode of specification which is intrinsically a property of the curve itself, and in no way defining its position upon the plane upon which it may happen to be drawn.

The s-ψ description is the one which is usually understood as the "Intrinsic Equation," and it is the system used by Whewell in his memoirs on the subject (*Camb. Phil. Trans.*, viii., p. 659; and ix., p. 150) and discussed in Boole's *Differential Equations*, pages 264–269.

The ρ-ψ specification was used by Euler.

537. To obtain the Intrinsic Equation from the Cartesian Equation.

When the Cartesian Equation is given as $y=f(x)$, then, supposing the initial tangent to be parallel to the x-axis, we have

$$\tan\psi=f'(x), \qquad\qquad (1)$$

and

$$\frac{ds}{dx}=\sqrt{1+[f'(x)]^2},$$

$$s=\int\sqrt{1+[f'(x)]^2}\,dx. \qquad\qquad (2)$$

And if after integration x be eliminated between equations (1) and (2), the required relation between s and ψ,

$$\text{say}\quad s=F(\psi),$$

will be obtained.

Conversely, if the equation $s=F(\psi)$ be given, and the Cartesian equation be desired, we have

$$\frac{dx}{ds}=\cos\psi,\quad \frac{dy}{ds}=\sin\psi;$$

whence

$$x+A=\int\cos\psi\, F'(\psi)\,d\psi, \qquad\qquad (1)$$

$$y+B=\int\sin\psi\, F'(\psi)\,d\psi, \qquad\qquad (2)$$

A and B being arbitrary constants.

And if after integration ψ be eliminated from equations (1) and (2), the Cartesian Equation of the curve will result.

538. **Illustrative Examples.**

Ex. 1. Intrinsic Equation of a circle.

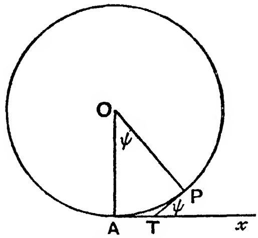


Fig. 117.

If ψ be the angle between the initial tangent at A and the tangent at P, the centre being O and the radius a, we have $P\hat{O}A = P\hat{T}x = \psi$, and therefore $s = a\psi$.

Ex. 2. Intrinsic Equation of a catenary.

In this case the equation of the curve referred to its axis and the tangent at the vertex as coordinate axes is

$$y + c = c\cosh\frac{x}{c}.$$

Hence $$\tan\psi = \frac{dy}{dx} = \sinh\frac{x}{c},$$

and $$\frac{ds}{dx} = \sqrt{1+\sinh^2\frac{x}{c}} = \cosh\frac{x}{c};$$

$$\therefore \quad s = c\sinh\frac{x}{c},$$

the constant of integration being zero if we measure s from the vertex where $x=0$; therefore $s = c\tan\psi$ is the intrinsic equation sought.

539. **Case when the Coordinates are expressed in terms of a Parameter.**

If the equations of the curve be given as

$$x = f(t), \quad y = \phi(t),$$

we have $$\tan\psi = \frac{dy}{dx} = \frac{\phi'(t)}{f'(t)}. \quad \text{...(1)}$$

Also $$\frac{ds}{dt} = \sqrt{\{f'(t)\}^2 + \{\phi'(t)\}^2},$$

and $$s = \int\sqrt{\{f'(t)\}^2 + \{\phi'(t)\}^2}\,dt. \quad \text{..................(2)}$$

If s be found in terms of t by integration from equation (2), then between this result and equation (1) we may eliminate t. The required relation between s and ψ will result.

540. Ex. 1. In the cycloid

$$x = a(t + \sin t),$$
$$y = a(1 - \cos t).$$

Hence $$\tan\psi = \frac{\sin t}{1+\cos t} = \tan\frac{t}{2}; \quad \therefore\ t = 2\psi.$$

Also $$\frac{ds}{dt} = a\sqrt{(1+\cos t)^2 + \sin^2 t} = 2a\cos\frac{t}{2};$$

whence $s = 4a\sin\frac{t}{2}$, s being measured from the vertex, where $t = 0$.

Hence $s = 4a\sin\psi$ is the equation required. See *Diff. Calc.*, Arts. 395, 397.

Ex. 2. In the epi- or hypo-cycloid

$$x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$
$$y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta,$$
$$\frac{dx}{d\theta} = -(a+b)\sin\theta + (a+b)\sin\frac{a+b}{b}\theta,$$
$$\frac{dy}{d\theta} = \quad (a+b)\cos\theta - (a+b)\cos\frac{a+b}{b}\theta,$$
$$\frac{ds}{d\theta} = \pm 2(a+b)\sin\frac{a}{2b}\theta,$$
$$s = \mp\frac{4b(a+b)}{a}\cos\frac{a}{2b}\theta + C.$$

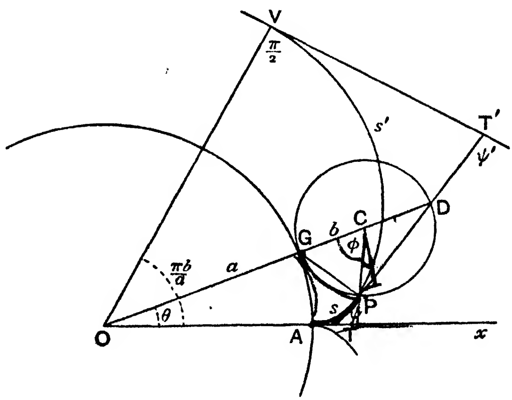


Fig. 118.

Also with the description of figure in Art. 405, *Diff. Calc.*,

$$a\theta = b\phi \quad \text{and} \quad \psi = \theta + \frac{\phi}{2} = \frac{a+2b}{2b}\theta,$$

$$s = \mp \frac{4b(a+b)}{a} \cos \frac{a}{a+2b}\psi + C.$$

If s be measured from the cusp, the tangent at the cusp being the initial line,

$$\text{arc } AP = s = \frac{4b(a+b)}{a}\left(1 - \cos\frac{a}{a+2b}\psi\right).$$

If we measure the arc from the vertex V, where $\theta = \dfrac{\pi b}{a}$,

$$\text{arc } VP = s' = \frac{4b(a+b)}{a} \cos \frac{a}{a+2b}\psi,$$

OA being retained as the initial line for the measurement of ψ. If we measure ψ from the tangent at the vertex, we must write

$$\frac{\pi}{2} + \frac{\pi b}{a} - \psi' \text{ for } \psi, \quad i.e.\ \frac{\pi}{2} - \frac{a}{a+2b}\psi' \text{ for } \frac{a}{a+2b}\psi,$$

and
$$s' = \frac{4b(a+b)}{a} \sin \frac{a}{a+2b}\psi'.$$

Hence the general intrinsic equation of such curves is

$$s = A \sin B\psi \quad \text{or} \quad s = A \cos B\psi.$$

In the case $s = A \sin B\psi$, s is measured from a vertex and ψ is measured from the tangent at that vertex.

In the case $s = A \cos B\psi$, s is measured from a vertex and ψ is measured from the tangent at the next cusp.

541. **To obtain the Intrinsic Equation from the Polar.**

Suppose the initial line parallel to the tangent at the point A

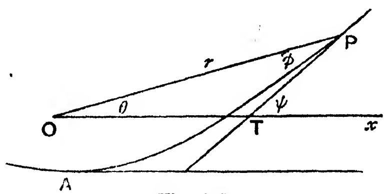


Fig. 119.

from which the arc is measured. Then, with the usual notation, we have

$$r = f(\theta), \text{ the equation to the curve, } \qquad (1)$$

$$\psi = \theta + \phi, \qquad (2)$$

$$\tan\phi = \frac{r\,d\theta}{dr} = \frac{f(\theta)}{f'(\theta)}, \qquad (3)$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\{f(\theta)\}^2 + \{f'(\theta)\}^2};$$

and therefore

$$s=\int\sqrt{\{f(\theta)\}^2+\{f'(\theta)\}^2}\,d\theta. \quad\text{.........................(4)}$$

If s be found by integration from (4), and θ, ϕ eliminated by means of equations (2) and (3), the required relation between s and ψ will be found. If the initial line of the polar equation be not that from which ψ is measured, equation (2) will need modification accordingly.

542. Ex. 1. Find the intrinsic equation of the cardioide

$$r=a(1-\cos\theta).$$

Here
$$\psi=\theta+\phi,$$

$$\tan\phi=\frac{a(1-\cos\theta)}{a\sin\theta}=\tan\frac{\theta}{2};$$

$$\therefore\ \phi=\frac{\theta}{2}\ \text{ and }\ \psi=\theta+\frac{\theta}{2}=\frac{3\theta}{2};$$

$$\therefore\ \theta=\frac{2}{3}\psi.$$

Also
$$\frac{ds}{d\theta}=a\sqrt{(1-\cos\theta)^2+\sin^2\theta}=2a\sin\frac{\theta}{2},$$

$$s=-4a\cos\frac{\theta}{2}+C.$$

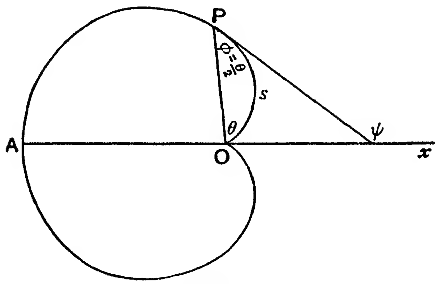


Fig. 120.

If we determine C so that $s=0$ when $\theta=0$, we have $C=4a$;

$$\therefore\ s=4a\left(1-\cos\frac{\psi}{3}\right),$$

the intrinsic equation sought.

If A be the vertex, the arc $AP=4a\cos\frac{\psi}{3}$.

If we measure ψ from the tangent at the vertex (Fig. 121), we must write for ψ,

$$\frac{3\pi}{2}-\psi',$$

and if arc $AP = s'$,
$$s' = 4a \cos\left(\frac{\pi}{2} - \frac{\psi'}{3}\right),$$
$$s' = 4a \sin\frac{\psi'}{3}.$$

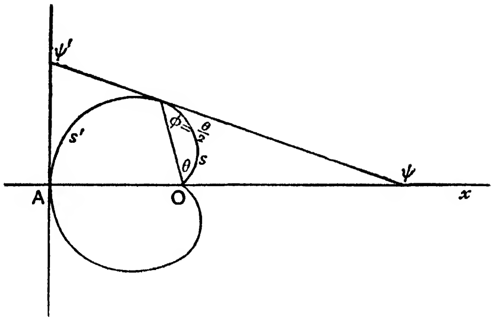


Fig. 121.

Ex. 2. Find the intrinsic equation of the first negative pedal of the Archimedean spiral $r = a\theta$.

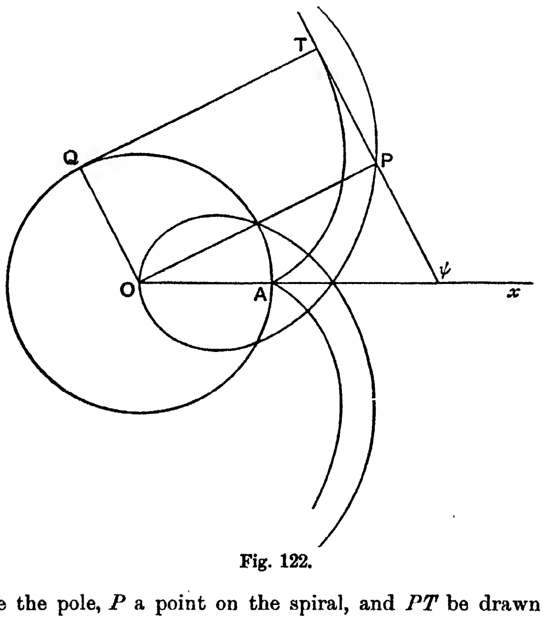


Fig. 122.

If O be the pole, P a point on the spiral, and PT be drawn perpendicular to OP touching the first negative pedal in T, then
$$PT = \frac{dr}{d\theta} = a.$$

Hence the normal TQ to the first negative pedal envelopes a circle with centre O and radius a. It is therefore an involute of the circle. If TQ touches this circle at Q, then $\rho = TQ = \text{arc } AQ$, where A is the cusp of the involute, *i.e.* $\rho = a\psi$, for $\psi = Q\hat{O}A$;

$$\therefore \frac{ds}{d\psi} = a\psi \quad \text{and} \quad s = \frac{a\psi^2}{2}.$$

(See *Diff. Calc.*, Art. 455.)

Otherwise: If $r = a\theta$ be the locus of P, r, θ being the polar coordinates of the foot of the perpendicular from the pole upon a tangent to the first negative pedal, the tangential polar equation of the pedal is $p = a\psi$;

$$\therefore \frac{ds}{d\psi} = p + \frac{d^2p}{d\psi^2} = a\psi; \quad \therefore s = \frac{a\psi^2}{2}.$$

543. **To obtain the Polar Equation from the Intrinsic.**

When the intrinsic equation $s = F(\psi)$ is given, and it is desired to get the equivalent polar equation, it is usually best to obtain the Cartesian coordinates of a point on the curve first, as above, from

$$x = \int \cos\psi\, F'(\psi)\, d\psi, \quad y = \int \sin\psi\, F'(\psi)\, d\psi,$$

and then, after integration, to form

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}$$

as functions of ψ, and finally to eliminate ψ, when the resulting equation will be the relation between r and θ.

If we attack the problem directly without the intervention of Cartesians, we have

$$\frac{(r^2 + r_1^{\,2})^{\frac{3}{2}}}{r^2 + 2r_1^{\,2} - rr_2} = \rho = F'(\psi) = F'(\theta + \phi)$$
$$= F'\left(\theta + \tan^{-1}\frac{r}{r_1}\right)$$

which is a troublesome second order differential equation; but one which, of course, theoretically furnishes the required relation between r and θ.

544. **Illustrative Examples.**

Ex. 1. Find the s, ψ relation for the equiangular spiral

$$r = ae^{\theta \cot \alpha}.$$

Here $\phi = \alpha$, $\psi = \theta + \alpha$,

$$s = a \operatorname{cosec} \alpha \int e^{\theta \cot \alpha} d\theta = \frac{a}{\cos \alpha} e^{\theta \cot \alpha}$$

the constant being determined so that s shall be measured from the point at which $\theta = -\infty$, *i.e.* from the pole;

$$\therefore\ s = \frac{a}{\cos\alpha} e^{(\psi-\alpha)\cot\alpha}.$$

Ex. 2. Conversely, find the polar equation corresponding to

$$s = \frac{a}{\cos\alpha} e^{(\psi-\alpha)\cot\alpha}.$$

We have $x\dfrac{\sin\alpha}{a} = \displaystyle\int \cos\psi\, e^{(\psi-\alpha)\cot\alpha} d\psi = \frac{e^{\overline{\psi-\alpha}\cot\alpha}}{\operatorname{cosec}\alpha}\cos(\psi-\alpha),$

$$y\frac{\sin\alpha}{a} = \int \sin\psi\, e^{(\psi-\alpha)\cot\alpha} d\psi = \frac{e^{\overline{\psi-\alpha}\cot\alpha}}{\operatorname{cosec}\alpha}\sin(\psi-\alpha),$$

the constants vanishing if we make $x=a$ and $y=0$, when $\psi=\alpha$;

$$\therefore\ \frac{r}{a} = e^{\overline{\psi-\alpha}\cot\alpha};$$

and $\qquad \tan\theta = \tan(\psi-\alpha); \qquad \therefore\ \psi = \theta+\alpha;$

$$\therefore\ r = ae^{\theta\cot\alpha}.$$

545. **Intrinsic Equation deduced from the Tangential Polar.**

When the tangential polar equation of the curve is given,

$$p = F(\psi), \quad \text{say},$$

we have at once

$$\frac{ds}{d\psi} = p + \frac{d^2p}{d\psi^2} = F(\psi) + F''(\psi),$$

and

$$s = F'(\psi) + \int F(\psi)\,d\psi,$$

the intrinsic equation required.

546. **Tangential Polar form deduced from the Intrinsic Equation.**

To get back to the tangential polar form from the intrinsic equation

$$s = f(\psi),$$

we have, of course, $\qquad \dfrac{d^2p}{d\psi^2} + p = f'(\psi). \quad \text{.......................} (1)$

To solve this differential equation we may either say at once

$$p = A\sin\psi + B\cos\psi + \frac{1}{D^2+1} f'(\psi),$$

and perform the operation indicated. (See Integral for Beginners, Chap. XVI.), or we may proceed thus:

(a) multiply (1) by $\cos\psi$ and then by $\sin\psi$, giving respectively,

$$\frac{d}{d\psi}\left(\frac{dp}{d\psi}\cos\psi+p\sin\psi\right)=f'(\psi)\cos\psi$$

and
$$\frac{d}{d\psi}\left(\frac{dp}{d\psi}\sin\psi-p\cos\psi\right)=f'(\psi)\sin\psi;$$

(b) integrating we have

$$\left.\begin{aligned}\frac{dp}{d\psi}\cos\psi+p\sin\psi&=\int_0^\psi f'(\psi)\cos\psi\,d\psi+A,\\ \frac{dp}{d\psi}\sin\psi-p\cos\psi&=\int_0^\psi f'(\psi)\sin\psi\,d\psi-B,\end{aligned}\right\}$$

where A and B are arbitrary constants;

(c) eliminating $\frac{dp}{d\psi}$,

$$p=\sin\psi\int_0^\psi f'(\psi)\cos\psi\,d\psi-\cos\psi\int_0^\psi f'(\psi)\sin\psi\,d\psi$$
$$+A\sin\psi+B\cos\psi;$$

and the tangential polar result is obtained.

The result may obviously be written as

$$p-A\sin\psi-B\cos\psi=\int_0^\psi f'(\omega)\sin(\psi-\omega)\,d\omega.$$

Moreover, if we choose our origin of measurement of p to be such that A and B both vanish, and suppose s to have been measured from a point where $\psi=0$, so that $f(0)=0$, we may integrate by parts and further reduce this equation to

$$p=\int_0^\psi f(\omega)\cos(\psi-\omega)\,d\omega.$$

547. **Intrinsic Equation deduced from the Pedal Equation.**

When the pedal equation (p, r) is given, say $p=f(r)$,

$$\frac{dr}{ds}=\cos\phi=\sqrt{1-\frac{p^2}{r^2}}.$$

Then s can be found in terms of r by integrating

$$s=\int\frac{r\,dr}{\sqrt{r^2-[f(r)]^2}}. \quad\text{.............................(1)}$$

Again,
$$\frac{ds}{d\psi}=\rho=\frac{r\,dr}{dp}=\frac{r}{f'(r)}. \quad\text{.............................(2)}$$

If r be eliminated between equations (1) and (2) we get a differential equation between s and ψ, whose solution furnishes the intrinsic equation sought.

548. Ex. Consider $p=r\sin\alpha$ (equiangular spiral).

$$s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}=\int\frac{r\,dr}{r\cos\alpha}=\frac{r}{\cos\alpha},$$

$$\frac{ds}{d\psi}=\frac{r\,dr}{dp}=\frac{r}{\sin\alpha}=s\cot\alpha,$$

$$\frac{ds}{s}=\cot\alpha\,d\psi,$$

$$\log s=\psi\cot\alpha+\text{constant},$$

$$s=Ce^{\psi\cot\alpha}.$$

549. **Pedal Equation from the Intrinsic.**

Conversely, if it be required to derive the pedal equation from the intrinsic equation $s=f(\psi)$, we have

$$r\frac{dr}{dp}=\rho=\frac{ds}{d\psi}=f'(\psi),\quad\text{.......................................}(1)$$

and $$p=\sin\psi\int^{\psi}f'(\psi)\cos\psi\,d\psi-\cos\psi\int^{\psi}f'(\psi)\sin\psi\,d\psi.\quad\text{...}(2)$$

Upon elimination of ψ we have a differential equation between $\frac{dr}{dp}$, r and p, which when solved gives the required *p-r* equation.

550. Ex. Starting with $s=Ce^{\psi\cot\alpha}$,

$$r\frac{dr}{dp}=\frac{ds}{d\psi}=C\cot\alpha\,e^{\psi\cot\alpha},$$

and $$p=\sin\psi\int C\cot\alpha\,e^{\psi\cot\alpha}\cos\psi\,d\psi-\cos\psi\int C\cot\alpha\,e^{\psi\cot\alpha}\sin\psi\,d\psi$$

$$=C\cot\alpha\,e^{\psi\cot\alpha}\frac{\sin\psi\cos(\psi-\alpha)-\cos\psi\sin(\psi-\alpha)}{\operatorname{cosec}\alpha},$$

i.e. $$p=C\cos\alpha\sin\alpha\,e^{\psi\cot\alpha};$$

$\therefore$ dividing, $$\frac{r}{p}\frac{dr}{dp}=\frac{1}{\sin^2\alpha},$$

i.e. $$r^2=\frac{p^2}{\sin^2\alpha},\quad\text{if } p \text{ and } r \text{ are taken to vanish together,}$$

i.e. $$p=r\sin\alpha.$$

551. Variations on these modes of procedure may of course be adopted to suit special cases.

552. WELL-KNOWN INTRINSIC EQUATIONS.

The following are the most common intrinsic equations of the "well-known" curves:

(1) For the circle, $s=a\psi$, *Diff. Calc.*, p. 273.

(2) For the catenary, $s=c\tan\psi$, " p. 273.

(3) For the cycloid, $s=4a\sin\psi$, " p. 340.

(4) For the epi- or hypo-cycloid, $\left\{\begin{array}{l} s=\dfrac{4b}{a}(a+b)\cos\dfrac{a}{a+2b}\psi; \\ \text{or, generally,} \\ \qquad s=A\sin B\psi \\ \text{or} \quad s=A\cos B\psi, \end{array}\right\}$ " p. 345.

(5) Involute of a circle, $\left\{s=\dfrac{a\psi^2}{2},\right.$ " p. 275.

(6) Parabola $y^2=4ax$, $\left\{\begin{array}{l}\rho=2a\sec^3\psi, \\ s=a[\sec\psi\tan\psi+\log(\sec\psi+\tan\psi)],\end{array}\right\}$ *Int. Calc.*, Art. 517.

(7) Evolute of a parabola $27ay^2=4(x-2a)^3$, $\left\{s=2a(\sec^3\psi-1),\right.$ *Diff. Calc.*, p. 275.

Semicubical parabola $3ay^2=2x^3$, $\left\{9s=4a(\sec^3\psi-1)\right\}$ *Int. Calc. for Beginners*, p. 151.

(8) Equiangular spiral, $\left\{s=Ae^{m\psi}.\right.$

(9) Tractory, $s=c\log\operatorname{cosec}\psi$, *Diff. Calc.*, p. 358.

(10) Cardioide $r=a(1+\cos\theta)$, $r=a(1-\cos\theta)$, $\left\{\begin{array}{l} s=4a\sin\left(\dfrac{\psi}{3}-\dfrac{\pi}{6}\right), \\ \text{included as a case of} \\ \quad\text{the epi-cycloid,} \\ s=4a\left(1-\cos\dfrac{\psi}{3}\right), \end{array}\right\}$ *Int. Calc.*, Art. 542.

(11) Catenary of equal strength, $y=a\log\sec\dfrac{x}{a}$, $\left\{\begin{array}{l} s=a\,\mathrm{gd}^{-1}\psi, \\ \textit{i.e. } s=a\log\tan\left(\dfrac{\pi}{4}+\dfrac{\psi}{2}\right), \end{array}\right\}$ *Int. Calc.*, Ex. 8, Art. 517.

553. Intrinsic Equation of the Evolute.

Let $s=f(\psi)$ be the equation of the given curve. Let s' be the length of the arc of the evolute measured from some fixed point A to any other point Q on the evolute. Let O and P be the points on the original curve corresponding to the points A, Q on the evolute; ρ_0, ρ the radii of curvature at O and P; ψ the angle the tangent QP makes with OA produced, or, which is the same thing, the angle the tangent PT makes with the tangent at O.

Then
$$s'=\rho-\rho_0=\frac{ds}{d\psi}-\rho_0$$
$$s'=f'(\psi)-\rho_0.$$

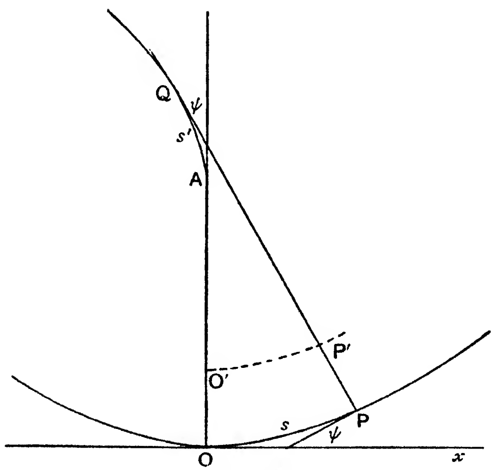


Fig. 123.

554. Intrinsic Equation of an Involute.

With the same figure, if the curve AQ be the original curve given by the equation $s'=f(\psi)$, we have

$$\rho=s'+\rho_0, \quad \rho=\frac{ds}{d\psi};$$
$$\frac{ds}{d\psi}=f(\psi)+\rho_0,$$
$$s=\int[f(\psi)+\rho_0]\,d\psi.$$

ρ_0 is now an arbitrary constant, and

$$s = \int f(\psi)\, d\psi + \rho_0 \psi + \text{const.}$$

The intrinsic equation of an involute

$$s = \rho_0 \psi + \int_0^\psi f(\psi)\, d\psi$$

is the particular involute whose radius of curvature at O is ρ_0. For any of the other involutes, the whole set of which form a family of parallel curves, replace ρ_0 by a where a is the radius of curvature of the parallel, corresponding to ρ_0 for the particular involute considered.

Then $$s = a\psi + \int_0^\psi f(\psi)\, d\psi.$$

The difference of the arcs OP, $O'P'$ of these parallel curves is therefore $(\rho_0 - a)\psi$; and if the involutes form closed ovals, the tangent making one complete revolution, the difference of their perimeters is $2\pi(\rho_0 - a)$.

555. In the case of an **involute of a circle**, already discussed in

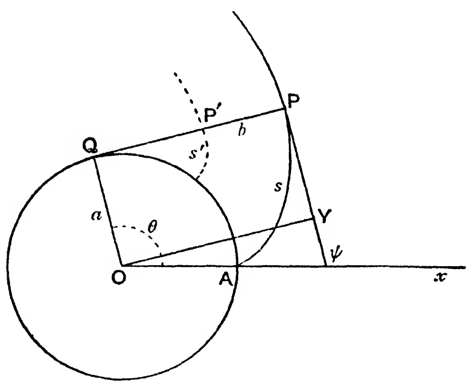


Fig. 124.

Art. 542, if a be the radius, O the centre, A the cusp of the involute, and Q the point of contact of the radius of curvature at P,

$$PQ = a\theta = a\psi \quad \text{and} \quad s = \int a\psi\, d\psi = \frac{a\psi^2}{2}.$$

For a parallel traced by a point at distance b from P

$$s' = \int (a\psi - b)\, d\psi$$

$$= \frac{a\psi^2}{2} - b\psi + C$$

$$= \frac{a}{2}\left(\psi - \frac{b}{a}\right)^2, \text{ if we measure } s' \text{ so that}$$

$$s' = 0 \text{ when } \psi = \frac{b}{a},$$

i.e. another involute of the circle.

556. In the case of the **epi- and hypo-cycloids**

$$s = A \sin B\psi,$$

the evolute is $$s' = AB \cos B\psi' - \rho_0,$$

or, dropping the accent, and writing $\rho_0 + s' = s$, *i.e.* changing the origin of measurement of s suitably,

$$s = AB \cos B\psi,$$

s being measured from the point where $\psi = \frac{\pi}{2B}$, or $s = AB \sin B\psi$ if we choose a suitable initial tangent, viz. that at the point from which s is measured.

Hence the evolute of an epi- or hypo-cycloid is a similar epi- or hypo-cycloid.

Putting $B = 1$ we have a case which shows that the evolute of a cycloid is an equal cycloid.

Supplying the values of A and B (Art. 540), the equations of the curve and of the evolute may be written

$$s = \frac{4b}{a}(a+b) \cos \frac{a}{a+2b}\psi, \quad s' = 4b\frac{a+b}{a+2b} \cos \frac{a}{a+2b}\psi';$$

with a different origin of measurement for s' and a different initial tangent, and we can compare the linear dimensions of the two curves, viz.

$$\frac{\text{linear dimensions of evolute}}{\text{linear dimensions of original curve}} = \frac{a}{a+2b};$$

e.g. in the case of a cardioide, for which $a = b$, the evolute is another cardioide of one-third the linear dimensions of the former.

557. **Whewell's Theorem.**

An interesting theorem is quoted by Boole from Whewell's *Memoir* (above referred to) with regard to the ultimate form to which the successive involutes of a given curve tend, the involutes being such as have equal "tails."

Whewell takes as his original curve $s = F(\psi)$, which he

supposes capable of expansion in powers of ψ, and s vanishing with ψ, so that

$$s = A_1\psi + A_2\psi^2 + A_3\psi^3 + \ldots,$$

and he further supposes the successive involutes to be defined as having the same "rectilinear tail" at starting.

Let P_0P be the original curve, and Q_0Q, R_0R, $S_0S, \ldots$ the successive involutes, and the several "tails" Q_0P_0, R_0Q_0,

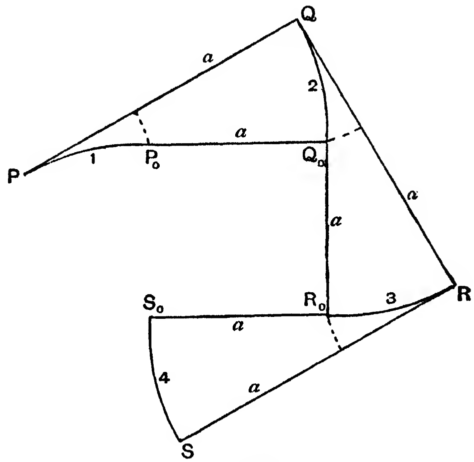


Fig. 125.

S_0R_0, ... all equal, say $= a$, and take s_1, s_2, s_3, ... the successive arcs. The ψ's are all equal if measured from the respective initial tangents.

Then for the arc Q_0Q, viz. the first involute,

$$\frac{ds_2}{d\psi} = a + s_1 = a + A_1\frac{\psi}{1!} + A_2\frac{\psi^2}{2!} + \ldots,$$

$$s_2 = a\psi + A_1\frac{\psi^2}{2!} + A_2\frac{\psi^3}{3!} + \ldots,$$

no constant being required, as each arc vanishes with ψ.

Similarly, $$s_3 = a\psi + a\frac{\psi^2}{2!} + A_1\frac{\psi^3}{3!} + A_2\frac{\psi^4}{4!} + \ldots,$$

$$s_4 = a\psi + a\frac{\psi^2}{2!} + a\frac{\psi^3}{3!} + A_1\frac{\psi^4}{4!} + \ldots.$$

Proceeding thus,

$$s_n = a\left(\psi + \frac{\psi^2}{2!} + \frac{\psi^3}{3!} + \dots + \frac{\psi^{n-1}}{(n-1)!}\right) + \left(A_1\frac{\psi^n}{n!} + A_2\frac{\psi^{n+1}}{(n+1)!} + \dots\right).$$

And when n is very large the terms in the first bracket (which are unaffected by the form of the original curve) approximate to $e^\psi - 1$.

And those in the second bracket have coefficients which are ultimately infinitesimally small.

Hence the involutes tend to the limiting form $s = a(e^\psi - 1)$, *i.e.* an equiangular spiral of angle $\frac{\pi}{4}$.

In a similar manner we note that if we start off with a curve in which $s = F(\phi)$, where F is an algebraic expression of the n^{th} degree,

$$\text{say} \quad s = a\frac{\psi^n}{n!} + b\frac{\psi^{n-1}}{(n-1)!} + \dots + j\psi + k,$$

then, since the radii of curvature of the curve and its successive evolutes are

$$\rho = \frac{ds}{d\psi},$$

$$\rho_1 = \frac{d\rho}{d\psi} = \frac{d^2 s}{d\psi^2},$$

$$\rho_2 = \frac{d\rho_1}{d\psi} = \frac{d^3 s}{d\psi^3}, \text{ etc.,}$$

it follows that $\rho_{n-1} = a$.

Hence the $(n-1)^{\text{th}}$ evolute is a circle.

$$\text{Therefore} \quad s = a\frac{\psi^n}{n!} + b\frac{\psi^{n-1}}{(n-1)!} + \dots + j\psi + k$$

is one of the $(n-1)^{\text{th}}$ involutes of a circle of radius a, or parallels to such involutes, the "tails" being the successive coefficients k, j, etc.

558. **Involute of a Catenary.**

Ex. The intrinsic equation of the catenary is $s = c \tan\psi$.
Hence the intrinsic equation of its evolute is

$$s = c\sec^2\psi - \rho_0,$$

and ρ_0 is the radius of curvature at the vertex $= c$

$$\left[\text{for } \rho = \frac{ds}{d\psi} = c\sec^2\psi = c, \text{ when } \psi = 0\right].$$

Hence the evolute is $s = c(\sec^2\psi - 1) = c\tan^2\psi$.

The intrinsic equation of an involute is

$$s=\int (c\tan\psi+A)\,d\psi$$

$$=c\log\sec\psi+A\psi+\text{constant},$$

and if s be so measured that $s=0$ when $\psi=0$, we have

$$s=c\log\sec\psi+A\psi.$$

559. **Tracing of a Curve from the Intrinsic Equation** $s=f(\psi)$.

(1) Generally it is best to obtain the Cartesian or polar form of equation if possible by the methods of Arts. 537, 543, and to trace the curve therefrom by the usual rules (*Diff. Calc.*, Chap. XII.).

(2) If this be not possible by reason of the failure to integrate the expressions occurring in the articles cited, find the curvature $\frac{d\psi}{ds}$, and examine how the curvature changes with ψ. Note also concavity or convexity to the origin according as $\frac{ds}{d\psi}$ is + or −. Note whether s becomes unreal for any values of ψ, and whether ρ changes sign for any values of ψ. Also the inflexions where $\frac{ds}{d\psi}=\infty$, and the cusps where $\frac{ds}{d\psi}=0$.

Tabulate corresponding values of ψ, s and ρ.

Observe whether a change of sign in ψ would alter the value of s. If not there is symmetry about the initial line from which ψ is measured.

Examine whether

$$\left.\begin{aligned} x&=\int_0^\psi \cos\chi f'(\chi)\,d\chi,\\ y&=\int_0^\psi \sin\chi f'(\chi)\,d\chi, \end{aligned}\right\}$$

even though not (as in the case considered) integrable in general terms, can be evaluated as definite integrals for any particular values of ψ. Approximate values of these integrals may lead to important information as to the position of some points through which the curve passes. For accurate plotting the tabulated values of these integrals for various values of ψ in general becomes necessary. For a general idea of the shape of the curve when close accuracy of plotting is not

necessary, an examination of the integrals and the behaviour of the integrand may furnish sufficient information.

560. Ex. Trace the curve $ks^2=\psi$, (k +ve). **Cornu's Spiral.***

Here $$\rho=\frac{ds}{d\psi}=\frac{1}{2ks}.$$

The curvature continuously increases with s. Hence, as s increases, the osculating circle at any point will contain the whole of the remainder of the curve; and ρ diminishes more and more slowly as s increases.

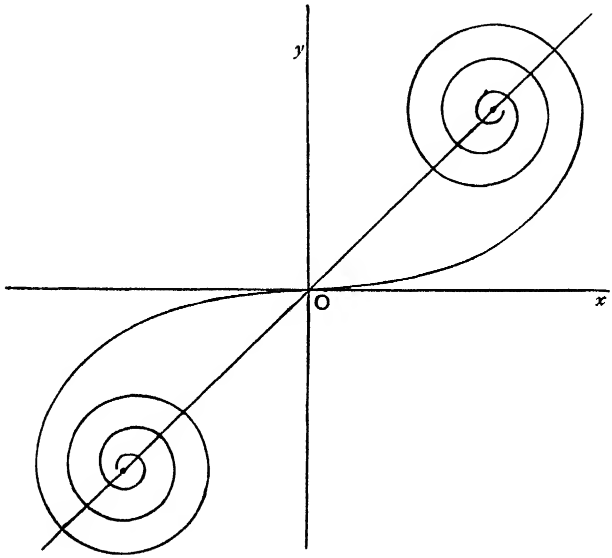


Fig. 126.

Negative values of ψ would give unreal values of s. Each value of ψ gives two values of s, one positive, one negative. It is to be inferred that the origin of measurement of s is a point of symmetry.

We have $$x=\int \cos\psi\, ds=\int_0^s \cos ks^2 ds,$$
$$y=\int \sin\psi\, ds=\int_0^s \sin ks^2 ds.$$

These integrals are not integrable in general terms.

But $\int_0^\infty \cos ks^2 ds=\frac{\sqrt{\pi}}{2\sqrt{2k}}$ is a known result (Art. 1163, Ch. XXVIII.), and $\int_0^\infty \sin ks^2 ds$ has the same value. These are known as Fresnel's integrals.

* *Journal de Physique*, t. iii., 1874, M. A. Cornu.

Hence, when s becomes very large the curve dwindles down to a point on the line $y=x$ after an infinite number of convolutions about the point. And the point is at a distance from the origin $=\frac{\sqrt{\pi}}{2\sqrt{k}}$. The value of ρ is infinite and changes sign when $s=0$. There is therefore a point of inflexion there.

Also
$$\frac{dx}{ds}=\cos ks^2, \quad \frac{dy}{ds}=\sin ks^2,$$

which show that the tangent is parallel to the initial line

$$\text{when } ks^2=0, \quad \pi, \quad 2\pi \ldots,$$

and perpendicular to it

$$\text{when } ks^2=\frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2} \ldots,$$

which, indeed, is obvious from the equation $ks^2=\psi$.

Taking k as unity for convenience,

	$\psi=0$,	1,	2,	3,	4,	5,	6,	∞,
give	$s=0$,	± 1,	$\pm 1{\cdot}414$,	$\pm 1{\cdot}732$,	± 2,	$\pm 2{\cdot}236$,	$\pm 2{\cdot}449 \ldots$	$\pm \infty$,
	$\rho=\infty$,	0·500	0·354,	0·289,	0·250,	0·224,	0·204 ...	0.

We are now in a position to form an idea of the curve which is shown in the figure.

This spiral is of considerable importance in the theory of light, the length and direction of the radius vector at any point giving a graphical representation of the amplitude of the resultant of a system of superposed vibrations.*

The values of Fresnel's Integrals

$$C=\int_0^v \cos\frac{\pi v^2}{2}\,dv, \quad S=\int_0^v \sin\frac{\pi v^2}{2}\,dv,$$

have been calculated for values of v from 0 to ∞ by Gilbert.† The tabulated values are necessary for accurate plotting.

The general methods of evaluating these integrals are discussed by Verdet (*Œuvres*, vol. v.), Fresnel (*Œuvres*, tom. i.), Knockenhauer (*Die undul. des Lichts*), Cauchy (*Comptes Rendus*, t. xv.) and others. See Preston, *Theory of Light*, page 220 onwards.

Incidentally the spiral exhibits graphically the march of these integrals, the abscissa and the ordinate representing the integrals and s being the independent variable, showing their oscillatory character.

Thus $x=\int_0^s \cos\frac{\pi s^2}{2}\,ds$ increases from $s=0$ to $s=\sqrt{1}$, decreases from $s=\sqrt{1}$ to $s=\sqrt{2}$, increases from $s=\sqrt{2}$ to $s=\sqrt{3}$, and so on. And similarly for y.

These integrals will be discussed more fully later.

* Preston, *Theory of Light*, Art. 141, onwards.

† *Mém. couronnés de l' Acad. de Bruxelles*, t. xxxi., 1863.

561. Length of Arc of First Positive Pedal Curve.

Let p be the perpendicular from the origin upon the tangent to any curve, and χ the angle this perpendicular makes with the initial line. We may then regard p, χ as the polar coordinates of a current point on the pedal curve.

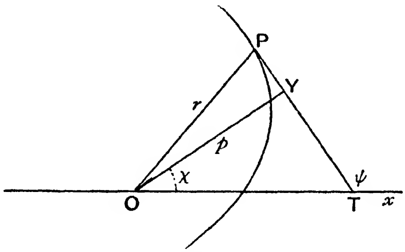


Fig. 127.

Hence the length of an arc s' of the pedal curve may be calculated by the formula

$$s'=\int\sqrt{p^2+\left(\frac{dp}{d\chi}\right)^2}\,d\chi. \quad\text{.......................(1)}$$

562. Ex. Apply the above method to find the length of any arc of the pedal of a circle with regard to a point on the circumference (*i.e.* a cardioide).

Here, if $2a$ be the diameter, we have from the figure,

$$p=OP\cos\frac{\chi}{2}=2a\cos^2\frac{\chi}{2}.$$

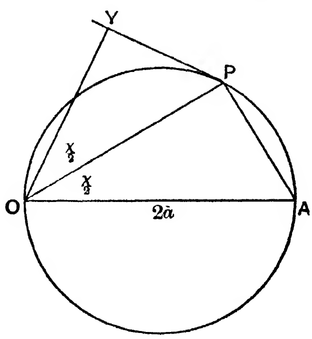


Fig. 128.

Hence, $\quad$ arc of pedal $=\displaystyle\int 2\sqrt{a^2\cos^4\frac{\chi}{2}+a^2\sin^2\frac{\chi}{2}\cos^2\frac{\chi}{2}}\,d\chi$

$$=\int 2a\cos\frac{\chi}{2}\,d\chi=4a\sin\frac{\chi}{2}+C.$$

The limits for the upper half of this curve are $\chi=0$ and $\chi=\pi$. Hence the whole perimeter of the pedal

$$=2\left[4a\sin\frac{\chi}{2}\right]_0^\pi$$

$$=8a.$$

563. **Arc of the Pedal Curve.**

Again, the tangent to the locus of Y, the foot of the perpendicular, makes with OY the same angle that the radius vector OP makes with the tangent at the corresponding point P of the original curve.

Thus $$\frac{dp}{ds'}=\frac{dr}{ds};$$

$$\therefore\quad s'=\int\frac{dp}{dr}\,ds=\int\frac{r}{\rho}\,ds,\quad\text{..........................(2)}$$

which again expresses the arc of the pedal in terms of elements of the original curve.

The result may be presented in various forms.

Thus $$s'=\int\frac{r}{\rho}\,ds=\int r\,d\psi,\quad\text{..........................(3)}$$

which is equivalent to (1) for

$$\psi=\frac{\pi}{2}+\chi\quad\text{and}\quad r^2=p^2+\left(\frac{dp}{d\psi}\right)^2.$$

Also $$s'=\int\frac{\sqrt{x^2+y^2}\,\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}\,dx,\quad\text{...............for Cartesians (4)}$$

or $$=\int\frac{r\left\{r^2+2\left(\dfrac{dr}{d\theta}\right)^2-r\dfrac{d^2r}{d\theta^2}\right\}}{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\,d\theta,\quad\text{.......for polars (5)}$$

or $$=\int\frac{r\dfrac{dp}{dr}}{\sqrt{r^2-p^2}}\,dr,\quad\text{............for pedal equations (6)}$$

(from equation 2).

564. **Arc of a First Negative Pedal.**

If the original curve be $r=f(\theta)$, then r,θ are the polar coordinates of the foot of the perpendicular from the pole upon

the tangent to the first negative pedal, whose tangential polar equation may therefore be written $p=f(\chi)$, χ being the angle the perpendicular to the tangent makes with the initial line

$$\left(\text{viz. } \chi=\psi-\frac{\pi}{2}\right).$$

Also
$$\frac{ds}{d\chi}=p+\frac{d^2p}{d\chi^2};$$

$$\therefore\ s=\frac{dp}{d\chi}+\int f(\chi)\,d\chi+\text{constant}.$$

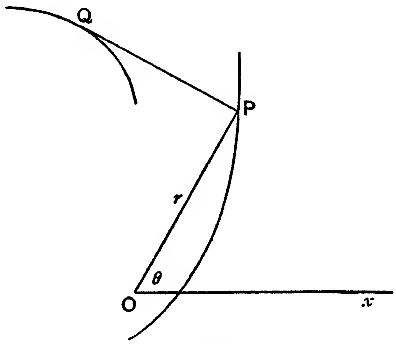


Fig. 129.

565. Ex. Find the intrinsic equation of the first negative pedal of an ellipse $\frac{l}{r}=1+e\cos\theta$, with regard to the pole.

Here
$$s=\frac{d}{d\chi}\frac{l}{1+e\cos\chi}+\int\frac{l}{1+e\cos\chi}\,d\chi;$$

$$\therefore\ s=\frac{el\sin\chi}{(1+e\cos\chi)^2}+\frac{l}{\sqrt{1-e^2}}\cos^{-1}\frac{e+\cos\chi}{1+e\cos\chi}+\text{const.}$$

If we choose to measure s from the point where $\chi=0$,

$$\text{the constant}=-\frac{l}{\sqrt{1-e^2}}\cos^{-1}1=0.$$

PROBLEMS.

1. Show that the whole length of the curve
$$y^2(a^2-y^2)=8a^2x^2 \quad \text{is} \quad \pi a\sqrt{2}.$$ [Oxford I. P., 1890.]

2. Find the whole length of the loop of the curve
$$3ay^2=x(x-a)^2.$$ [Oxford I. P., 1889.]

3. Show that the arcs of an equiangular spiral, measured from the pole to the different points of its intersection with another equiangular spiral having the same pole but a different angle, will form a series in geometrical progression. [TRINITY, 1884.]

4. Show that the length of an arc of the curve $y^n = x^{m+n}$ can be found in finite terms in the cases when $\frac{n}{2m}$ or $\frac{n}{2m}+\frac{1}{2}$ is an integer.

5. Evaluate the expressions,

$$\text{(i)} \int y\frac{dx}{ds}\,ds, \qquad \text{(ii)} \int x\frac{dy}{ds}\,ds, \qquad \text{(iii)} \int\left(\frac{x}{r^2}\frac{dy}{ds}-\frac{y}{r^2}\frac{dx}{ds}\right)ds,$$

wherein the line-integrals are taken round the perimeter of a closed curve. [ST. JOHN'S, 1890.]

6. If s be the length of the curve

$$r = a\tanh\frac{\theta}{2}$$

between the origin and $\theta = 2\pi$, and A be the area between the same points, show that $A = a(s - a\pi)$. [OXFORD I., 1888.]

7. Show that if the arc of the curve

$$r = a\tanh^n\frac{\theta}{2n} \quad (n \text{ being integral}),$$

measured from the origin, be called s, and if A be the corresponding area swept out by the radius vector from the origin,

$$A = \frac{a^2\theta}{2} - na^2\left[\left(\frac{r}{a}\right)^{\frac{1}{n}} + \frac{1}{3}\left(\frac{r}{a}\right)^{\frac{3}{n}} + \dots + \frac{1}{2n-1}\left(\frac{r}{a}\right)^{\frac{2n-1}{n}}\right],$$

$$\begin{cases} s + r = a\theta - 2na\left[\left(\frac{r}{a}\right)^{\frac{1}{n}} + \frac{1}{3}\left(\frac{r}{a}\right)^{\frac{3}{n}} + \dots + \frac{1}{n-2}\left(\frac{r}{a}\right)^{\frac{n-2}{n}}\right], \\ \hfill \text{if } n \text{ be odd and } > 2, \\ s + r = 2na\left[\log\cosh\frac{\theta}{2n} - \left\{\frac{1}{2}\left(\frac{r}{a}\right)^{\frac{2}{n}} + \frac{1}{4}\left(\frac{r}{a}\right)^{\frac{4}{n}} + \dots + \frac{1}{n-2}\left(\frac{r}{a}\right)^{\frac{n-2}{n}}\right\}\right], \\ \hfill \text{if } n \text{ be even,} \end{cases}$$

the results for $r = a\tanh\frac{\theta}{2}$ giving $2A = a(s-r) = a(2s - a\theta)$.

8. Show that the length of an arc of the Cissoid of Diocles

$$r = a\frac{\sin^2\theta}{\cos\theta}$$

is $a\sqrt{3}(z - \tanh^{-1}z)$ taken between limits θ_1 and θ_2 where

$$3z^2\cos\theta = 1 + 3\cos^2\theta.$$

9. Show that the intrinsic equation of the semicubical parabola

$$3ay^2 = 2x^3 \quad \text{is} \quad 9s = 4a(\sec^3\psi - 1).$$

10. In a certain curve

$$x = e^\theta \sin\theta, \quad y = e^\theta \cos\theta.$$

Show that $s = e^\theta\sqrt{2} + C$.

Also that for the curve

$$x = e^{a\theta}\sin b\theta, \quad y = e^{a\theta}\cos b\theta, \quad s = e^{a\theta}\sqrt{a^2+b^2} + C.$$

Name these curves.

11. Show that the length of an arc of the curve

$$x\sin\theta + y\cos\theta = f'(\theta),$$
$$x\cos\theta - y\sin\theta = f''(\theta),$$

is given by $s = f(\theta) + f''(\theta) + C$.

12. Trace the curve $y^2 = \dfrac{x}{3a}(a-x)^2$, and find the length of that part of the evolute which corresponds to the loop.

[ST. JOHN'S, 1881 AND 1891.]

13. Show that the curve whose pedal equation is $p^2 = r^2 - a^2$ has for its intrinsic equation $s = a\dfrac{\psi^2}{2}$.

What curve is this?

14. The coordinates of a point on a plane curve are given by the relations

$$x = a[(1-\theta^2)\cos\theta + 2\theta\sin\theta - 1],$$
$$y = a[(1-\theta^2)\sin\theta - 2\theta\cos\theta];$$

prove that

$$3sa^{\frac{1}{2}} = (2a+\rho)(\rho-a)^{\frac{1}{2}},$$

ρ being the radius of curvature at the point and s the arcual distance from the origin. [OXFORD II. P., 1888.]

15. The evolute of a parabola whose vertex is A meets the axis in C, and the parabola in Q. Find the perimeter of the figure bounded by AC, the parabolic arc AQ, and the arc of the evolute CQ. [OXFORD I. P., 1889.]

16. Prove that the length of the first negative pedal, taken with respect to the origin, of the loop of the folium of Descartes

$$x^3 + y^3 - 3axy = 0$$

is equal to

$$6a - a\{\pi - \sqrt{2}\log(\sqrt{2}+1)\}.$$

17. Find the length of the arc between two consecutive cusps of the curve

$$(c^2 - a^2)p^2 = c^2(r^2 - a^2).$$

[OXFORD I. P., 1889.]

18. Show that the length of the arc of the hyperbola $xy=a^2$ between the limits $x=b$ and $x=c$ is equal to the arc of the curve $p^2(a^4+r^4)=a^4r^2$, between the limits $r=b$, $r=c$. [OXFORD I. P., 1888.]

19. By means of the formula $s=\int \frac{r\,dr}{\sqrt{r^2-p^2}}$, find the length of the curve $r=a\sin^2\frac{\theta}{2}$.

[COLLEGES α, 1887.]

20. If s be the arc of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ measured from the end of the major axis to a point whose eccentric angle is ϕ, prove that

$$s+ae^2\cos\phi\sin\theta=\int_0^\theta \sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta,$$

where
$$\theta=\tan^{-1}\left(\frac{a}{b}\tan\phi\right).$$
[COLLEGES α, 1883.]

21. Show that the circumference of an ellipse can be expressed either as

$$4a\int_0^{\frac{\pi}{2}}(1-e^2\sin^2\theta)^{\frac{1}{2}}\,d\theta$$

or as
$$4a(1-e^2)\int_0^{\frac{\pi}{2}}(1-e^2\sin^2\theta)^{-\frac{3}{2}}d\theta,$$

where a is the semi-major axis, e the eccentricity. [TRINITY, 1887.]

22. Show that the three-cusped hypocycloid has equations of the forms

(i) $p=b\cos 3\psi$,

(ii) $r^4+8br^3\cos 3\theta+18b^2r^2=27b^4$.

Show that the length of an arc of the inverse of this curve with respect to the centre is proportional to

$$\tan^{-1}(2\sqrt{2}\sin 3\psi).$$
[ST. JOHN'S, 1887.]

23. Prove that the intrinsic equation of the curve

$$r^{\frac{1}{5}}=a^{\frac{1}{5}}\cos\frac{1}{5}\theta \quad \text{is} \quad \frac{s}{a}=\frac{5}{16}\psi+\frac{5}{4}\sin\frac{\psi}{3}+\frac{5}{32}\sin\frac{2\psi}{3},$$

where s and ψ are measured from the point a, 0, and the tangent at that point. [ST. JOHN'S, 1889.]

24. A circle of perimeter S and area A rolls externally in its plane entirely round an oval curve of perimeter S and area B. Prove that its centre describes an oval of perimeter $2S$ and area $3A+B$. [OXFORD I. P., 1918.]

25. Find the centroid of a sector bounded by two radii vectores, and an arc of the curve whose polar equation is

$$r^2 = a^2(1 - \sin 2\theta)(1 + \sin 2\theta)^{-1},$$

and show that an arc of this curve is expressible as

$$\frac{5a}{2}\int \frac{\cos^2\chi\, d\chi}{(1+\sqrt{5}\sin\chi)\sqrt{1-5\sin^2\chi}}.$$

[MATZ, *Educ. Times.*]

26. A rod moves always to pass through a fixed point and have one extremity on a straight line distant h from the point. Show that the arc of the curve traced out by its centre of instantaneous rotation, as the rod moves from the perpendicular position to one inclined at 45° to the line, is

$$\tfrac{1}{4}\{\log(\sqrt{5}+2) + 2\sqrt{5}\}\, h.$$ [MATH. TRIPOS, 1883.]

27. On the tangent at any point P of a curve, PT is taken equal to the radius vector of P; show how to find the length of any arc of the locus of T. For example, take the equiangular spiral and verify the result geometrically. [ST. JOHN'S, 1884.]

28. Find the arc of the curve enveloped by the line

$$x\cos\phi + y\sin\phi = (a\cos^2\phi + b\sin^2\phi)^{-3}$$

between the points corresponding to $\phi = 0$, $\phi = \dfrac{\pi}{2}$. [ST. JOHN'S, 1891.]

29. Find the whole area of the curve

$$x = a\sin\theta - b\sin 2\theta,$$
$$y = a\cos\theta - b\cos 2\theta,$$

and show that the whole length of its perimeter is equal to that of an ellipse whose semiaxes are $a+2b$, $a-2b$. [COLLEGES α, 1885.]

30. Prove that if s be the arc of the curve

$$\left.\begin{aligned} r &= a\sec\alpha, \\ \theta &= \tan\alpha - \alpha, \end{aligned}\right\}$$

where α is a variable parameter, measured from the initial line to a point P on the curve, and if A be the area bounded by the curve, the initial line, and the radius vector to P, then

$$9A^2 = 2as^3.$$

Find the area swept out in any portion of its progress by the intercept of the tangent to the curve between the curve and the first positive pedal with regard to the origin. [TRINITY, 1890.]

31. If a curve be given by $r^{2n} = \sin^2\phi + m^2\cos^2\phi$, where r is the radius vector and ϕ the angle it makes with the tangent, show that

$$m \tan \frac{1}{m}(\phi - n\theta) = \tan \phi,$$

θ being the angle the radius vector makes with the initial line (which is to be appropriately chosen).

Obtain also a formula for the rectification of the curve. (The result is not obtainable in *finite terms.*) [I. C. S., 1898.]

32. Consider the nature of the curves

$$\text{(i)}\ s\psi^2 = a, \qquad \text{(ii)}\ \frac{\psi}{2\pi} = \sin\frac{s}{a}, \qquad \text{(iii)}\ s = l\sin m\psi,$$

when $m < 1$ and when $m > 1$.

33. Given a closed oval of continuous curvature without any singularities: a series of parallel curves is drawn. Prove that if A denote the area of any one of them and l its perimeter, then

$$4\pi A - l^2$$

is the same for all. [I. C. S., 1895.]

34. In the equation of the curve $r = a + \epsilon u$, a and ϵ are constants, the latter being small; and u is a function of θ finite for all values of θ and periodic, with a period 2π. Show that if A denote the area of the curve, then its length is $2\sqrt{\pi A}$ accurately as far as small quantities of the first order inclusive. [I. C. S., 1896.]

35. The area of an ellipse differs from that of its auxiliary circle by 10 per cent. of the area of the latter. Show that the perimeter of the ellipse differs from that of the auxiliary circle by 4·93 per cent. approximately of the perimeter of the latter. [I. C. S., 1910.]

36. Assuming that for the catenary formed by a hanging elastic wire

$$\frac{x}{c} = u + k\,\mathrm{sh}\,u, \quad \frac{y}{c} = \mathrm{ch}\,u + \tfrac{1}{2}k\,\mathrm{ch}^2 u,$$

prove that

$$\frac{s}{c} = \tfrac{1}{2}ku + \mathrm{sh}\,u + \tfrac{1}{4}k\,\mathrm{sh}\,2u,$$

reducing to the common catenary when $k = 0$ and approximating to a parabola when k is large. [B.A. Hon. Lond., 1899.]

37. In the cycloid $y = \dfrac{s^2}{8a}$. Show that the only curve for which *both* x and y are finite integral functions of s is a straight line. [Oxf. I. P., 1913.]

38. Find the Cartesian equation (choosing convenient axes of coordinates) of the curve in which

$$\rho^2/a^2 = (d\rho/ds)^2 + 1. \qquad \text{[Oxf. I. P., 1917.]}$$

39. Find the intrinsic equation of the curve $27ay^2 = 4x^3$. Prove that the involutes of the curve $27ay^2 = 4x^3$ are given by the equations

$$x = a\tan^2\psi + c\cos\psi - 2a,$$
$$y = -2a\tan\psi + c\sin\psi,$$

c being an arbitrary constant.

What happens when $c = 0$? [Oxf. I. P., 1915.]

40. Show that the length of a quadrant of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is equal to $\frac{3a}{2}$, and find the length of one quadrant of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

[Math. Tripos, Part I., 1910.]

41. Trace the form of the curve

$$x(1+t^2) = 1 - t^2, \quad y(1+t^2) = 2t,$$

as t increases from $-\infty$ to ∞, and show that its area is π.

Find also the length of any arc of the curve in terms of t.

[Math. Trip., Part I., 1910.]

42. Show that the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the point of intersection of the parabola and $3y = 8x$ is

$$a(\log 2 + \tfrac{15}{16}). \qquad \text{[Math. Trip. I., 1908.]}$$

43. Prove that the perimeter of an ellipse of small eccentricity e and semiaxes a, b is equal to

$$\tfrac{1}{2}\pi\{3(a+b) - 2\sqrt{ab}\},$$

neglecting e^7 and higher powers. [Math. Trip. I., 1917.]

44. Prove that the length of an ellipse may be expressed by $\iint \frac{dS}{\rho}$ taken over the area, where dS is an element of the area of the ellipse and ρ the radius of curvature of the similar, similarly situated and concentric ellipse passing through the element dS.

[Colleges, 1892.]

45. Find the intrinsic equations of a circle, a catenary, and a cycloid, and trace the curves $s = a\phi^3$, $s\phi = a$ and $s^2 = a^2\phi$.

At any point P of a cycloid the tangent is produced to a length PT equal to the arc measured from the vertex, and at T a perpendicular is drawn equal to the radius of curvature at P. Prove that the locus of the extremity of this perpendicular is the same cycloid moved parallel to its axis through a distance equal to twice the diameter of the generating circle. [St. John's College, 1882.]

CHAPTER XVII.

RECTIFICATION (II.).

CENTRAL CONIC, LIMAÇON, LEMNISCATE, TROCHOIDS, ETC. APPLICATION OF ELLIPTIC FUNCTIONS.

566. We have reserved for a separate chapter the consideration of those curves whose rectification needs the employment of Elliptic Integrals.

567. **Rectification of the Ellipse. Arc measured from the End of the MINOR AXIS.**

If θ be the eccentric angle of a point x, y on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

we have

$$x=a\cos\theta, \qquad y=b\sin\theta,$$
$$dx=-a\sin\theta\,d\theta, \quad dy=b\cos\theta\,d\theta.$$

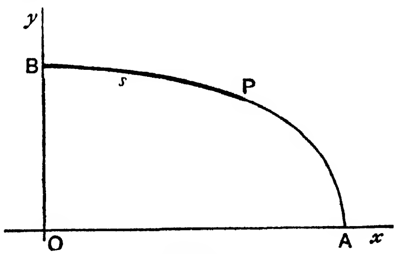


Fig. 130.

Hence $$ds^2=(a^2\sin^2\theta+b^2\cos^2\theta)\,d\theta^2,$$

and $$s=a\int_\theta^{\frac{\pi}{2}}(1-e^2\cos^2\theta)^{\frac{1}{2}}\,d\theta$$

gives the arc BP from the end B of the minor axis to any point P on the curve.

Putting $\theta=\frac{\pi}{2}-\chi$,

$$s=a\int_0^{\chi}\sqrt{1-e^2\sin^2\chi}\,d\chi=aE(\chi,e).$$

(See Chapter XI.)

568. This integral is Legendre's elliptic integral of the second kind, and is not expressible in terms of the ordinary circular or inverse circular functions. But its value can be found for specific values of e and χ from the tables calculated for the function E. Thus, for instance, the tables for E corresponding to $e=\frac{1}{2}$ give

$$\left.\begin{aligned}E(10^\circ)&=\ \cdot17431\\E(20^\circ)&=\ \cdot34733\\E(30^\circ)&=\ \cdot51788\\E(40^\circ)&=\ \cdot68506\\E(50^\circ)&=\ \cdot84832\\E(60^\circ)&=1\cdot00756\\E(70^\circ)&=1\cdot16318\\E(80^\circ)&=1\cdot31606\\E(90^\circ)&=1\cdot46746\end{aligned}\right\}\text{Values extracted from tables given in Bertrand, \textit{Calc. Intég.}, p. 717.}$$

Hence, taking an ellipse with a 20-inch major axis and eccentricity $\frac{1}{2}$, the arcs for eccentric angles 80°, 70°, 60°, ... 0°, measured from B, the end of the minor axis, are: 1·74, 3·47, 5·18, 6·85, 8·48, 10·08, 11·63, 13·16, 14·67 inches to two places of decimals.

The student should construct a quadrant of such an ellipse on squared paper, and by careful stepping with dividers round the perimeter verify this calculation approximately.

The total perimeter of the ellipse in any case is $4aE_1$, where E_1 is the complete elliptic integral. And in the present case $4\times14\cdot6746=58\cdot7$ inches very approximately.

The circumference of the auxiliary circle $=20\pi=62\cdot8318$, *i.e.* 4·1 inches longer than that of the ellipse.

569. **Approximation.**

If an approximate value be required, we may expand the radical $\sqrt{1-e^2\sin^2\chi}$, and in cases where the eccentricity is small the series is rapidly convergent.

We then have

$$s=a\int_0^\chi\left(1-\frac{1}{2}e^2\sin^2\chi-\frac{1}{2}\cdot\frac{1}{4}e^4\sin^4\chi-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}e^6\sin^6\chi-\dots\right)d\chi.$$

For a quadrant the limits are 0 and $\frac{\pi}{2}$, and the arc of the quadrant

$$=a\left(\frac{\pi}{2}-\frac{1}{2}e^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\cdot\frac{1}{4}e^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}e^6\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\dots\right)$$

$$=\frac{\pi a}{2}\left(1-\frac{1}{2^2}e^2-\frac{1^2\,.\,3}{2^2\,.\,4^2}e^4-\frac{1^2\,.\,3^2\,.\,5}{2^2\,.\,4^2\,.\,6^2}e^6-\dots\text{ to }\infty\right).$$

The first three terms give for the above ellipse a perimeter of 58·7 approximately.

570. **Other modes of procedure** may be adopted.

Cartesians.

Keeping x for the independent variable, we have

$$\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y};$$

$$\therefore\ \left(\frac{ds}{dx}\right)^2=1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}=1+(1-e^2)\frac{x^2}{a^2-x^2}=\frac{a^2-e^2x^2}{a^2-x^2}.$$

Hence $$s=\int_0^x\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}\,dx.$$

If we now put $x=a\sin\chi$, where χ is, as before, the complement of the eccentric angle, this reduces at once to

$$s=a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

as before.

571. Taking the **central pedal equation**

$$\frac{a^2b^2}{p^2}=a^2+b^2-r^2,$$

we get $$s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}=\int\frac{r\sqrt{a^2+b^2-r^2}}{\sqrt{(a^2-r^2)(r^2-b^2)}}\,dr.$$

Putting $$r^2=a^2\sin^2\chi+b^2\cos^2\chi,$$

$$r\,dr=(a^2-b^2)\sin\chi\cos\chi\,d\chi,$$

$$a^2+b^2-r^2=a^2\cos^2\chi+b^2\sin^2\chi=a^2(1-e^2\sin^2\chi),$$

and $$(a^2-r^2)(r^2-b^2)=(a^2-b^2)^2\sin^2\chi\cos^2\chi;$$

$$\therefore\ s=a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi.$$

572. Taking the **focal p-r equation**

$$\frac{b^2}{p^2}=\frac{2a}{r}-1,$$

$$s=\int\frac{r\,dr}{\sqrt{r^2-\dfrac{b^2r}{2a-r}}}=\int\frac{r\sqrt{2a-r}\,dr}{\sqrt{2ar^2-b^2r-r^3}}=\int\frac{\sqrt{2ar-r^2}}{\sqrt{2ar-b^2-r^2}}dr.$$

Putting $r=a(1+e\sin\chi)$ this reduces at once to

$$s=a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

as before.

573. It appears then that $aE(\chi, e)$, *i.e.*

$$a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

represents the length of the arc of an ellipse measured from the end of the minor axis to a point, on the curve, whose eccentric angle is $\frac{\pi}{2}-\chi$, the semi-major axis being a and the eccentricity e. (See Art. 567.)

This may be written as

$$\int_0^\chi\sqrt{a^2\cos^2\chi+b^2\sin^2\chi}\,d\chi,$$

or as

$$\int_0^\chi\sqrt{l^2+2lm\cos 2\chi+m^2}\,d\chi,$$

where $l+m=a$ and $l-m=b$. And it is useful to be able to recognise these forms at once, when they appear, as representing an arc of an ellipse. They occur in many other rectifications.

574. **March of the Second Elliptic Function.**

The form
$$s=a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi$$

for an ellipse gives a very clear idea of the "march" of the "second elliptic function" corresponding to any given modulus e, and it is easy to construct a graph of the relation between χ and s by measuring off ordinates equal to the arc of the ellipse and abscissae proportional to the complement of the eccentric angle.

Taking $a=1$, the figure (Fig. 131) shows the march of the

function for the values $e=0$, which gives a straight line, viz. $s=\chi$;

$$e=\tfrac{1}{2}, \text{ which gives } s=\int_0^\chi \sqrt{1-\tfrac{1}{4}\sin^2\chi}\,d\chi=E(\chi, \tfrac{1}{2}),$$

and $e=1$, which gives $s=\sin\chi$, the curve of sines.

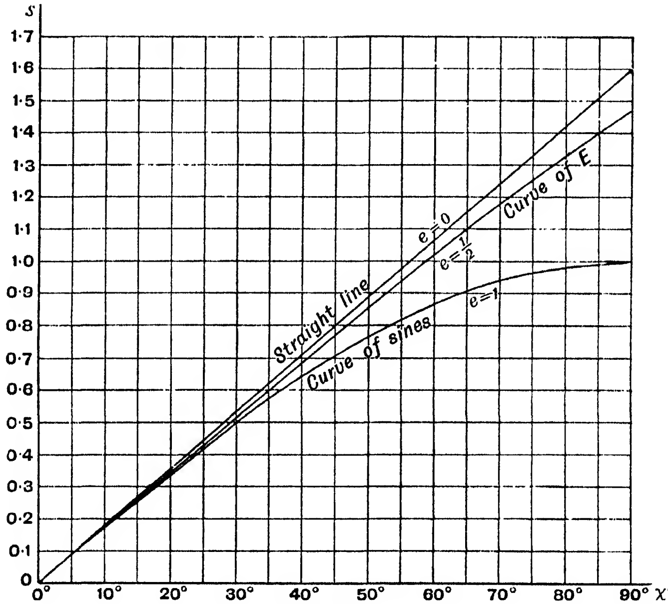


Fig. 131.

It will be seen that for the first 15° the difference of the ordinates is so small that there is no appreciable difference between ordinates in the drawings, in fact for $e=0$, $s=\cdot26180$; for $e=\frac{1}{2}$, $s=\cdot26106$; and for $e=1$, $s=\cdot25882$, for $\chi=15°$, which only gives a difference of ordinate of $\cdot0030$ between the greatest and least, and the curve $s=E(\chi)$ lies between these extremes. There is much more rapid deviation of $s=E\left(\chi, \sin\frac{\pi}{6}\right)$ from the curve $s=\sin\chi$ after $\chi=\frac{\pi}{4}$.

575. Arc measured from the End of the MAJOR AXIS. FAGNANO'S THEOREM.

Another method of proceeding gives the length of the arc AQ measured from the end of the *major axis*, and incidentally

a comparison of the two methods establishes a remarkable result with regard to the difference of two arcs, one measured from A, the other from B. This theorem is known as Fagnano's theorem, being discovered by Giulio, Count de Fagnano (1682-1760).* It shows that two arcs of an ellipse can be found in an infinite number of ways, whose difference can be expressed by a certain straight line, and really establishes in a particular case the addition formula for elliptic integrals of the second kind.

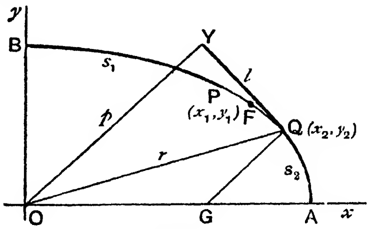


Fig. 132.

Take the central tangential polar equation

$$p^2 = a^2 \cos^2 \psi + b^2 \sin^2 \psi,$$

ψ being the angle between the perpendicular upon the tangent and the major axis; we have

$$\frac{ds}{d\psi} = p + \frac{d^2 p}{d\psi^2},$$

i.e.

$$s = \frac{dp}{d\psi} + \int p \, d\psi.$$

Let Q be the point of contact, whose coordinates are obviously by comparison of the equation, $x \cos \psi + y \sin \psi = p$, with the equation $\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$,

$$x_2 = \frac{a^2 \cos \psi}{p}, \quad y_2 = \frac{b^2 \sin \psi}{p}.$$

Also $\frac{dp}{d\psi} = -QY$, the negative sign occurring, because in this case Y is on the "forward drawn" tangent from Q, and p is diminishing as ψ is increasing.

* Cajori, *History of Mathematics*, p. 241.

Also

$$\int p\,d\psi = \int\sqrt{a^2\cos^2\psi + b^2\sin^2\psi}\,d\psi = a\int_0^\psi \sqrt{1-e^2\sin^2\psi}\,d\psi,$$

which is the same integral as obtained in Art. 567 for the arc BP, ψ being in that case a different angle, viz. the complement of the eccentric angle of P.

Hence, if these angles be taken the same in magnitude,

$$\text{arc } AQ + \text{tangent } QY = a\int_0^\psi \sqrt{1-e^2\sin^2\psi}\,d\psi,$$

and $\quad$ arc BP

$$= a\int_0^\psi \sqrt{1-e^2\sin^2\psi}\,d\psi.$$

Thus, $\quad$ arc $BP -$ arc $AQ \quad$ = tangent QY.

This is Fagnano's result.

576. **Algebraic Relation between the Abscissae of P and Q.**

Now $\quad QY = -\dfrac{dp}{d\psi} = \dfrac{(a^2-b^2)\sin\psi\cos\psi}{p} = \dfrac{a^2e^2}{p}\sin\psi\cos\psi.$

Also the coordinates of Q being

$$x_2 = \frac{a^2}{p}\cos\psi, \quad y_2 = \frac{b^2}{p}\sin\psi,$$

and those of P being

$$x_1 = a\sin\psi, \qquad y_1 = b\cos\psi,$$

we have $\qquad QY = e^2 x_2 \dfrac{x_1}{a}, \; \left(\text{or } e^2\dfrac{a^2}{b^3}y_1y_2\right).$

Hence $\quad$ arc $BP -$ arc $AQ = \dfrac{e^2}{a}x_1x_2, \; \left(\text{or } e^2\dfrac{a^2}{b^3}y_1y_2\right)$

This result is symmetrical as regards x_1, x_2, and therefore

$$\text{arc } BQ - \text{arc } AP = \frac{e^2}{a}x_1x_2,$$

as is, of course, immediately obvious otherwise.

Also $\dfrac{e^2}{a}x_1x_2 =$ tangent PY', if OY' be the perpendicular on the tangent at P from O. Hence $QY = PY'$.

Again, $\quad (a^2-x_1^2)(a^2-x_2^2) = (a^2 - a^2\sin^2\psi)\left(a^2 - \dfrac{a^4\cos^2\psi}{p^2}\right)$

$$= \frac{a^4\cos^2\psi}{p^2}b^2\sin^2\psi = (1-e^2)x_1^2x_2^2;$$

$$\therefore\; e^2x_1^2x_2^2 - a^2(x_1^2+x_2^2) + a^4 = 0.$$

577. The corresponding relation between y_1 and y_2 is

$$a^2e^2y_1^2y_2^2+b^4(y_1^2+y_2^2)-b^6=0,$$

that is $$e'^2y_1^2y_2^2-b^2(y_1^2+y_2^2)+b^4=0,$$

where $$\frac{1}{e^2}+\frac{1}{e'^2}=1,$$

e' being the "imaginary" eccentricity.

578. The Fagnano Points.

It will be noticed also that

$$\frac{x_1x_2}{a^3}=\frac{y_1y_2}{b^3}.$$

Hence, at the point F on the arc AB at which P and Q coincide when ϕ is suitably chosen,

$$\frac{x^2}{a^3}=\frac{y^2}{b^3}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a+b}=\frac{1}{a+b},$$

and the coordinates of the point are therefore

$$x=\sqrt{\frac{a^3}{a+b}},\quad y=\sqrt{\frac{b^3}{a+b}},$$

and this is called the "Fagnano Point,"* for the first quadrant.

579. **Properties.**

At this point F,

$$\text{arc } BF-\text{arc } AF=\frac{e^2x^2}{a}=\frac{a^2-b^2}{a^3}\cdot\frac{a^3}{a+b}=a-b$$

$$=\text{the difference of the semiaxes.}$$

And the length of the projection of the radius vector OF on the tangent at F is also $=a-b$.

580. The expression for QY, viz. $\dfrac{a^2e^2\sin\psi\cos\psi}{\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}}$, may be written as

$$\frac{a^2-b^2}{\sqrt{a^2\operatorname{cosec}^2\psi+b^2\sec^2\psi}},\quad i.e.\ \frac{a^2-b^2}{\sqrt{(a+b)^2+(a\cot\psi-b\tan\psi)^2}},$$

and therefore QY attains its maximum when $\tan\psi=\sqrt{\dfrac{a}{b}}$, viz. $a-b$. The Fagnano point is therefore the point for which QY has a maximum value. QY varies continuously from zero to $a-b$ in travelling from B or A to F.

* Greenhill's *Elliptic Functions*, p. 178 onward.

581. If we seek for a point Q upon the quadrantal arc AB of an ellipse such that QY, the projection of OQ upon the tangent at Q, is of given length l, where $0<l<a-b$, there will be two solutions, viz. the points P and Q, whose positions are given by the equations

$$r^2=l^2+p^2 \quad \text{and} \quad \frac{a^2b^2}{p^2}=a^2+b^2-r^2,$$

r being the radius vector to either of the required points, viz. OP or OQ.

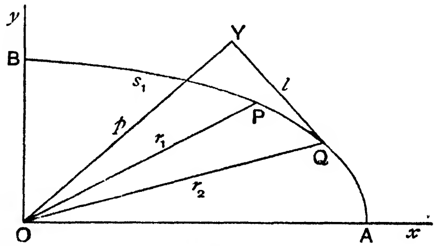


Fig. 133.

Eliminating p we have

$$(r^2-a^2-b^2)(r^2-l^2)+a^2b^2=0,$$

$$r^4-(a^2+b^2+l^2)r^2+l^2(a^2+b^2)+a^2b^2=0, \quad \text{..........(1)}$$

with roots r_1^2, r_2^2, such that

$$r_1^2+r_2^2=a^2+b^2+l^2, \quad \text{......................(2)}$$

and equal roots when $l=a-b$ and $r^2=a^2-ab+b^2$.

If we differentiate equation (2),

$$r_1\,dr_1+r_2\,dr_2=l\,dl.*$$

If we call BP, s_1, and BQ, s_2, and remember that

$$r\frac{dr}{ds}=\text{projection of radius vector on the tangent,}$$

viz. l in both cases,

$$ds_1+ds_2=dl,$$

i.e. $$s_1+s_2=l+C, \quad \text{......................(3)}$$

where C is a constant.

Taking the case when $r_1=b$, that is P at B, we have $r_2^2=a^2+l^2$, and therefore r_2 must $=a$ and $l=0$, for $r_2 \ngtr a$, so that Q is at A; then $s_1=0$, $s_2=\text{arc } AB$, $l=0$ simultaneously;

$$\therefore\ C=\text{arc } AB;$$

$$\therefore\ \text{arc } BP+\text{arc } BQ=l+\text{arc } BA, \quad i.e.\ \text{arc } BP-\text{arc } AQ=l,$$

* See Bertrand, *Calc. Intég.*, p. 380.

which is Fagnano's result, and the points P, Q, in which the arc AP must be divided to give a definite value l for QY, are determined by equation (1).

Examples.

1. Show that if coaxial ellipses be drawn with a given centre such that the areas enclosed between them and their respective director circles is constant, the locus of the Fagnano points is a circle of the same area.

2. Show that the locus of the Fagnano points for similar and similarly situated concentric ellipses is a pair of straight lines.

3. Show that the locus of the Fagnano points which lie on confocal ellipses is
$$(x^{\frac{2}{3}}+y^{\frac{2}{3}})^2(x^{\frac{2}{3}}-y^{\frac{2}{3}})=c^2,$$
$2c$ being the distance between the foci.

4. Show that if F be the Fagnano point on an ellipse of semiaxes $OA=a$, $OB=b$,
$$\left.\begin{array}{l} 2\,\text{arc}\,BF=aE_1+a-b, \\ 2\,\text{arc}\,AF=aE_1-a+b, \end{array}\right\}$$
where E_1 is the complete elliptic integral of the second kind
$$\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\phi}\,d\phi.$$

5. Show that the central perpendicular upon the tangent at a Fagnano point is a geometric mean between the semiaxes, and equal to the semi-diameter conjugate to the radius to the Fagnano point. Further, that the radius of curvature at this point is also equal to the perpendicular, and that the normals at the corresponding point on the evolute pass through the centre. Finally, that the arc of the evolute is at such a point divided in the ratio
$$b^{\frac{3}{2}}:a^{\frac{3}{2}}.$$

6. Show that if a straight rod LM of length $a+b$ slides with its ends on two axes Ox, Oy at right angles and carries a point F whose distance from L and M are respectively a and b, which thus describes an ellipse, then at the instant when LM is tangential to the path of F, F is a Fagnano point on the described ellipse, and the circle on LM for diameter passes through the point on the normal at F where that normal touches the evolute.

7. Show that the tangents at the points $P(x_1, y_1)$, $Q(x_2, y_2)$ on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which are related to each other so that $\frac{x_1x_2}{a^3}=\frac{y_1y_2}{b^3}$ intersect on a confocal hyperbola which passes through the Fagnano points.

[Many properties of these points will be found in Greenhill's *Elliptic Functions*, pages 182, 183.]

582. Properties of the Locus traced by a Pointer which pulls taut an Inextensible String passing round a given Oval.

Taking the case of any oval curve, let A be the point from which s is measured; PQ, $P'Q'$, the tangents at contiguous

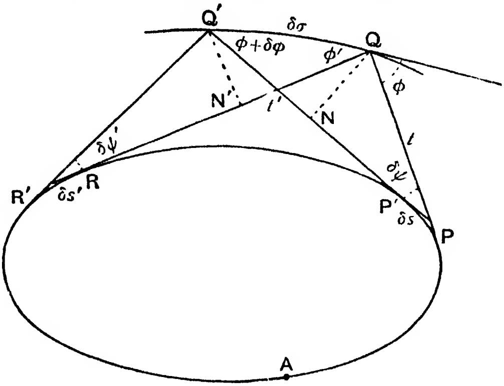


Fig. 134.

points (s, ψ) $(s+\delta s, \psi+\delta\psi)$ of the oval; and let a length $PQ=t$ be measured upon the forward drawn tangent at P, $P'Q'=t+\delta t$ upon the tangent at P'. Let the tangent to the locus of Q make an angle ϕ with the tangent at P to the oval. Draw QN perpendicular to $P'Q'$, and let the arc $QQ'=\delta\sigma$.

Then, to the first order,

$$QN=t\,\delta\psi, \quad Q'N=\delta\sigma\cos\phi,$$

and

$$t+\delta t+\delta s=t\cos\delta\psi+NQ'$$
$$=t+\delta\sigma\cos\phi;$$
$$\therefore\ \delta t+\delta s=\cos\phi\,\delta\sigma. \quad\text{.........................(1)}$$

If QR, $Q'R'$ of lengths t', $t'+\delta t'$ be the other tangents from Q, Q' which can be drawn to the oval, and s', $s'+\delta s'$ be the arcs APR, APR' respectively, and if ϕ' be the angle which QR makes with the tangent QQ' to the Q-locus and $\delta\psi'$ the difference of the angles of contingence at R, R', we have in the same way, $Q'N'$ being the perpendicular upon QR,

$$Q'N'=t'\delta\psi', \quad QN'=\delta\sigma\cos\phi',$$
$$t'+\delta s'=\delta\sigma\cos\phi'+t'+\delta t',$$

to the first order;

$$\therefore\ \delta t'-\delta s'=-\cos\phi'\,\delta\sigma. \quad\text{.....................(2)}$$

If the Q-locus be such that the tangent at Q always bisects the exterior angle between the tangents from Q to the oval,

$$\phi=\phi' \quad \text{and} \quad QN=Q'N'=\delta\sigma \sin\phi \text{ to the first order.}$$

Therefore $\left.\begin{aligned} \delta t+\delta s+\delta t'-\delta s'=0, \\ t\,\delta\psi=t'\,\delta\psi' \end{aligned}\right\}$ and

These equations give

$$\frac{1}{t}\frac{dt}{d\psi}+\frac{1}{t'}\frac{dt'}{d\psi'}=\frac{\rho'}{t'}-\frac{\rho}{t},$$

i.e.
$$\frac{d\log t}{d\psi}+\frac{d\log t'}{d\psi'}=\frac{\rho'}{t'}-\frac{\rho}{t}, \qquad\qquad (3)$$

and also
$$t+t'+s-s'=\text{constant}. \qquad\qquad (4)$$

Equation (4) expresses that in such case

$$QP+QR-\text{arc } PR=\text{constant},$$

i.e.
$$QP+QR+\text{arc } PAR=\text{constant}.$$

In this case the Q-locus is an oval traced by a pencil at Q which draws taut a loop of string placed round the original oval.

583. Dr. Graves's Theorem.

The case when the original oval is an ellipse and the Q-locus is a confocal, when the necessary property holds, viz. that the tangent to the Q-locus bisects the exterior angle between QP, QR, gives the well-known theorem due to Dr. Graves, viz.

If two tangents be drawn to an ellipse from any point of a confocal ellipse, the excess of the sum of these two tangents over the intercepted arc is constant.*

Incidentally, we have a method of drawing an ellipse confocal to a given one.

584. If the Q-locus be such that its tangent bisects the *interior* angle between the tangents QP, QR, as it would do in the case of an ellipse and a confocal hyperbola, and if we measure s and s' in opposite directions from the

* Salmon's *Conic Sections*, p. 357; Graves's *Translation of Chasles's Memoirs*.

point A, where the Q-locus meets the oval, we have, in the same way,

$$QN = \delta\sigma \sin\phi = t\,d\psi, \qquad QN' = \delta\sigma \sin\phi' = t'\,d\psi',$$
$$NQ' = \delta\sigma \cos\phi, \qquad N'Q' = \delta\sigma \cos\phi';$$

and
$$\left.\begin{aligned} t+\delta s+\delta\sigma\cos\phi &= t+\delta t, \\ t'+\delta s'+\delta\sigma\cos\phi' &= t'+\delta t', \end{aligned}\right\}\text{to the first order;}$$

and when $\phi=\phi'$, we have $dt-dt'=ds-ds'$, and $t\,d\psi=t'\,d\psi'$,

so that
$$\frac{d\log t}{d\psi}-\frac{d\log t'}{d\psi}=\frac{\rho}{t}-\frac{\rho'}{t'},$$

and also
$$t-s=t'-s'+\text{const.};$$

also, as $t,\ t',\ s,\ s'$ all vanish at A,
$$t-s=t'-s',$$

i.e. tangent QP $-$ arc AP $=$ tangent QR $-$ arc AR.

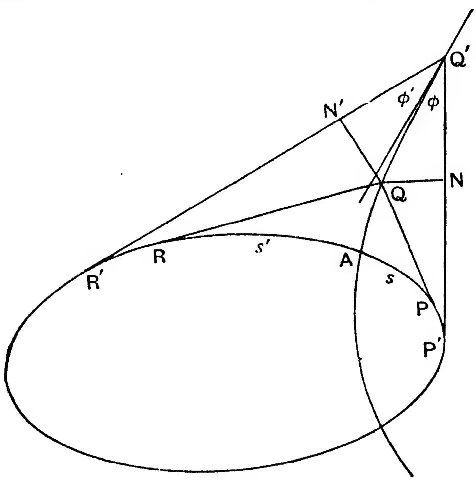


Fig. 135.

MACCULLAGH'S THEOREM.

For the case of the ellipse and the confocal hyperbola, where the condition $\phi=\phi'$ is necessarily satisfied, we have the following result.

If tangents QP, QR be drawn from a point Q on a hyperbola to a confocal ellipse cutting the hyperbola at A, the difference of the tangents is equal to the difference of the arcs AP, AR. This theorem is due to MacCullagh.*

* Salmon's *Conic Sections*, p. 358; Chasles, *Comptes Rendus*, Tom. xvii.

585. **Deductions.**

If we draw tangents to the ellipse at the extremities of the axes, the particular confocal to the ellipse which passes through the corners of the rectangle formed cuts the ellipse in the Fagnano points, and if Q be the intersection of tangents

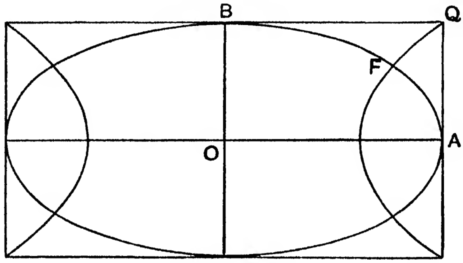


Fig. 136.

at A and B, and F the point in the first quadrant where the confocals cut, MacCullagh's theorem gives

$$QB-QA=\text{arc } FB-\text{arc } FA,$$

and if the semiaxes be a and b, we have

$$\text{arc } FB-\text{arc } FA=a-b,$$

which is Fagnano's result.

586. From the theorem of Dr. Graves it appears that if Q_1, Q_2 be any two points on the confocal and Q_1P_1, Q_1R_1;

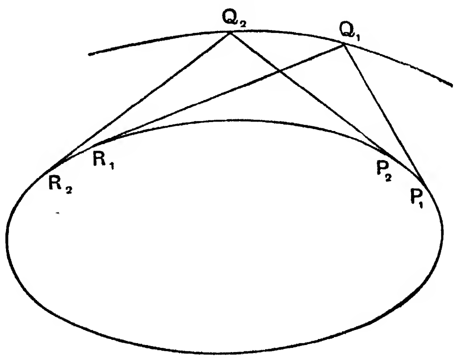


Fig. 137.

Q_2P_2, Q_2R_2 are the corresponding pairs of tangents to the original ellipse,

$$Q_1P_1+Q_1R_1-\text{arc } P_1R_1=Q_2P_2+Q_2R_2-\text{arc } P_2R_2;$$

and therefore that the difference of the arcs P_1R_1, P_2R_2 is

$$(Q_1P_1+Q_1R_1)-(Q_2P_2+Q_2R_2)$$

and is therefore rectifiable in terms of known lines.

The particular value of the constant to which

$$QP+QR-\text{arc } PR$$

is equal may be found by taking Q at a specified point on the confocal, *e.g.* where it cuts the conjugate axis.

And a similar result follows also from MacCullagh's theorem.

587. Exactly in the same way, if Q be a point on the ellipse and QP, QP' be tangents to the same branch of the hyperbola, it will be clear that

$$QP-\text{arc } AP=QP'-\text{arc } AP',$$

for the tangent at Q still satisfies the requisite condition, namely that the internal bisector of the angle PQP' is a tangent

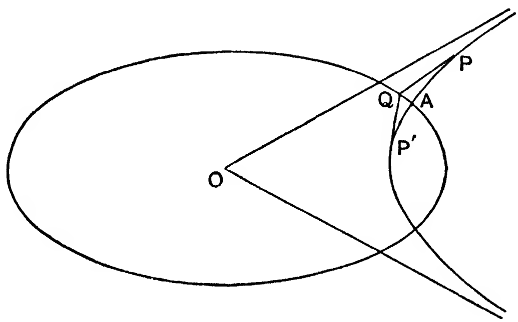


Fig. 138.

to the ellipse. And the difference of the arcs AP, AP' is therefore expressible as the difference of two straight lines and is rectifiable. Moreover, if Q_1 be another point on the ellipse, such that tangents Q_1P_1, Q_1P_1' can be drawn to the same branch of the confocal hyperbola, the difference of the arcs PP_1, $P'P_1'$ is rectifiable. In order that the point Q should be such that tangents can be drawn to the same branch of the hyperbola, such point must obviously lie in one of the regions between the asymptotes in which the hyperbola lies. In the limiting case in which QP is an asymptote, the difference of the infinite portion of the

asymptote QP and the infinite arc AP is finite and equal to the difference of QP' and the arc AP', Q being now at the point of intersection of the asymptote with the ellipse.

588. **Rectification of the hyperbola** $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Let C be the centre, CA the semimajor axis, s the length of an arc AP measured from A in the first quadrant, CY the perpendicular p upon the tangent at P.

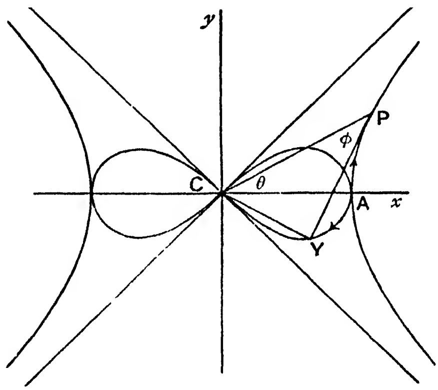


Fig. 139.

Then $p=x\cos\psi+y\sin\psi$ touches the curve if

$$p^2=a^2\cos^2\psi-b^2\sin^2\psi=a^2(1-e^2\sin^2\psi).$$

In the case of the hyperbola, when P lies in the first quadrant, ψ is the angle xCY and is negative, and as s increases from 0 to ∞ whilst P travels along the arc from A, Y travels from A towards C along the first positive pedal curve $r^2=a^2\cos^2\theta-b^2\sin^2\theta$, which becomes a Lemniscate of Bernoulli when $b=a$, *i.e.* when the hyperbola is rectangular. The angle ψ therefore remains negative, and as its actual magnitude is increasing ψ is algebraically decreasing and an increment $d\psi$ is negative. When P has travelled to ∞ along this branch of the curve the limiting position of YP is an asymptote. The tangents at the node of the pedal are therefore the perpendiculars to the asymptotes of

the hyperbola, coinciding with them in the case of the rectangular hyperbola and its pedal $r^2=a^2\cos 2\theta$.

Let us find the length of the arc AP from A to a point P for which $\psi=-\chi$.

We have $$\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2} \quad \text{and} \quad t=\frac{dp}{d\psi};$$

therefore, integrating,

$$s-t=\int_0^\psi p\,d\psi.$$

Now $t\equiv\dfrac{dp}{d\psi}$ is the projection of the radius vector OP upon the tangent $=PY$, and is positive.

$$\therefore\ PY=\frac{-ae^2\sin\psi\cos\psi}{\sqrt{1-e^2\sin^2\psi}}=\frac{ae^2\sin\chi\cos\chi}{\sqrt{1-e^2\sin^2\chi}},$$

and $$\int_0^\psi p\,d\psi=a\int_0^{-\chi}\sqrt{1-e^2\sin^2\psi}\,d\psi=-a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi;$$

$$\therefore\ \text{arc } AP=PY-a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

i.e. $$=\frac{ae^2\sin\chi\cos\chi}{\sqrt{1-e^2\sin^2\chi}}-a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

or $$PY-\text{arc } AP=a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi. \quad\text{.................(1)}$$

This integral is not of the Legendrian form at present, e being essentially greater than unity.

If P be allowed to travel to ∞, χ ultimately becomes

$$\tan^{-1}\frac{a}{b}\quad\left(i.e.\ \frac{\pi}{2}-\tan^{-1}\frac{b}{a}\right).$$

Hence the excess of the infinite asymptote $C\infty$ over the infinite arc $A\infty$ is

$$a\int_0^{\tan^{-1}\frac{a}{b}}\sqrt{1-e^2\sin^2\chi}\,d\chi.$$

It is easy to reduce the integral in equation (1) to two integrals of Legendre's standard form.

Let $e\sin\chi=\sin\omega$.

Then $c\cos\chi\, d\chi=\cos\omega\, d\omega$, and

$$\int_0^\chi \sqrt{1-e^2\sin^2\chi}\, d\chi$$

$$=\frac{1}{e}\int_0^\omega \frac{\cos^2\omega\, d\omega}{\sqrt{1-\frac{1}{e^2}\sin^2\omega}}$$

$$=e\int_0^\omega \frac{\left(\frac{1}{e^2}-1\right)+\left(1-\frac{1}{e^2}\sin^2\omega\right)}{\sqrt{1-\frac{1}{e^2}\sin^2\omega}}\, d\omega$$

$$=e\left[-\frac{e^2-1}{e^2}\int_0^\omega \frac{d\omega}{\sqrt{1-\frac{1}{e^2}\sin^2\omega}}+\int_0^\omega \sqrt{1-\frac{1}{e^2}\sin^2\omega}\, d\omega\right]$$

$$=e\left(-\cos^2 a\int_0^\omega \frac{d\omega}{\sqrt{1-\sin^2 a\sin^2\omega}}+\int_0^\omega \sqrt{1-\sin^2 a\sin^2\omega}\, d\omega\right),$$

where $\qquad \cot a=\frac{b}{a}, \quad i.e.\ e^2=\frac{a^2+b^2}{a^2}=\operatorname{cosec}^2 a,$

and a is the complement of the half angle between the asymptotes.

Hence,

$$\text{Arc } AP=PY+ae[\cos^2 a\, F(\omega, \sin a)-E(\omega, \sin a)],$$

F and E being the Legendrian standard integrals of the first and second species, whose values are tabulated for particular values of the modulus $\sin a$, ω being $\sin^{-1}\left(\frac{\sin\chi}{\sin a}\right)$ in the upper limit and PY, written in terms of ω, being

$$\frac{a}{\sin a}\tan\omega\sqrt{1-\sin^2 a\sin^2\omega}\equiv ae\tan\omega\,\Delta\,\left(\text{Mod. }\frac{1}{e}\right),$$

where $\qquad \Delta=\sqrt{1-\frac{1}{e^2}\sin^2\omega},$

i.e. $\qquad \text{Arc}=ae\{\tan\omega\Delta+\cos^2 a\, F(\omega, \sin a)-E(\omega, \sin a)\}. \quad\ldots\ldots(2)$

589. In a rectangular hyperbola $a=\frac{\pi}{4}$, $e=\sqrt{2}$, and we have

$$\text{Arc}=a\sqrt{2}\left[\tan\omega\sqrt{1-\tfrac{1}{2}\sin^2\omega}+\tfrac{1}{2}F\left(\omega, \frac{1}{\sqrt{2}}\right)-E\left(\omega, \frac{1}{\sqrt{2}}\right)\right]$$

EXAMPLES.

1. In the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, put $a=b\tan\alpha$, $\Delta=\sqrt{1-\sin^2\alpha\sin^2\phi}$, and show that we may take $x=b\tan\alpha\sec\phi\,\Delta$, $y=b\cos\alpha\tan\phi$, and that

$$\frac{ds}{d\phi}=\frac{b\cos\alpha}{\Delta\cos^2\phi},\quad t=\frac{b}{\cos\alpha}\Delta\tan\phi,$$

and $$s=b\sec\alpha\tan\phi\Delta+b\cos\alpha\, F(\phi, \sin\alpha)-b\sec\alpha\, E(\phi, \sin\alpha).$$

2. From the polar equation $r^2=a^2\sec 2\theta$ deduce the rectification of the rectangular hyperbola, viz.

$$s=a\sqrt{2}[\Delta\tan\omega+\tfrac{1}{2}F-E].$$

3. If PQ be a chord of one branch of a hyperbola, touching a confocal ellipse at F, and the confocal cutting that branch of the hyperbola at A and B, and if PR, QS be the other tangents from P and Q to the ellipse, show that the elliptic arcs AR, BS exceed the elliptic arc AFB by the excess of the tangents PR, QS over the chord PQ, *i.e.* that

$$\text{arc } AR+\text{arc } BS-\text{arc } AFB$$

is rectifiable in terms of known lines.

In particular, examine what happens:

(1) When F is the vertex of the confocal ellipse.

(2) When F is at B.

(3) When PR and QS are at right angles to PQ and F the vertex of the ellipse.

590. Another Method of Treatment for the Central Conics. Use of Hyperbolic Functions.

In the case of the central conics it is instructive to consider another mode of treatment of the rectification.

The relation $$x+\iota y=c\sin(u+\iota v)$$

gives $$x=c\sin u\cosh v,\quad y=c\cos u\sinh v$$

Then $v=$const. is the equation to the ellipse

$$\frac{x^2}{c^2\cosh^2 v}+\frac{y^2}{c^2\sinh^2 v}=1,$$

and $u=$const. is the equation to the hyperbola

$$\frac{x^2}{c^2\sin^2 u}-\frac{y^2}{c^2\cos^2 u}=1,$$

and different constant values of v and u give confocal ellipses and hyperbolae.

Now $$\frac{dx}{c}=\cos u\cosh v\,du+\sin u\sinh v\,dv,$$

$$\frac{dy}{c}=-\sin u\sinh v\,du+\cos u\cosh v\,dv.$$

Hence

$$\frac{ds^2}{c^2}=(\cos^2 u\cosh^2 v+\sin^2 u\sinh^2 v)(du^2+dv^2)$$

$$=\{(1-\sin^2 u)\cosh^2 v+\sin^2 u(\cosh^2 v-1)\}(du^2+dv^2)$$

$$=(\cosh^2 v-\sin^2 u)(du^2+\ v^2).$$

Hence, for any of the family of the ellipses v=const.,

$$\frac{ds}{c}=\sqrt{\cosh^2 v-\sin^2 u}\,du\quad(\ =\text{const.});$$

and for any of the family of hyperbolae u=const.,

$$\frac{ds}{c}=\sqrt{\cosh^2 v-\sin^2 u}\,dv\ (u=\text{const.}).$$

591. **In the case of the ellipse** $x^2/a^2+y^2/b^2=1$,

$$a=c\cosh v,\quad b=c\sinh v,\quad c^2=a^2-b^2=a^2e^2,$$

where e is the eccentricity, and $\therefore\ e=\operatorname{sech} v$.

And $$ds=a\sqrt{1-e^2\sin^2 u}\,du,$$

$$s=a\int_0^u\sqrt{1-e^2\sin^2 u}\,du=aE(u,e).$$

In the case of the hyperbola $x^2/a^2-y^2/b^2=1$,

$$a=c\sin u,\quad b=c\cos u,\quad\text{and}\quad c^2=a^2+b^2=a^2e^2,\quad e=\operatorname{cosec} u.$$

With the notation of Art. 589, in which

$$\psi=-\chi,\quad \sin\chi=\sin u\sin\omega,$$

we have

$$\cos\chi=\sqrt{1-\sin^2 u\sin^2\omega}=\Delta\quad\text{and}\quad t\equiv PY=c\tan\omega\,\Delta.$$

The line $x\cos\psi+y\sin\psi=p$ is tangential, provided that

$$p^2=a^2\cos^2\psi-b^2\sin^2\psi$$

$$=c^2\sin^2 u\,\Delta^2-c^2\cos^2 u\sin^2 u\sin^2\omega=c^2\sin^2 u\cos^2\omega;$$

$$\therefore\ p=c\sin u\cos\omega.$$

The point of contact P is given by

$$x=\frac{a^2\cos\psi}{p}=c\sin u\,\Delta\sec\omega,\quad y=-\frac{b^2\sin\psi}{p}=c\cos^2 u\tan\omega,$$

and, as these are to be $c\sin u\cosh v$, $c\cos u\sinh v$, we have

$$\cosh v=\Delta\sec\omega,\quad \sinh v=\cos u\tan\omega.$$

It follows that $\cosh v\,dv=\cos u\sec^2\omega\,d\omega$,

i.e.

$$dv=\frac{\cos u\,d\omega}{\Delta\cos\omega}.$$

Again,

$$\sqrt{\cosh^2 v-\sin^2 u}=\sqrt{\Delta^2\sec^2\omega-\sin^2 u}=\cos u\sec\omega.$$

Hence

$$\begin{aligned}\frac{s}{c}&=\int\sqrt{\cosh^2 v-\sin^2 u}\,dv\\&=\cos^2 u\int\frac{\sec^2\omega}{\Delta}d\omega\\&=\Delta\tan\omega+\cos^2 uF-E\quad(\text{mod.}\sin u)\end{aligned}$$

by Legendre's fourth formula, p. 399;

$$\therefore\ \text{Arc}=PY+ae\left(1-\frac{1}{e^2}\right)F(\omega,\sin u)-aeE(\omega,\sin u),$$

the same result as before.

592. **The Lemniscate.**

The equation is $$r^2=a^2\cos 2\theta;$$

we have at once $$\frac{dr}{r\,d\theta}=-\tan 2\theta;$$

whence $$\frac{ds}{d\theta}=r\sec 2\theta=\frac{a}{\sqrt{\cos 2\theta}},$$

$$s=a\int_0^\theta\frac{d\theta}{\sqrt{\cos 2\theta}}.$$

Put $$\cos 2\theta=\cos^2\phi;\quad\therefore\ d\theta=\frac{\sin\phi\cos\phi\,d\phi}{\sin 2\theta}$$

$$\begin{aligned}s&=a\int_0^\phi\frac{\sin\phi\cos\phi\,d\phi}{\cos\phi\sqrt{1-\cos^4\phi}}=a\int_0^\phi\frac{d\phi}{\sqrt{2-\sin^2\phi}}\\&=\frac{a}{\sqrt{2}}\int_0^\phi\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}=\frac{a}{\sqrt{2}}F\left(\phi,\frac{1}{\sqrt{2}}\right),\end{aligned}$$

or

$$=\frac{a}{\sqrt{2}}\,\text{am}^{-1}\phi.$$

Hence
$$\operatorname{am}\frac{s\sqrt{2}}{a}=\phi,$$
$$\operatorname{cn}\frac{s\sqrt{2}}{a}=\cos\phi=\frac{r}{a}.$$

Hence
$$s=\frac{a}{\sqrt{2}}\operatorname{cn}^{-1}\frac{r}{a},\quad \text{mod. }\frac{1}{\sqrt{2}}.$$

Here s is measured from the vertex.

We might have expressed θ from the beginning in terms of r, and then
$$\theta=\frac{1}{2}\cos^{-1}\frac{r^2}{a^2},$$
$$\frac{d\theta}{dr}=-\frac{r}{\sqrt{a^4-r^4}},$$
$$\frac{ds}{dr}=-\frac{a^2}{\sqrt{a^4-r^4}}\qquad s=a^2\int_r^a\frac{dr}{\sqrt{a^4-r^4}};$$
then putting $r=a\cos\phi$ the work proceeds as before.

For the whole length of the arc, we have
$$\frac{4a}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}=2a\sqrt{2}\,F_1,\quad \text{mod. }\frac{1}{\sqrt{2}}.$$

The tables for F_1 (Bertrand, *C.I.* p. 716) give $F_1=1{\cdot}85407$, whence $\quad$ whole arc $=2a\sqrt{2}\times1{\cdot}85407=a\times5{\cdot}2441.$

We might, however, proceed as follows:
$$s=4a\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{\cos 2\theta}}.$$

Putting $2\theta=\omega$, we have
$$s=2a\int_0^{\frac{\pi}{2}}(\cos\omega)^{-\frac{1}{2}}\,d\omega=2a\frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})}.$$

It will be shown later (Art. 872) that
$$\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi},$$
where n is less than unity. Borrowing this theorem for present purposes,
$$\Gamma(\tfrac{3}{4})\Gamma(\tfrac{1}{4})=\frac{\pi}{\sin\frac{\pi}{4}}=\pi\sqrt{2};$$
$$\therefore\ \text{Perimeter}=2a\frac{[\Gamma(\frac{1}{4})]^2\sqrt{\pi}}{2\pi\sqrt{2}}=\frac{a}{\sqrt{2\pi}}[\Gamma(\tfrac{1}{4})]^2=ka,\text{ say.}$$

The values of the Γ functions are calculated. Tables of these values are given in Bertrand's *Calcul Intégral*, pages 285, 286, to seven places of decimals from $\operatorname{Log}\Gamma(1)$ to $\operatorname{Log}\Gamma(2)$. As the values of $\Gamma(x)$ from $\Gamma(1)$ to $\Gamma(2)$ are all fractional, 10 is added to their ordinary logarithms for convenience of tabulation, as is usual in tables of logarithms of sines and cosines. (See Chambers's *Mathematical Tables.*)

Now $$\Gamma(\tfrac{5}{4})=\tfrac{1}{4}\Gamma(\tfrac{1}{4}),$$
and $$L\,\Gamma(\tfrac{1}{4})=L\,\Gamma(\tfrac{5}{4})+\log 4,$$
where L denotes the tabular logarithm,

$$\begin{array}{rr}
= & 9{\cdot}9573211 \\
+ & {\cdot}6020600 \\ \hline
 & 10{\cdot}5593811 \\
2\log\Gamma(\tfrac{1}{4})= & 1{\cdot}1187622 \\
\log\sqrt{2\pi}= & {\cdot}3990899 \\ \hline
\log k= & {\cdot}7196723 \\
\log 5{\cdot}2441= & {\cdot}7196710 \\ \hline
 & 13 \\
\text{Difference for } 1= & 8 \\ \hline
 & 50 \\
 & 50
\end{array}$$

from the tables of $L\,\Gamma(x)$.

$$\begin{array}{ll}
\log 2 & ={\cdot}3010300 \\
\log \pi & ={\cdot}4971499 \\ \hline
\log 2\pi & ={\cdot}7981799 \\
\log \sqrt{2\pi} & ={\cdot}3990899
\end{array}$$

Hence $k=5{\cdot}244116$.

Hence the whole perimeter of $r^2=a^2\cos 2\theta$ is, as before,
$$5{\cdot}244116\times a.$$

593. Incidentally, it may be remarked that the equation
$$r=a\operatorname{cn}\frac{s\sqrt{2}}{a}$$

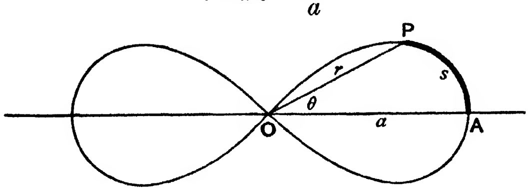


Fig. 140.

for a lemniscate gives a very good idea of the graph of the functions cn and cn^{-1} for the case mod. $\dfrac{1}{\sqrt{2}}$, and we can readily

draw a graph, taking, for instance, as unit length $\frac{a}{\sqrt{2}}$ on the x-axis, and any convenient unit on the y-axis, say a, and constructing the curve with abscissa s and ordinate r.

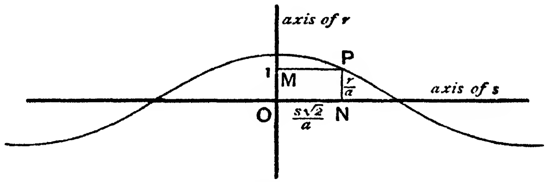


Fig. 141.

The ordinate shows the march of the function $\operatorname{cn} x$, the abscissa the march of $\operatorname{cn}^{-1} x$.

EXAMPLES.

1. Find the length of the arc of a lemniscate $r^2 = a^2 \cos 2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{6}$.

Here

$$s = \frac{a}{\sqrt{2}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}, \quad \text{and } k = \frac{1}{\sqrt{2}}, \quad \cos^2\phi = \cos\frac{\pi}{3} = \tfrac{1}{2}, \quad \phi = \frac{\pi}{4},$$

and from the tables for $F\left(\phi, \frac{1}{\sqrt{2}}\right)$, (Bertrand, *Calcul Intégral*, p. 716.)

$$\int_0^{\frac{\pi}{4}} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}} = \cdot 82602\,;$$

$$\therefore\ s = a\sqrt{2} \times \cdot 41301$$

$$= \cdot 5841 a.$$

2. Find the area of the curve $y^2 = \frac{1}{1 - x^4}$ for the portion in the first quadrant. What connection is there between this problem and the evaluation of the perimeter of the lemniscate?

3. Draw a careful polar graph of the lemniscate $r^2 = 25 \cos 2\theta$, taking one inch as unit of length, and deduce a Cartesian graph of

$$y = 5 \operatorname{cn} \frac{x\sqrt{2}}{5} \quad \left(\text{mod. } \frac{1}{\sqrt{2}}\right).$$

4. Show that the difference between the lengths of the asymptote and the infinite arc of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ in the first quadrant is

$$t - s = \frac{\pi a}{2}\left[\frac{1}{2} \cdot \frac{1}{e} + \frac{1 \cdot 1^2}{2^2 \cdot 4} \cdot \frac{1}{e^3} + \frac{1 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6}\frac{1}{e^5} + \frac{1 \cdot 1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} \cdot \frac{1}{e^7} + \dots\right].$$

594. **The Limaçon** $r=a+b\cos\theta$.

Here $\dfrac{dr}{d\theta}=-b\sin\theta$ and $\left(\dfrac{ds}{d\theta}\right)^2=a^2+2ab\cos\theta+b^2$;

$$\therefore\quad s=\int_0^\theta\sqrt{a^2+2ab\cos\theta+b^2}\,d\theta$$

$$=\int_0^\theta\sqrt{(a+b)^2-4ab\sin^2\frac{\theta}{2}}\,d\theta \qquad (\text{Let } \theta=2\phi.)$$

$$=2(a+b)\int_0^\phi\sqrt{1-k^2\sin^2\phi}\,d\phi,\ \text{where}\ k^2=\frac{4ab}{(a+b)^2},$$

$$=2(a+b)\,E\left(\phi,\ \frac{2\sqrt{ab}}{a+b}\right).$$

An obvious modification will be necessary if a and b be of opposite sign.

This curve very well illustrates the march of the second elliptic integral E. The arc AP measured from the vertex

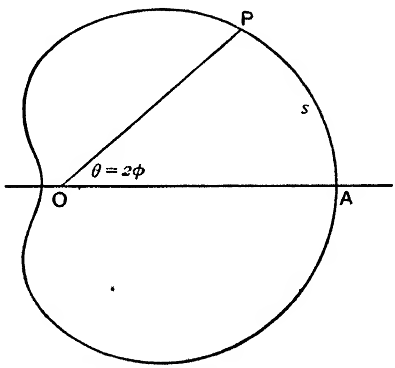


Fig. 142. For the case $a>b$.

is proportional to E, whilst ϕ is half the angle AOP. See also Art. 574.

The result shows that the arc AP of the limaçon is equal to the arc of an ellipse of semi-major axis $2(a+b)$ and eccentricity $\dfrac{2\sqrt{ab}}{a+b}$, measured from the end of the semi-minor axis to a point on the ellipse for which the complement of the eccentric angle is $\dfrac{\theta}{2}$ (compare Art. 573). The semiaxes of the ellipse in question are then $2(a+b)$ and $2(a-b)$.

This would also be evident upon writing

$$\int_0^\theta \sqrt{a^2+2ab\cos\theta+b^2}\,d\theta$$

as
$$\int_0^\theta \sqrt{(a+b)^2\cos^2\frac{\theta}{2}+(a-b)^2\sin^2\frac{\theta}{2}}\,d\theta$$

$$=\int_0^\phi \sqrt{(2a+b)^2\cos^2\phi+(2a-b)^2\sin^2\phi}\,d\phi, \quad \text{where} \quad \theta=2\phi.$$

595. Ex. Consider the case of the limaçon in which $\frac{a}{b}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$ for the portion from $\theta=0$ to $\theta=\frac{\pi}{3}$.

Here $\quad \frac{a+b}{a-b}=\frac{2}{\sqrt{3}}$, and $k^2=\frac{4ab}{(a+b)^2}=\frac{1}{4}$, $\quad k=\frac{1}{2}=\sin\frac{\pi}{6}$,

$$s=2(a+b)\int_0^{\frac{\pi}{6}} \sqrt{1-\tfrac{1}{4}\sin^2\phi}\,d\phi$$

$$=8a(2-\sqrt{3})\times \cdot 51788, \quad \text{from the tables for } E(\phi, \tfrac{1}{2}),$$
$$=1\cdot 11012\times a.$$

The limaçon is of course the focal inverse of a conic, and when $a=b$ the cardioide is the inverse of a parabola.

596. **Trochoidal Curves.** (See *Diff. Calc.*, p. 344.)

If a be the radius of the fixed circle, b that of the rolling circle and the carried point P be at a distance mb from the centre of the rolling circle,

$$\left.\begin{aligned} x&=(a+b)\cos\theta-mb\cos\frac{a+b}{b}\theta,\\ y&=(a+b)\sin\theta-mb\sin\frac{a+b}{b}\theta.\end{aligned}\right\}$$

Hence
$$\left.\begin{aligned} \frac{dx}{d\theta}&=-(a+b)\sin\theta+m(a+b)\sin\frac{a+b}{b}\theta,\\ \frac{dy}{d\theta}&=(a+b)\cos\theta-m(a+b)\cos\frac{a+b}{b}\theta;\end{aligned}\right\}$$

$$\therefore \quad \left(\frac{ds}{d\theta}\right)^2=(a+b)^2(1+m^2)-2m(a+b)^2\cos\frac{a\theta}{b}$$

$$=(a+b)^2(1+m)^2\left[1-\frac{4m}{(1+m)^2}\cos^2\frac{a\theta}{2b}\right].$$

Let
$$\frac{a\theta}{2b}=\frac{\pi}{2}+\chi.$$

Then $s=\frac{2b}{a}(a+b)(1+m)\int_0^\chi\sqrt{1-k^2\sin^2\chi}\,d\chi$, where $k=\frac{2\sqrt{m}}{1+m}$,

$$=\frac{2b}{a}(a+b)(1+m)\,E(\chi,k),$$

where s is measured from the point at which $\chi=0$, *i.e.* $\theta=\frac{b\pi}{a}$, *i.e.* from a vertex V, as in the case of the epicycloid (Art. 540).

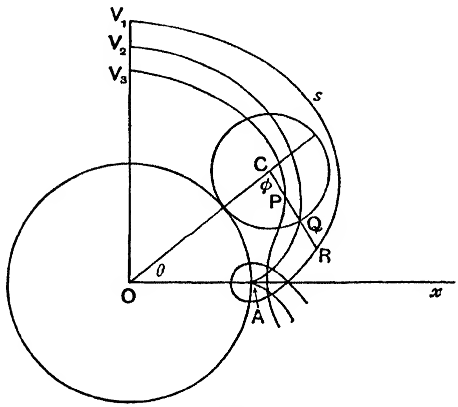


Fig. 143.

Hence again we can find the length of any desired portion by means of the tables for Legendre's elliptic integrals of the second form; or, which comes to the same thing, such length can be expressed as being equal to the corresponding arc of an ellipse, measured from the end of the minor axis, the semi-major axis being $\frac{2b}{a}(a+b)(1+m)$, the eccentricity being $a=\frac{2\sqrt{m}}{1+m}$, and χ being the complement of the eccentric angle at the end of the elliptic arc.

For a circle, when $m=0$,

$$s=\frac{2b}{a}(a+b)\,\chi=(a+b)\left(\theta-\frac{\pi b}{a}\right)+\text{const.}$$

For the epicycloid, when $m=1$.

$$s=\frac{4b}{a}(a+b)\sin\chi=\frac{-4b}{a}(a+b)\cos\frac{a\theta}{2b}+\text{const.}$$

which agrees with the result of Art. 540.

We might use this curve, like the ellipse and the limaçon, to construct a graph showing the march of

$$\int_0^{\chi} \sqrt{1-k^2 \sin^2 \chi}\, d\chi$$

for any modulus $$k=\frac{2\sqrt{m}}{1+m}.$$

597. **The Cassinian Oval.**

The bipolar equation of this curve is $r_1 r_2 = b^2$. (See *Diff. Calc.*, Art. 458.)

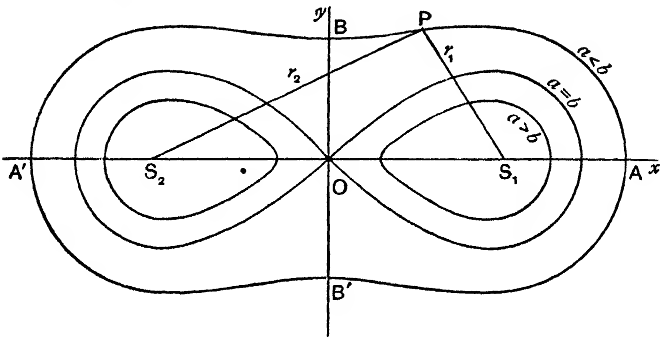


Fig. 144.

If S_1, S_2 be the foci, $S_1S_2=2a$, and if the line of foci be taken as x-axis and its centre O as origin, the equivalent polar equation is

$$r^4-2a^2r^2\cos 2\theta+a^4=b^4.$$

Three cases arise :

(1) $a>b$, two separate twin ovals with vertices distant $\sqrt{a^2+b^2}$, $\sqrt{a^2-b^2}$ from O.

(2) $a=b$, reducing to Bernoulli's lemniscate.

(3) $a<b$, one single oval lying outside the lemniscate, which may or may not possess inflexions.

The equation may be written

$$r^2+\frac{a^4-b^4}{r^2}=2a^2\cos 2\theta.$$

Take an auxiliary angle θ' such that

$$r^2+\frac{b^4-a^4}{r^2}=2b^2\cos 2\theta'.$$

Then $$r^2=a^2\cos 2\theta+b^2\cos 2\theta',$$

$$\frac{r^4-b^4}{r^2}=a^2\cos 2\theta-b^2\cos 2\theta';$$

$$\therefore\ a^4-b^4=a^4\cos^2 2\theta-b^4\cos^2 2\theta',$$

or $$a^4\sin^2 2\theta=b^4\sin^2 2\theta',$$

i.e. the auxiliary angle θ' is such that

$$a^2\sin 2\theta=b^2\sin 2\theta'.$$

Differentiating the original equation, we have

$$\frac{r\,d\theta}{dr}=-\frac{r^2-a^2\cos 2\theta}{a^2\sin 2\theta};$$

$$\therefore\ \left(\frac{ds}{dr}\right)^2=\frac{b^4}{a^4}\frac{1}{\sin^2 2\theta}\quad\text{or}\quad\frac{1}{\sin^2 2\theta'};$$

$$\therefore\ s=\frac{b^2}{a^2}\int_r^{\sqrt{a^2+b^2}}\frac{dr}{\sin 2\theta}\quad\text{or}\quad\int_r^{\sqrt{a^2+b^2}}\frac{dr}{\sin 2\theta'};$$

$$\therefore\ \frac{as}{b}=\frac{b}{a}\int_r^{\sqrt{a^2+b^2}}\frac{dr}{\sqrt{1-u^2}}\quad\text{or}\quad\frac{a}{b}\int_r^{\sqrt{a^2+b^2}}\frac{dr}{\sqrt{1-v^2}},$$

where $$u\equiv\cos 2\theta,\quad v\equiv\cos 2\theta'.$$

We shall adopt the first or the second forms according as a is $>$ or $<$ than b.

Let $$\lambda=\frac{\sqrt{a^4-b^4}}{a^2},\ (a \nless b);\quad =\cos 2\alpha,\ \text{where}\ \frac{b^2}{a^2}=\sin 2\alpha;$$

$$\mu=\frac{\sqrt{b^4-a^4}}{b^2},\ (a \ngtr b);\quad =\cos 2\beta,\ \text{where}\ \frac{a^2}{b^2}=\sin 2\beta.$$

In the case $a \nless b$, $u\equiv\cos 2\theta\equiv\dfrac{r^4+a^4-b^4}{2a^2r^2}$,

so $$\frac{r^2}{a^2}+\lambda^2\frac{a^2}{r^2}=2u;$$

$$\therefore\ \frac{r}{a}+\lambda\frac{a}{r}=\sqrt{2}\sqrt{u+\lambda},$$

$$\frac{r}{a}-\lambda\frac{a}{r}=\sqrt{2}\sqrt{u-\lambda};$$

$$\therefore\ r=\frac{a}{\sqrt{2}}(\sqrt{u+\lambda}+\sqrt{u-\lambda}),$$

$$dr=\frac{a}{2\sqrt{2}}\left(\frac{du}{\sqrt{u+\lambda}}+\frac{du}{\sqrt{u-\lambda}}\right),$$

$$\therefore \frac{a}{b}s=\frac{b}{2\sqrt{2}}\left[\int_u^1 \frac{du}{\sqrt{(1-u^2)(u+\lambda)}}+\int_u^1 \frac{du}{\sqrt{(1-u^2)(u-\lambda)}}\right]$$

$$=\frac{b}{2}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha},\ \cos\alpha\right)+\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha},\ \sin\alpha\right)\right]$$

where $\sin 2\alpha=\frac{b^2}{a^2}$ (Art. 388, 4).

In the case $a \not> b$, $v\equiv\cos 2\theta'=\frac{r^4+b^4-a^4}{2b^2r^2}$,

$$\frac{r^2}{b^2}+\mu^2\frac{b^2}{r^2}=2v,$$

and the work proceeds precisely as before, interchanging a and b, u and v, θ and θ', λ and μ, α and β, on the right-hand side of the values of $\frac{as}{b}$.

$$\therefore \frac{a}{b}s=\frac{a}{2}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta},\ \cos\beta\right)+\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta},\ \sin\beta\right)\right],$$

where $\theta'=\frac{1}{2}\sin^{-1}(\sin 2\beta\sin 2\theta)$ and $\sin 2\beta=\frac{a^2}{b^2}$.

The arc is in both cases measured from the vertex, where

$$r=\sqrt{a^2+b^2}.$$

598. In the case of the **Lemniscate,**

$$a=b,\quad r^2=2a^2\cos 2\theta=c^2\cos 2\theta, \text{ say};$$

then $\theta=\theta'$, and either case gives

$$s=\frac{a}{2}\cdot 2\operatorname{sn}^{-1}\left(\sqrt{2}\sin\theta,\ \frac{1}{\sqrt{2}}\right)$$

$$=a\operatorname{cn}^{-1}\left(\sqrt{1-2\sin^2\theta},\ \frac{1}{\sqrt{2}}\right)=\frac{c}{\sqrt{2}}\operatorname{cn}^{-1}\left(\sqrt{\cos 2\theta},\ \frac{1}{\sqrt{2}}\right)$$

$$=\frac{c}{\sqrt{2}}\operatorname{cn}^{-1}\left(\frac{r}{c},\ \frac{1}{\sqrt{2}}\right),\quad \text{as in Art. 592.}$$

599. It is a very instructive process to perform the same rectification first expressing θ in terms of r. We have

$$\sin^2 2\theta=1-\left(\frac{r^4+a^4-b^4}{2a^2r^2}\right)^2$$

$$=(r^2+a^2+b^2)(-r^2+a^2+b^2)(r^2-a^2+b^2)(r^2+a^2-b^2)/4a^4r^4$$

$$=[(a^2+b^2)^2-r^4][r^4-(a^2\sim b^2)^2]/4a^4r^4.$$

Let $$r=\sqrt{a^2+b^2}\,u \quad\text{and}\quad \lambda^2=\frac{a^2\sim b^2}{a^2+b^2},$$
the positive value to be taken.

$$\sin 2\theta=(a^2+b^2)\sqrt{(1-u^4)(u^4-\lambda^4)}/2a^2u^2,$$

and $$dr=\sqrt{a^2+b^2}\,du\,;$$

$$\therefore\ s=\frac{2b^2}{\sqrt{a^2+b^2}}\int_u^1\frac{u^2du}{\sqrt{(1-u^4)(u^4-\lambda^4)}}.$$

Again, $$\begin{aligned}(1-u^4)(u^4-\lambda^4)&=[(1-u^2)(u^2-\lambda^2)][(1+u^2)(u^2+\lambda^2)]\\&=[(1+\lambda^2)u^2-(u^4+\lambda^2)][(1+\lambda^2)u^2+(u^4+\lambda^2)]\\&=(1+\lambda^2)^2u^4-(u^4+\lambda^2)^2\\&=(1+\lambda^2)^2u^4(1-v^2),\end{aligned}$$

where $$\frac{u^4+\lambda^2}{u^2}=(1+\lambda^2)v.$$

This transformation gives

$$u^2+\frac{\lambda^2}{u^2}=(1+\lambda^2)v\,;$$

$$\therefore\ u+\frac{\lambda}{u}=\sqrt{(1+\lambda^2)v+2\lambda},$$

$$u-\frac{\lambda}{u}=\sqrt{(1+\lambda^2)v-2\lambda},$$

$$2u=\sqrt{(1+\lambda^2)v+2\lambda}+\sqrt{(1+\lambda^2)v-2\lambda},$$

$$\frac{4du}{\sqrt{1+\lambda^2}}=\frac{dv}{\sqrt{v+\frac{2\lambda}{1+\lambda^2}}}+\frac{dv}{\sqrt{v-\frac{2\lambda}{1+\lambda^2}}}\,;$$

$$\therefore\ s=\frac{2b^2}{\sqrt{a^2+b^2}}\cdot\int_v^1\frac{1}{(1+\lambda^2)\sqrt{1-v^2}}\cdot\frac{\sqrt{1+\lambda^2}}{4}\times\left\{\frac{1}{\sqrt{v+\frac{2\lambda}{1+\lambda^2}}}+\frac{1}{\sqrt{v-\frac{2\lambda}{1+\lambda^2}}}\right\}dv\,;$$

$$\therefore\ s=\frac{b^2}{2\sqrt{a^2+b^2}\sqrt{1+\lambda^2}}\times\left\{\int_v^1\frac{dv}{\sqrt{(1-v^2)\left(v+\frac{2\lambda}{1+\lambda^2}\right)}}+\int_v^1\frac{dv}{\sqrt{(1-v^2)\left(v-\frac{2\lambda}{1+\lambda^2}\right)}}\right\}.$$

Now an integral of form $I=\int_v^1\frac{dv}{\sqrt{(1-v^2)(v+c)}}$ can be converted at once into the standard Legendrian form as follows (Art. 388, 4):

Put $v+c=(1+c)\cos^2\phi$.

Then

$$I=\int_\phi^0 \frac{-2(1+c)\sin\phi\cos\phi\,d\phi}{\sqrt{\{(1+c)-(1+c)\cos^2\phi\}\{(1-c)+(1+c)\cos^2\phi\}(1+c)\cos^2\phi}}$$

$$=2\int_0^\phi \frac{d\phi}{\sqrt{2-(1+c)\sin^2\phi}}$$

$$=\sqrt{2}\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{1+c}{2}\sin^2\phi}},$$

and as in our case $c=\pm\frac{2\lambda}{1+\lambda^2}$, it is numerically less than unity and $\frac{1+c}{2}$ is positive and less than unity;

$$\therefore\quad \phi=\mathrm{am}(I/\sqrt{2}),\ \mathrm{mod.}\ \sqrt{\frac{1+c}{2}},$$

$$\cos\phi=\mathrm{cn}(I/\sqrt{2})\quad\text{and}\quad I=\sqrt{2}\,\mathrm{cn}^{-1}\sqrt{\frac{v+c}{1+c}}.$$

Hence, finally, we have

$$s=\frac{b^2}{\sqrt{2}\sqrt{a^2+b^2}\sqrt{1+\lambda^2}}\left\{\mathrm{cn}^{-1}\left(\sqrt{\frac{v+\frac{2\lambda}{1+\lambda^2}}{1+\frac{2\lambda}{1+\lambda^2}}},\ \sqrt{\frac{1}{2}\left(1+\frac{2\lambda}{1+\lambda^2}\right)}\right)\right.$$

$$\left.+\,\mathrm{cn}^{-1}\left(\sqrt{\frac{v-\frac{2\lambda}{1+\lambda^2}}{1-\frac{2\lambda}{1+\lambda^2}}},\ \sqrt{\frac{1}{2}\left(1-\frac{2\lambda}{1+\lambda^2}\right)}\right)\right\}$$

$$=\frac{b^2}{\sqrt{2}\sqrt{(a^2+b^2)+(a^2\sim b^2)}}$$

$$\times\left\{\mathrm{cn}^{-1}\left(\frac{u+\frac{\lambda}{u}}{1+\lambda},\ \frac{1+\lambda}{\sqrt{2(1+\lambda^2)}}\right)+\mathrm{cn}^{-1}\left(\frac{u-\frac{\lambda}{u}}{1-\lambda},\ \frac{1-\lambda}{\sqrt{2(1+\lambda^2)}}\right)\right\}$$

$$=\frac{b^2}{\sqrt{2}\sqrt{(a^2+b^2)+(a^2\sim b^2)}}$$

$$\times\left\{\mathrm{cn}^{-1}\left(\frac{r+\frac{\sqrt{a^4\sim b^4}}{r}}{\sqrt{(a^2+b^2)}+\sqrt{(a^2\sim b^2)}}\right)+\mathrm{cn}^{-1}\left(\frac{r-\frac{\sqrt{a^4\sim b^4}}{r}}{\sqrt{(a^2+b^2)}-\sqrt{(a^2\sim b^2)}}\right)\right\},$$

the respective moduli being

$$\frac{\sqrt{a^2+b^2}+\sqrt{a^2\sim b^2}}{\sqrt{2}\sqrt{(a^2+b^2)+(a^2\sim b^2)}}\quad\text{and}\quad\frac{\sqrt{a^2+b^2}-\sqrt{a^2\sim b^2}}{\sqrt{2}\sqrt{(a^2+b^2)+(a^2\sim b^2)}}.$$

For the twin-loop curve $a>b$,

$$s=\frac{b^2}{2a}\left\{\mathrm{cn}^{-1}\frac{r+\frac{\sqrt{a^4-b^4}}{r}}{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}+\mathrm{cn}^{-1}\frac{r-\frac{\sqrt{a^4-b^4}}{r}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}\right\},$$

with respective moduli

$$\frac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{2a}, \quad \frac{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}{2a}.$$

For the single-loop curve $a<b$,

$$s=\frac{b}{2}\left\{\mathrm{cn}^{-1}\frac{r+\dfrac{\sqrt{b^4-a^4}}{r}}{\sqrt{b^2+a^2}+\sqrt{b^2-a^2}}+\mathrm{cn}^{-1}\frac{r-\dfrac{\sqrt{b^4-a^4}}{r}}{\sqrt{b^2+a^2}-\sqrt{b^2-a^2}}\right\},$$

with respective moduli

$$\frac{\sqrt{b^2+a^2}+\sqrt{b^2-a^2}}{2b}, \quad \frac{\sqrt{b^2+a^2}-\sqrt{b^2-a^2}}{2b}.$$

600. The expressions written in this rectification are less simple than when written in terms of θ, as in Art. 597, but can readily be reduced.

In the case $a>b$, let $\sin 2\alpha=\dfrac{b^2}{a^2}$; then $r^4-2a^2r^2\cos 2\theta+a^4\cos^2 2\alpha=0$.

Also $\quad \cos 2\alpha=\sqrt{1-\dfrac{b^4}{a^4}}, \quad \sin\alpha=\dfrac{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}{2a},$

$$\cos\alpha=\frac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{2a},$$

and

$$\mathrm{cn}^{-1}\left(\frac{r+\dfrac{\sqrt{a^4-b^4}}{r}}{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}\right)=\mathrm{cn}^{-1}\frac{r+\dfrac{a^2\cos 2\alpha}{r}}{2a\cos\alpha}$$

$$=\mathrm{cn}^{-1}\left(\frac{\sqrt{\cos 2\alpha+\dfrac{r^4+a^4\cos^2 2\alpha}{2a^2r^2}}}{\sqrt{2}\cos\alpha}\right)$$

$$=\mathrm{cn}^{-1}\frac{\sqrt{\cos 2\alpha+\cos 2\theta}}{\sqrt{2}\cos\alpha}$$

$$=\mathrm{cn}^{-1}\sqrt{\frac{\cos^2\alpha-\sin^2\theta}{\cos^2\alpha}}=\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha}\right).$$

Similarly,

$$\mathrm{cn}^{-1}\left(\frac{r-\dfrac{\sqrt{a^4-b^4}}{r}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}\right)=\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha}\right).$$

Hence $a>b$,

$$s=\frac{b^2}{2a}\left[\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha}, \cos\alpha\right)+\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha}, \sin\alpha\right)\right],$$

as before.

Also for the case $a<b$, since

$$r^2+\frac{b^4-a^4}{r^2}=2b^2\cos 2\theta' \quad \text{(Art. 597)},$$

$$r^2+\frac{b^4\cos^2 2\beta}{r^2}=2b^2(1-2\sin^2\theta');$$

$$\therefore \left(r+\frac{b^2}{r}\cos 2\beta\right)^2 = 2b^2(1-2\sin^2\theta')+2b^2\cos 2\beta$$
$$=4b^2(\cos^2\beta-\sin^2\theta');$$

$$\therefore \frac{r+\dfrac{b^2\cos 2\beta}{r}}{2b\cos\beta}=\sqrt{1-\frac{\sin^2\theta'}{\cos^2\beta}};$$

$$\therefore \operatorname{cn}^{-1}\frac{r^2+b^2\cos 2\beta}{2rb\cos\beta}=\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right).$$

Similarly, $$\operatorname{cn}^{-1}\frac{r^2-b^2\cos 2\beta}{2rb\sin\beta}=\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta}, \sin\beta\right);$$

$$\therefore\ a<b, \qquad s=\frac{b}{2}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right)+\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta}, \sin\beta\right)\right],$$

where $\theta'=\frac{1}{2}\sin^{-1}(\sin 2\beta\sin 2\theta)$, the result of Art. 597.

601. **Serret's Method of Rectification of a Cassinian.**

A different method of rectification of a Cassinian Oval is given by Serret* connecting two arcs measured from different vertices of the curve, and expressing these arcs directly in terms of θ.

In the twin-oval case $a>b$, let A and B be the vertices of one of the ovals, and let a radius vector OQP be drawn

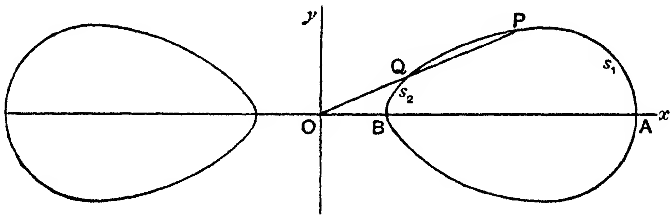


Fig. 145.

cutting that oval in Q and P. Let the vertex A be the one furthest from the centre O. Let arcs AP, BQ be called s_1, s_2 respectively. Let $b^2=a^2\sin 2\alpha$.

Then $$r^4-2a^2r^2\cos 2\theta+a^4=b^4.$$

Solving, $$r^2=a^2\cos 2\theta\pm a^2\sqrt{\cos^2 2\theta-\cos^2 2\alpha},$$

the upper sign giving OP^2, the lower OQ^2.

* *Calcul Intégral*, p. 265.

Now, as before, $$\frac{ds_1}{dr} = -\frac{b^2}{a^2}\frac{1}{\sin 2\theta},$$

and $$\frac{ds_1}{r\,d\theta} = +\frac{b^2}{r^2 - a^2\cos 2\theta} = +\frac{b^2}{a^2\sqrt{\cos^2 2\theta - \cos^2 2a}};$$

$$\therefore\ \frac{ds_1}{d\theta} = \frac{b^2}{a}\frac{\sqrt{\cos\ \ \theta + \sqrt{\cos^2 2\theta - \cos^2 2a}}}{\sqrt{\cos^2\ \ \theta - \cos^2 2a}},$$

the positive sign being taken as s_1 increases with θ.

Similarly $$\frac{ds_2}{d\theta} = \frac{b^2}{a}\frac{\sqrt{\cos 2\theta - \sqrt{\cos^2 2\theta - \cos^2 2a}}}{\sqrt{\cos^2 2\theta - \cos^2 2a}};$$

$$\therefore\ \left(\frac{ds_1}{d\theta} + \frac{ds_2}{d\theta}\right)^2 = \frac{b^4}{a^2}\frac{2(\cos 2\theta + \cos 2a)}{\cos^2 2\theta - \cos^2 2a} = \frac{b^4}{a^2}\frac{1}{\cos 2\theta - \cos 2a}$$

and $$\left(\frac{ds_1}{d\theta} - \frac{ds_2}{d\theta}\right)^2 = \frac{b^4}{a^2}\frac{2(\cos 2\theta - \cos 2a)}{\cos^2 2\theta - \cos^2 2a} = \frac{2b^4}{a^2}\frac{1}{\cos 2\theta + \cos 2a}.$$

Hence

$$s_1 + s_2 = \frac{b^2}{a}\sqrt{2}\int_0^\theta \frac{d\theta}{\sqrt{\cos 2\theta - \cos 2a}} = \frac{b^2}{a}\int_0^\theta \frac{d\theta}{\sqrt{\sin^2 a - \sin^2\theta}},$$

$$s_1 - s_2 = \frac{b^2}{a}\sqrt{2}\int_0^\theta \frac{d\theta}{\sqrt{\cos 2\theta + \cos 2a}} = \frac{b^2}{a}\int_0^\theta \frac{d\theta}{\sqrt{\cos^2 a - \sin^2\theta}}.$$

In these integrals put $\left.\begin{array}{l}\sin\theta = \sin a \sin\phi \\ \sin\theta = \cos a \sin\psi\end{array}\right\}$ respectively.
and

Then $$s_1 + s_2 = \frac{b^2}{a}\int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 a \sin^2\phi}},$$

$$s_1 - s_2 = \frac{b^2}{a}\int_0^\psi \frac{d\phi}{\sqrt{1 - \cos^2 a \sin^2\psi}},$$

i.e. $$\phi = \operatorname{am}\frac{a}{b^2}(s_1 + s_2),\ \text{mod.}\ \sin a,$$

$$\psi = \operatorname{am}\frac{a}{b^2}(s_1 - s_2),\ \text{mod.}\ \cos a;$$

$$\therefore\ \frac{\sin\theta}{\sin a} = \operatorname{sn}\frac{a}{b^2}(s_1 + s_2);\quad \frac{\sin\theta}{\cos a} = \operatorname{sn}\frac{a}{b^2}(s_1 - s_2);$$

$$\therefore\ s_1 + s_2 = \frac{b^2}{a}\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\sin a},\ \sin a\right),$$

$$s_1 - s_2 = \frac{b^2}{a}\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\cos a},\ \cos a\right);$$

$$\therefore \quad s_1 = \frac{b^2}{2a}\left[\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha}, \sin\alpha\right) + \mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha}, \cos\alpha\right)\right],$$

$$s_2 = \frac{b^2}{2a}\left[\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha}, \sin\alpha\right) - \mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha}, \cos\alpha\right)\right],$$

the former of these being the result previously obtained.

Reducing in the case of Bernoulli's Lemniscate, we have

$$\alpha = \frac{\pi}{4}, \quad r^2 = 2a^2\cos 2\theta,$$

$$s_1 = a\,\mathrm{sn}^{-1}\sqrt{2}\sin\theta$$

$$= a\,\mathrm{cn}^{-1}\sqrt{\cos 2\theta}, \text{ mod. } \frac{1}{\sqrt{2}},$$

$$= a\,\mathrm{cn}^{-1}\frac{r}{a\sqrt{2}}, \text{ as in Art. 598.}$$

602. **The Single-loop Case.**

In the one-loop case $a < b$, the same method cannot be adopted, and M. Serret considers the arcs traversed by a pair of perpendicular radii vectores OP, OQ, starting from the ends

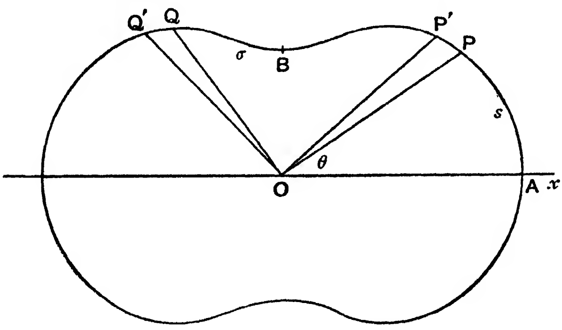


Fig. 146.

A, B of the two perpendicular axes. Let the arcs AP, BQ be respectively s and σ, and let $a^2 = b^2\sin 2\beta$. Then, solving as before,

$$r^4 - 2a^2r^2\cos 2\theta + a^4\cos^2 2\theta = a^4(\cos^2 2\theta + \cot^2 2\beta)$$

and $$r^2 = a^2\cos 2\theta \pm a^2\sqrt{\cos^2 2\theta + \cot^2 2\beta},$$

and the positive sign must now be taken.

Also, as before,

$$\frac{ds}{r\,d\theta} = \frac{b^2}{r^2 - a^2\cos 2\theta}, \quad \frac{ds}{d\theta} = \frac{b^2}{a}\frac{\sqrt{\cos 2\theta + \sqrt{\cos^2 2\theta + \cot^2 2\beta}}}{\sqrt{\cos^2 2\theta + \cot^2 2\beta}}.$$

Writing $\theta+\frac{\pi}{2}$ for θ,

$$\frac{d\sigma}{d\theta}=\frac{b^2}{a}\frac{\sqrt{-\cos 2\theta+\sqrt{\cos^2 2\theta+\cot^2 2\beta}}}{\sqrt{\cos^2 2\theta+\cot^2 2\beta}};$$

$$\therefore\ \left(\frac{ds}{d\theta}+\frac{d\sigma}{d\theta}\right)^2=\frac{2b^4}{a^2}\frac{\sqrt{\cos^2 2\theta+\cot^2 2\beta}+\cot 2\beta}{\cos^2 2\theta+\cot^2 2\beta},$$

and $$\left(\frac{ds}{d\theta}-\frac{d\sigma}{d\theta}\right)^2=\frac{2b^4}{a^2}\frac{\sqrt{\cos^2 2\theta+\cot^2 2\beta}-\cot 2\beta}{\cos^2 2\theta+\cot^2 2\beta}.$$

In each of these change the variable to θ', where

$$\sin 2\theta=\frac{\sin 2\theta'}{\sin 2\beta},\ \text{and therefore}\ \cos 2\theta\,d\theta=\frac{\cos 2\theta'\,d\theta'}{\sin 2\beta}.$$

Then

$$\cos^2 2\theta+\cot^2 2\beta=1+\cot^2 2\beta-\frac{\sin^2 2\theta'}{\sin^2 2\beta}=\frac{\cos^2 2\theta'}{\sin^2 2\beta}.$$

Then

$$\left(\frac{ds}{d\theta'}+\frac{d\sigma}{d\theta'}\right)^2=\frac{2b^4}{a^2}\frac{\cos 2\theta'+\cos 2\beta}{\cos^2 2\theta'}\frac{\cos^2 2\theta'}{\sin 2\beta}\frac{1}{1-\dfrac{\sin^2 2\theta'}{\sin^2 2\beta}}$$

$$=\frac{2b^4}{a^2}\frac{\cos 2\theta'+\cos 2\beta}{\sin^2 2\beta-\sin^2 2\theta'}\sin 2\beta$$

$$=\frac{2b^4}{a^2}\frac{\sin 2\beta}{\cos 2\theta'-\cos 2\beta}=\frac{b^4}{a^2}\frac{\sin 2\beta}{\sin^2\beta-\sin^2\theta'}.$$

Similarly

$$\left(\frac{ds}{d\theta'}-\frac{d\sigma}{d\theta'}\right)^2=\frac{2b^4}{a^2}\frac{\sin 2\beta}{\cos 2\theta'+\cos 2\beta}=\frac{b^4}{a^2}\frac{\sin 2\beta}{\cos^2\beta-\sin^2\theta'},$$

i.e. $$s+\sigma=\frac{b^2}{a}\sqrt{\sin 2\beta}\int\frac{d\theta'}{\sqrt{\sin^2\beta-\sin^2\theta'}},$$

$$s-\sigma=\frac{b^2}{a}\sqrt{\sin 2\beta}\int\frac{d\theta'}{\sqrt{\cos^2\beta-\sin^2\theta'}}.$$

In these integrals put respectively

$$\sin\theta'=\sin\beta\sin\phi\quad\text{and}\quad\sin\theta'=\cos\beta\sin\psi,$$

and remembering that $\sin 2\beta=\dfrac{a^2}{b^2}$,

$$s+\sigma=b\int_0^\phi\frac{d\phi}{\sqrt{1-\sin^2\beta\sin^2\phi}},$$

$$s-\sigma=b\int_0^\psi\frac{d\psi}{\sqrt{1-\cos^2\beta\sin^2\psi}};$$

$$\therefore\ \phi = \mathrm{am}\,\frac{s+\sigma}{b},\quad \psi = \mathrm{am}\,\frac{s-\sigma}{b};$$

$$\therefore\ \frac{\sin\theta'}{\sin\beta} = \mathrm{sn}\,\frac{s+\sigma}{b},\quad \frac{\sin\theta'}{\cos\beta} = \mathrm{sn}\,\frac{s-\sigma}{b};$$

$$s+\sigma = b\,\mathrm{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta}, \sin\beta\right);\quad s-\sigma = b\,\mathrm{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right);$$

whence

$$s = \frac{b}{2}\left[\mathrm{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta}, \sin\beta\right) + \mathrm{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right)\right],$$

$$\sigma = \frac{b}{2}\left[\mathrm{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta}, \sin\beta\right) - \mathrm{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right)\right],$$

where $$\theta' = \tfrac{1}{2}\sin^{-1}(\sin 2\beta \sin 2\theta).$$

The first of these was established in Art. 597.

603. **The Elastica or Lintearia.**

This curve is of considerable importance in various branches of Physics. It is (1) the form assumed by a uniform originally straight elastic rod bent into a bow by a bow-string, or by equal thrusts at its extremities, *i.e.* it may take the form *ABC* or

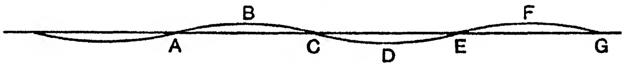


Fig. 147.

ABCDE, etc., according as the string is tied at *A* and *C*, *A* and *E*, etc. This is called an undulating elastica. When the bending is slight, the form is approximately the curve of cosines (E. J. Routh, *Anal. Statics*, vol. ii. p. 281, "Bending of Rods").

(2) It is the form assumed by a flexible thin rectangular sheet, two of whose opposite edges are fixed horizontally at

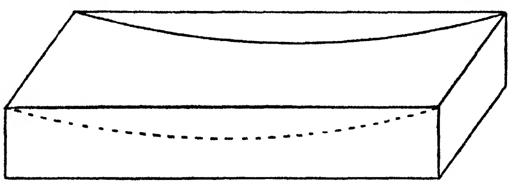


Fig. 148.

the same height, the flexible rectangular sheet forming the base of a rectangular box with vertical sides into which water is poured, the material being supposed impermeable for water

and the base fitting the sides so closely as to prevent appreciable escape of water. From this property the second name arises (*lintearius* = made of linen).

(3) The curve also occurs in the case of water drawn up by capillary action against a partially immersed vertical plate.

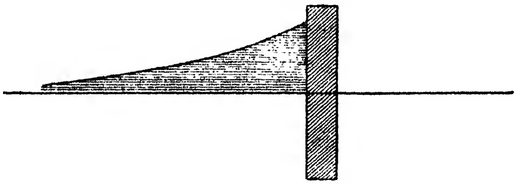


Fig. 149.

The curve may assume various shapes according to the physical circumstances occurring. It may undulate, or there may be any number of complete convolutions forming loops and nodes. Such cases are exhibited in the accompanying figures.

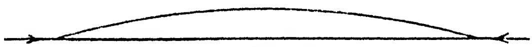


Fig. 150.

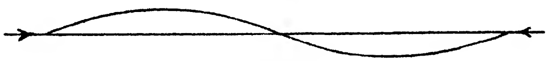


Fig. 151.

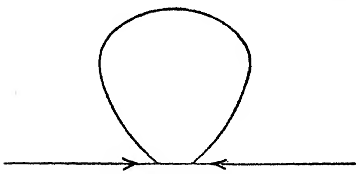


Fig. 152.

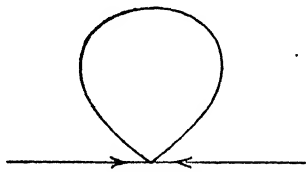


Fig. 153.

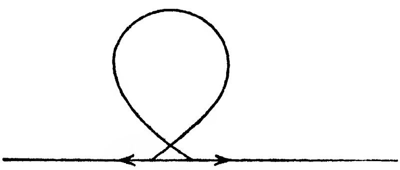


Fig. 154.

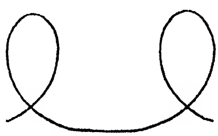


Fig. 155.

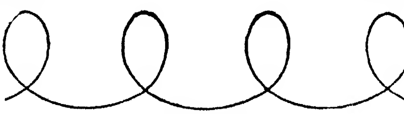


Fig. 156.

604. The determination of the nature of this curve is due to James Bernoulli (1654-1705).

For much detailed information as to the curve and its physical properties, the student may consult W. H. Besant, *Hydromechanics*, pages 168-171, p. 194, p. 201, etc.; G. M. Minchin, *Statics*, vol ii. p. 204; E. J. Routh, *Analytical Statics*, vol. ii. p. 283, etc., "Bending of Rods"; Sir A. G. Greenhill, *Elliptic Functions*, p. 87; and the article on Capillarity in the *Encyclopaedia Britannica*, by the late Sir J. Clerk-Maxwell.

605. The stress couple at any point being $\frac{K}{\rho}$, where ρ is the radius of curvature and K a certain constant called the flexural rigidity, we have as the geometrical property of the curve,

$$\frac{K}{\rho} = Ty,$$

where y is the ordinate from any point to the line of thrust and T the thrust, or string tension if the bow is bent as in the ordinary case by a bow-string.

Hence the equation to be considered is $\rho y = c^2$, c being a constant, and two cases arise accordingly as the curve is

(1) undulating, (2) nodal.

606. **Rectification of the Bow.**

Taking the bow-string as x axis, its mid-point O as origin, and a perpendicular through O as the y-axis, let y be the

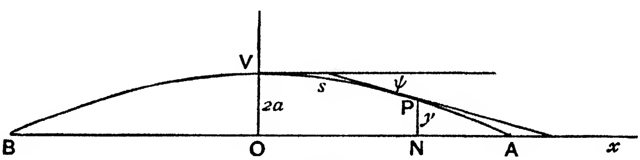


Fig. 157.

ordinate of any point P, and let ψ be the acute angle the tangent makes with the tangent at the vertex V of the arc, and let arc $VP = s$. Let $\psi = a$ when P is at A, and let $OV = 2a$

Then
$$\rho y = c^2;$$
$$\therefore \frac{c^2}{\rho} = y.$$

Differentiating, $$\frac{c^2}{\rho^2}\frac{d\rho}{d\psi} = -\frac{dy}{d\psi} = +\rho\sin\psi \quad \left(\rho = +\frac{ds}{d\psi}\right);$$

$$\therefore \frac{c^2}{\rho^3}d\rho = \sin\psi\, d\psi,$$

and integrating, $$\frac{c^2}{\rho^2} = 2(\cos\psi - \cos\alpha),$$

for $\psi = \alpha$ when $y=0$ and $\rho = \infty$, *i.e.* at A.

Hence $$s = \frac{c}{\sqrt{2}}\int_0^\psi \frac{d\psi}{\sqrt{\cos\psi - \cos\alpha}}$$

$$= \frac{c}{2}\int_0^\psi \frac{d\psi}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\psi}{2}}}.$$

Let $$\sin\frac{\psi}{2} = \sin\frac{\alpha}{2}\sin\chi;$$

$$\therefore \cos\frac{\psi}{2}\,d\psi = 2\sin\frac{\alpha}{2}\cos\chi\, d\chi;$$

$$\therefore s = c\int_0^\chi \frac{d\chi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\chi}}$$

$$= cF\left(\chi, \sin\frac{\alpha}{2}\right)$$

and $$\chi = \operatorname{am}\frac{s}{c};$$

$$\therefore \sin\frac{\psi}{2} = \sin\frac{\alpha}{2}\operatorname{sn}\frac{s}{c};\ \text{mod. } \sin\frac{\alpha}{2}.$$

And the intrinsic equation of the curve is therefore

$$s = c\,\operatorname{sn}^{-1}\left(\frac{\sin\frac{\psi}{2}}{\sin\frac{\alpha}{2}}, \sin\frac{\alpha}{2}\right). \quad \text{..................(1)}$$

The student should note the analogous result in Kinetics in Art. 389, viz. the case of the oscillating motion of a simple circular pendulum. For a comparison of the two results, see Greenhill, *Elliptic Functions*, p. 87.

The ordinate y is given by

$$y=\frac{c^2}{\rho}=2c\sqrt{\sin^2\frac{a}{2}-\sin^2\frac{\psi}{2}}$$

$$=2c\sin\frac{a}{2}\cos\chi=2c\sin\frac{a}{2}\,.\,\mathrm{cn}\frac{s}{c};$$

$$\therefore\ y=2c\sin\frac{a}{2}\,\mathrm{cn}\left(\frac{s}{c},\ \sin\frac{a}{2}\right).\quad\text{......................(2)}$$

To find the abscissa x, we have

$$\frac{dx}{ds}=\cos\psi;$$

$$\therefore\ \frac{dx}{d\psi}=\cos\psi\,\frac{ds}{d\psi}$$

and

$$\frac{dx}{d\chi}=\cos\psi\,\frac{ds}{d\chi}=c\,\frac{1-2\sin^2\frac{a}{2}\sin^2\chi}{\sqrt{1-\sin^2\frac{a}{2}\sin^2\chi}},$$

and adding $\dfrac{ds}{d\chi}$,

$$\frac{d(x+s)}{d\chi}=2c\sqrt{1-\sin^2\frac{a}{2}\sin^2\chi};$$

$$\therefore\ x+s=2c\int_0^{\chi}\sqrt{1-\sin^2\frac{a}{2}\sin^2\chi}\,d\chi,$$

i.e.

$$x=2cE\left(\chi,\ \sin\frac{a}{2}\right)-s.\quad\text{..........................(3)}$$

We thus have for the bow, or undulatory elastica, $\rho y=c^2$,

$$\left.\begin{aligned}
s&=c\,\mathrm{sn}^{-1}\left(\frac{\sin\frac{\psi}{2}}{\sin\frac{a}{2}},\ \sin\frac{a}{2}\right),\\
x&=2cE\left(\sin^{-1}\frac{\sin\frac{\psi}{2}}{\sin\frac{a}{2}},\ \sin\frac{a}{2}\right)-s,\\
y&=2c\sin\frac{a}{2}\,\mathrm{cn}\left(\frac{s}{c},\ \sin\frac{a}{2}\right).
\end{aligned}\right\}$$

607. **Rectification of the Elastica in the case when there are several Convolutions, viz. the Nodal Elastica.**

Taking the y-axis to pass through a vertex V as before and the line of terminal thrusts as the x-axis and ψ the angle

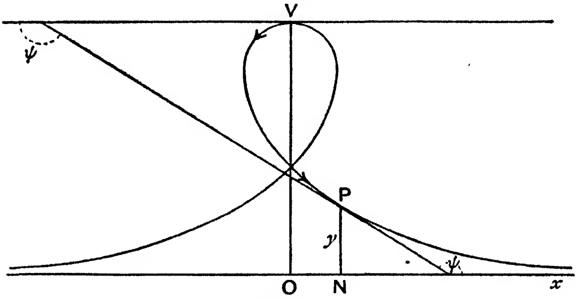


Fig. 158.

which the tangent at P has turned through in passing from V to P, we have again $\frac{c^2}{\rho}=y$.

$$\frac{c^2}{\rho^2}\frac{d\rho}{d\psi}=-\frac{dy}{d\psi}=\rho\sin\psi,$$

$$\frac{c^2}{\rho^3}\,d\rho=\sin\psi\,d\psi,$$

and integrating $\frac{c^2}{\rho^2}=2\cos\psi+\text{a constant}=2\cos\psi+A$, say. We have not, however, in this case, as we had before, any point at which ρ is infinite. Let $2a$ be the ordinate of the vertex. Then at V, $\rho=\frac{c^2}{2a}$.

$$\therefore \text{ putting } \quad \rho=\frac{c^2}{2a}, \quad \text{when} \quad \psi=0, \quad A=\frac{4a^2}{c^2}-2;$$

$$\therefore \frac{c^2}{\rho^2}=\frac{4a^2}{c^2}-2(1-\cos\psi)$$

$$=4\left(\frac{a^2}{c^2}-\sin^2\frac{\psi}{2}\right),$$

$\frac{a}{c}$ being >1, as ρ cannot be ∞ by supposition, and

$$\frac{ds}{d\psi}=\frac{c}{2}\frac{1}{\sqrt{\frac{a^2}{c^2}-\sin^2\frac{\psi}{2}}};$$

$$\therefore\ s=\frac{c^2}{2a}\int_0^{\psi}\frac{d\psi}{\sqrt{1-\frac{c^2}{a^2}\sin^2\frac{\psi}{2}}},\quad \text{or putting } \psi=2\chi,$$

$$=\frac{c^2}{a}\int_0^{\chi}\frac{d\chi}{\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}}\qquad\left(\frac{c}{a}<1\right)$$

$$=\frac{c^2}{a}F\left(\chi,\ \frac{c}{a}\right),$$

and $\chi=\operatorname{am}\dfrac{as}{c^2}$.

Hence the intrinsic equation is

$$s=\frac{c^2}{a}\operatorname{am}^{-1}\frac{\psi}{2}.\qquad\text{.........................(1)}$$

Also $y=\dfrac{c^2}{\rho}=2a\sqrt{1-\dfrac{c^2}{a^2}\sin^2\dfrac{\psi}{2}}=2a\,\Delta\left(\dfrac{\psi}{2}\right)=2a\,\Delta(\chi)$;

$$\therefore\ y=2a\operatorname{dn}\frac{as}{c^2}.\qquad\text{...(2)}$$

Again, $\dfrac{dx}{ds}=-\cos\psi,$

$$\frac{dx}{d\psi}=-\cos\psi\,\frac{ds}{d\psi},$$

$$\frac{dx}{d\chi}=-\frac{c^2}{a}\frac{(1-2\sin^2\chi)}{\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}}$$

$$=-a\frac{\left[\left(\frac{c^2}{a^2}-2\right)+2\left(1-\frac{c^2}{a^2}\sin^2\chi\right)\right]}{\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}};$$

$$\therefore\ x=a\left(2-\frac{c^2}{a^2}\right)\int_0^{\chi}\frac{d\chi}{\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}}-2a\int_0^{\chi}\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}\,d\chi;$$

$$\therefore\ x=\left(2\frac{a^2}{c^2}-1\right)s-2aE\left(\chi,\ \frac{c}{a}\right).\qquad\text{..............................(3)}$$

Hence, in the nodal case of $\rho y = c^2$,

$$\left.\begin{aligned} s &= \frac{c^2}{a}\,\mathrm{am}^{-1}\frac{\psi}{2}, \\ x &= \left(2\frac{a^2}{c^2} - 1\right)s - 2aE\left(\frac{\psi}{2},\ \frac{c}{a}\right), \\ y &= 2a\,\mathrm{dn}\,\frac{as}{c^2}. \end{aligned}\right\}$$

Compare with this case the result and process of Art. 390 for a revolving pendulum.

608. In the case of an infinitely long rod, imagining the elastica to touch the line of thrust at ∞, we have

$$\rho = \infty \quad \text{when} \quad \psi = \pi,$$

and
$$\frac{c^2}{\rho^2} = 2(1 + \cos\psi) = 4\cos^2\frac{\psi}{2};$$

$$\therefore\ \frac{ds}{d\psi} = \frac{c}{2}\sec\frac{\psi}{2} \quad \text{and} \quad s = c\log\tan\left(\frac{\psi}{4} + \frac{\pi}{4}\right),$$

s being still measured from the vertex.

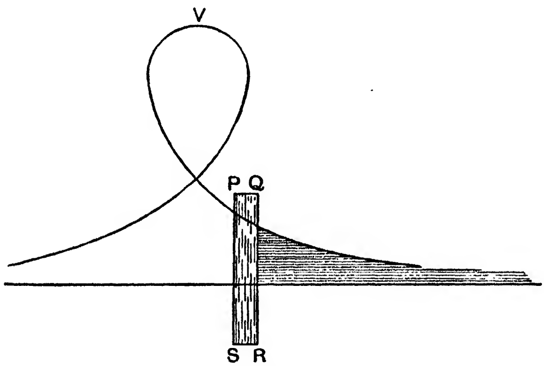


Fig. 159.

This species of elastica is called the Capillary curve (see Besant, *Hydromechanics*, p. 201), the shaded portion in Fig. 159 representing the water raised above the normal level by capillary action due to the presence of a partially immersed vertical plate *PQRS*. In this case $\rho = \frac{c}{2}$ at the vertex, and $c = a$, the modulus of the elliptic functions occurring in the second case becoming unity.

609. **Cotes' Spirals.**

These Spirals are defined by the pedal equation

$$\frac{1}{p^2}=\frac{A}{r^2}+B. \quad \text{(See } \textit{Diff. Calc.}\text{, Art. 454.)}$$

There are five varieties:

(1) $B=0$, an Equiangular Spiral.

(2) $A=1$, in which case B is essentially positive (as $r>p$); the curve is the Reciprocal Spiral (*Diff. Calc.*, Art. 452); and the other three are reducible to the polar forms

$$u=a\sin n\theta, \quad u=a\sinh n\theta \quad \text{and} \quad u=a\cosh n\theta.$$

(1) The rectification of an **equiangular spiral** has been effected in Art. 449, *Diff. Calc.*

(2) In the **reciprocal** spiral $r=\dfrac{a}{\theta}$, we have $\dot r=-\dfrac{a}{\theta^2}$, and

$$\frac{ds}{d\theta}=\frac{a}{\theta}\sqrt{1+\frac{1}{\theta^2}},$$

giving

$$\begin{aligned}
s&=a\int\frac{\sqrt{1+\theta^2}}{\theta^2}\,d\theta. \qquad (\text{Let } \theta=\tan\phi.)\\
&=a\int\cot^2\phi\,.\sec^3\phi\,d\phi=a\int\frac{d\phi}{\sin^2\phi\cos\phi}\\
&=a\int\frac{d\sin\phi}{\sin^2\phi(1-\sin^2\phi)}\\
&=a\int\left[\frac{1}{\sin^2\phi}+\frac{1}{2}\left(\frac{1}{1-\sin\phi}+\frac{1}{1+\sin\phi}\right)\right]d\sin\phi\\
&=-a\operatorname{cosec}\phi+\frac{a}{2}\log\frac{1+\sin\phi}{1-\sin\phi}\\
&=-a\frac{\sqrt{1+\theta^2}}{\theta}+\frac{a}{2}\log\frac{\sqrt{1+\theta^2}+\theta}{\sqrt{1+\theta^2}-\theta}.
\end{aligned}$$

The **remaining three** are rectifiable by the aid of elliptic functions. For instance, take the first, viz. $u=a\sin n\theta$ for the case $n>1$.

$$s=\int\frac{1}{u^2}\sqrt{\left(\frac{du}{d\theta}\right)^2+u^2}\,d\theta \quad \text{(Art. 511)};$$

$$\therefore\ as=\int_{\frac{\pi}{2n}}^{\theta}\frac{\sqrt{\sin^2 n\theta+n^2\cos^2 n\theta}}{\sin^2 n\theta}\,d\theta,$$

measuring s from the vertex at $\theta=\frac{\pi}{2n}$. (See figure of curve in Art. 387, *Diff. Calc.*)

Let $$n\theta=\phi;$$

$$\therefore\ as=\frac{1}{n}\int_{\frac{\pi}{2}}^{\phi}\frac{\sqrt{n^2-(n^2-1)\sin^2\phi}}{\sin^2\phi}\,d\phi$$

$$=\int_{\frac{\pi}{2}}^{\phi}\frac{\Delta}{\sin^2\phi}\,d\phi,$$

where $\Delta=\sqrt{1-\kappa^2\sin^2\phi}$ and $\kappa^2=\frac{n^2-1}{n^2}$;

$$\therefore\ as=\Big[-\Delta\cot\phi\Big]_{\frac{\pi}{2}}^{\phi}+\int_{\frac{\pi}{2}}^{\phi}\cot\phi\,\frac{-\kappa^2\sin\phi\cos\phi}{\Delta}\,d\phi$$

$$=-\Delta\cot\phi+\int_{\frac{\pi}{2}}^{\phi}\frac{(1-\kappa^2)-(1-\kappa^2\sin^2\phi)}{\Delta}\,d\phi$$

$$=-\Delta\cot\phi+(1-\kappa^2)\int_{\frac{\pi}{2}}^{\phi}\frac{d\phi}{\Delta}-\int_{\frac{\pi}{2}}^{\phi}\Delta\,d\phi$$

$$=-\frac{1}{n}\cot n\theta\sqrt{\sin^2 n\theta+n^2\cos^2 n\theta}+\frac{1}{n^2}\left\{F(n\theta,\kappa)-F_1\left(\frac{\pi}{2},\kappa\right)\right\}$$
$$-\left\{E(n\theta,\kappa)-E_1\left(\frac{\pi}{2},\kappa\right)\right\},$$

where $$\kappa^2=1-\frac{1}{n^2}.$$

610. **Bi-Polar Curves; Plane Elliptic Coordinates.**

Let S, H be fixed points, and let the distances of a moving point P from S and H be r_1 and r_2 respectively. Let $SH=2c$; O the mid-point of SH, PN a perpendicular from P upon SH; $ON=x$, $NP=y$; also let $r_1+r_2=2\xi$, $r_1-r_2=2\eta$.

Then ξ, η may be called the elliptic coordinates of P; for $\xi=$const. and $\eta=$const. give families of confocal ellipses and hyperbolae.

Let Δ be the area of the triangle SPH.

Then

$$16\Delta^2=(2c+r_1+r_2)(-2c+r_1+r_2)(2c-r_1+r_2)(2c+r_1-r_2),$$

i.e. $\Delta^2=(\xi^2-c^2)(c^2-\eta^2)$,

where ξ is necessarily $\nless c$ and $\eta\ngtr c$.

Hence $$cy=\sqrt{(\xi^2-c^2)(c^2-\eta^2)}.$$

Also, if m be the length of the median OP,

$$m^2+c^2=\frac{r_1^2+r_2^2}{2}=\xi^2+\eta^2;$$

$$\therefore\ x^2=m^2-y^2=(\xi^2+\eta^2-c^2)-\frac{(\xi^2-c^2)(c^2-\eta^2)}{c^2}=\frac{\xi^2\eta^2}{c^2};$$

$$\therefore\ cx=\xi\eta.$$

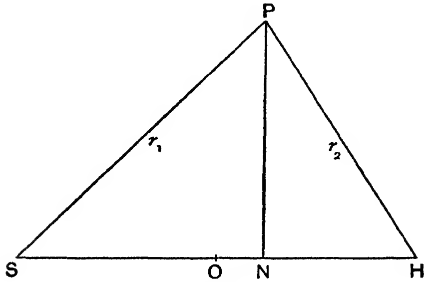


Fig. 160.

Thus the Cartesian coordinates of P are given by

$$cx=\xi\eta,\quad cy=\sqrt{\xi^2-c^2}\sqrt{c^2-\eta^2}; \quad\text{(1)}$$

$$\therefore\ c\,dx=\eta\,d\xi+\xi\,d\eta,$$

$$c\,dy=\xi\sqrt{\frac{c^2-\eta^2}{\xi^2-c^2}}\,d\xi-\eta\sqrt{\frac{\xi^2-c^2}{c^2-\eta^2}}\,d\eta.$$

And therefore, if ds be an element of the arc of the Bi-Polar curve traced by P for any relation between r_1 and r_2,

$$c^2ds^2=\left(\eta^2+\xi^2\frac{c^2-\eta^2}{\xi^2-c^2}\right)d\xi^2+\left(\xi^2+\eta^2\frac{\xi^2-c^2}{c^2-\eta^2}\right)d\eta^2$$

$$=c^2(\xi^2-\eta^2)\left(\frac{d\xi^2}{\xi^2-c^2}+\frac{d\eta^2}{c^2-\eta^2}\right),$$

and $$s=\int\sqrt{\xi^2-\eta^2}\sqrt{\frac{d\xi^2}{\xi^2-c^2}+\frac{d\eta^2}{c^2-\eta^2}}. \quad\text{(2)}$$

If we put $\qquad \xi=c\cosh v,\quad \eta=c\sin u,$

we have $$s=c\int\sqrt{\cosh^2 v-\sin^2 u}\sqrt{du^2+dv^2}. \quad\text{(3)}$$

Moreover, $$x=c\cosh v\sin u,$$
$$y=c\sinh v\cos u,$$
and $$x+\iota y=c\sin(u+\iota v),$$
the transformation used in Art. 590 for the rectification of the central conics.

The (u, v) system and the (ξ, η) system are therefore connected, and either may be regarded as "elliptic" coordinates. Moreover, we have a definite interpretation of u, v as used in Art. 590, viz.

$$v=\cosh^{-1}\frac{r_1+r_2}{2c},\quad u=\sin^{-1}\frac{r_1-r_2}{2c},$$

and they are thus expressed in terms of the bi-polar determination of a point.

Ex. Employ Formula (3) in the case

$$\sin u=m\cosh v.$$

To what curve does this equation refer?

611. If we wish to express the result of Art. 610 in terms of the original radii vectores r_1, r_2, we have

$$\xi^2-\eta^2=r_1r_2,$$

and
$$\frac{d\xi^2}{\xi^2-c^2}+\frac{d\eta^2}{c^2-\eta^2}=\frac{(dr_1+dr_2)^2}{(r_1+r_2)^2-4c^2}+\frac{(dr_1-dr_2)^2}{4c^2-(r_1-r_2)^2}$$

$$=\frac{(dr_1+dr_2)^2[4c^2-(r_1-r_2)^2]+(dr_1-dr_2)^2[(r_1+r_2)^2-4c^2]}{(2c+r_1+r_2)(-2c+r_1+r_2)(2c-r_1+r_2)(2c+r_1-r_2)}$$

$$=4\,\frac{r_1r_2(dr_1^2+dr_2^2)+dr_1dr_2(a^2-r_1^2-r_2^2)}{16\sigma(\sigma-a)(\sigma-r_1)(\sigma-r_2)},$$

where $2c=a$ and $2\sigma=a+r_1+r_2$;

$$\therefore\ s=\frac{1}{2}\int\sqrt{r_1r_2}\,\frac{\sqrt{r_1r_2(dr_1^2+dr_2^2)+(a^2-r_1^2-r_2^2)dr_1dr_2}}{\sqrt{\sigma(\sigma-a)(\sigma-r_1)(\sigma-r_2)}}.\quad(4)$$

List of Well-known Bi-Polar Equations.

612. The principal bi-polar cases of well-known curves are:

Name.	Bi-Polar Equation.	Form of Equation in Elliptic Coordinates.
1. Ellipse	$r_1+r_2=2a$	$\xi=a$
2. Hyperbola	$r_1-r_2=2a$	$\eta=a$
3. Cartesian oval	$l_1r_1+l_2r_2=n$	$\frac{\xi}{a}+\frac{\eta}{b}=1$
4. Circle	$r_1=\kappa r_2$	$\eta=m\xi$
5. Circle	$r_1^2+r_2^2=\kappa^2$	$\xi^2+\eta^2=\frac{\kappa^2}{2}$
6. Straight line	$r_1^2-r_2^2=\kappa^2$	$\xi\eta=\frac{\kappa^2}{4}$
7. Cassinian oval	$r_1r_2=\kappa^2$	$\xi^2-\eta^2=\kappa^2$

613. Ex. 1. Rectify the ellipse $r_1+r_2=2a$.

Here $$\xi=a, \quad d\xi=0.$$

$$s=\int_0^\eta \sqrt{\frac{a^2-\eta^2}{c^2-\eta^2}}\,d\eta \quad (\eta \text{ increasing}) \quad (\eta<c<a)$$

$$=aE\left(\theta, \frac{c}{a}\right), \quad \text{where } \eta=c\sin\theta \quad \text{(cf. Art. 567).}$$

Ex. 2. Rectify the hyperbola $r_1-r_2=2a$.

Here $$\eta=a, \quad d\eta=0.$$

$$s=\int_c^\xi \sqrt{\frac{\xi^2-a^2}{\xi^2-c^2}}\,d\xi \quad (\xi>c>a) \quad \text{(cf. Art 388, Case 6),}$$

$$=\tan\omega\sqrt{c^2-a^2\sin^2\omega}+\frac{c^2-a^2}{c}F\left(\omega, \frac{a}{c}\right)-cE\left(\omega, \frac{a}{c}\right),$$

where $$\frac{\xi^2-c^2}{\xi^2-a^2}=\sin^2\omega \quad \text{(cf. Art. 588).}$$

Ex. 3. Consider the case of the Bernoulli's Lemniscate $r_1r_2=c^2$.

Here $$\xi^2-\eta^2=c^2 \quad \text{and} \quad \frac{d\xi}{\eta}=\frac{d\eta}{\xi}.$$

Hence $$\frac{d\xi^2}{\xi^2-c^2}+\frac{d\eta^2}{c^2-\eta^2}=\frac{d\eta^2}{\eta^2+c^2}+\frac{d\eta^2}{c^2-\eta^2}=\frac{2c^2d\eta^2}{c^4-\eta^4};$$

$$\therefore \quad s=c^2\sqrt{2}\int_\eta^c \frac{d\eta}{\sqrt{c^4-\eta^4}} \quad \text{(cf. Art. 388, Case 2),}$$

$$=c\,\mathrm{cn}^{-1}\left(\frac{\eta}{c}, \frac{1}{\sqrt{2}}\right)$$ (cf. *Diff. Calc.*, Art. 458, and *Int. Calc.*, Art. 592).

614. **Use of Bi-Angular Coordinates.**

It is sometimes desirable to express an element of arc of a bi-polar curve in terms of the bi-angular coordinates θ_1, θ_2 which r_2, r_1 respectively make with the line joining the poles.

Let $f(r_1, r_2)=$const. be the bi-polar equation of a curve, c the distance between the poles S, H. Let the angles of the triangle SHP be θ_2, θ_1, θ_3; so that r_1, θ_2 are the polar coordinates of P with SH for initial line, r_2, θ_1 the polar coordinates with HS for initial line. Let the normal PG cut the line SH at G and the circumcircle of SHP at Q. Let

$$S\hat{P}Q=\chi_1, \quad H\hat{P}Q=\chi_2,$$

and let $$SQ=\rho_2, \quad HQ=\rho_1, \quad PQ=N.$$

Then $$r_1\frac{d\theta_2}{ds}=\cos\chi_1=\frac{c^2+\rho_1^2-\rho_2^2}{2c\rho_1},$$

$$-r_2\frac{d\theta_1}{ds}=\cos\chi_2=\frac{c^2+\rho_2^2-\rho_1^2}{2c\rho_2}.$$

Hence multiplying by ρ_1, ρ_2 respectively, and then adding and subtracting,

$$\rho_1 r_1 \frac{d\theta_2}{ds} - \rho_2 r_2 \frac{d\theta_1}{ds} = c, \quad \text{.........................(i)}$$

$$\rho_1 r_1 \frac{d\theta_2}{ds} + \rho_2 r_2 \frac{d\theta_1}{ds} = \frac{\rho_1^2 - \rho_2^2}{c}. \quad \text{.................(ii)}$$

Now *PSQH* being cyclic,

$$\rho_1 r_1 + \rho_2 r_2 = Nc.$$

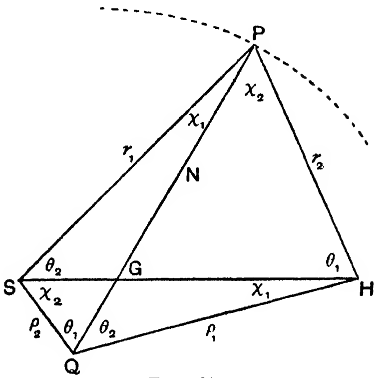


Fig. 161.

Hence these results may be respectively written

$$c\,ds = (Nc - \rho_2 r_2)\,d\theta_2 - (Nc - \rho_1 r_1)\,d\theta_1 \quad \text{...........(iii)}$$
$$= \rho_1 r_1\,d\theta_1 - \rho_2 r_2\,d\theta_2 - Nc(d\theta_1 - d\theta_2),$$

and $$\frac{\rho_2^2 - \rho_1^2}{c}\,ds = \rho_1 r_1\,d\theta_1 + \rho_2 r_2\,d\theta_2 + Nc\,d\theta_3, \quad \text{.........(iv)}$$

for $$d\theta_1 + d\theta_2 + d\theta_3 = 0.$$

The last equation (iv) is due to Mr. Roberts (*vide* Professor Williamson's *Integral Calculus*, p. 501, for a somewhat different proof).

Again, in travelling along the curve $f(r_1, r_2) = \text{const.}$,

$$f_{r_1}\,dr_1 + f_{r_2}\,dr_2 = 0 \quad \left(\text{where } f_{r_1} \text{ stands for } \frac{\partial f}{\partial r_1}, \text{ etc.}\right),$$

i.e. $$f_{r_1} \sin\chi_1 - f_{r_2} \sin\chi_2 = 0.$$

Hence $$(a)\ \frac{SG}{HG} = \frac{r_1 \sin\chi_1}{r_2 \sin\chi_2} = \frac{r_1 f_{r_2}}{r_2 f_{r_1}}$$

(see *Diff. Calc.*, p. 181, Ex. 32);

$$(b)\ \frac{\rho_2}{\rho_1} = \frac{\sin\chi_1}{\sin\chi_2} = \frac{f_{r_2}}{f_{r_1}}.$$

In cases in which $f(r_1, r_2)$ is homogeneous in r_1 and r_2 and of degree n, and if for convenience we write the constant as

$$c\frac{a^{n-1}}{n}, \quad \text{so that} \quad f(r_1, r_2)=c\frac{a^{n-1}}{n},$$

we have, by the theorems of Ptolemy and Euler,

$$\frac{\rho_1}{f_{r_1}}=\frac{\rho_2}{f_{r_2}}=\frac{r_1\rho_1+r_2\rho_2}{r_1f_{r_1}+r_2f_{r_2}}=\frac{Nc}{nf}=\frac{N}{a^{n-1}}=\nu, \text{ say.}$$

Then $$\rho_1=f_{r_1}\nu, \quad \rho_2=f_{r_2}\nu, \quad N=a^{n-1}\nu.$$

The quantities ν and N can be obtained in terms of r_1, r_2 as follows:

$$N^2=(r_1\rho_1+r_2\rho_2)\frac{r_1r_2+\rho_1\rho_2}{r_1\rho_2+r_2\rho_1}$$

(Hobson's *Trigonometry*, p. 203);

$$\therefore \quad a^{2n-2}\nu^2=\frac{r_1f_{r_1}+r_2f_{r_2}}{r_1f_{r_2}+r_2f_{r_1}}(r_1r_2+\nu^2f_{r_1}f_{r_2});$$

$$\therefore \quad \nu^2=\frac{r_1r_2(r_1f_{r_1}+r_2f_{r_2})}{a^{2n-2}(r_1f_{r_2}+r_2f_{r_1})-f_{r_1}f_{r_2}(r_1f_{r_1}+r_2f_{r_2})},$$

and ν is therefore found in terms of r_1, r_2 and the constant a.

And as $$\rho_1=\nu f_{r_1}, \quad \rho_2=\nu f_{r_2}, \quad N=a^{n-1}\nu,$$

ρ_1, ρ_2, N are also known in terms of r_1, r_2.

Also, since $$\frac{r_1}{\sin\theta_1}=\frac{r_2}{\sin\theta_2}=\frac{c}{\sin(\theta_1+\theta_2)}$$

and $$f(r_1, r_2)=c\frac{a^{n-1}}{n},$$

we have theoretically the means of expressing r_1, r_2, ρ_1, ρ_2 and N either in terms of θ_1 or in terms of θ_2, as required.

Hence the rectification of the curve depends upon the integration of either of the formulae

$$cs=\int\rho_1r_1\,d\theta_2-\int\rho_2r_2\,d\theta_1$$

or $$\frac{s}{c}=\int\frac{\rho_1r_1}{\rho_1^2-\rho_2^2}\,d\theta_2+\int\frac{\rho_2r_2}{\rho_1^2-\rho_2^2}\,d\theta_1,$$

or $$=\int\frac{\rho_2r_2}{\rho_2^2-\rho_1^2}\,d\theta_2+\int\frac{\rho_1r_1}{\rho_2^2-\rho_1^2}\,d\theta_1+\int\frac{Nc}{\rho_2^2-\rho_1^2}\,d\theta_3.$$

615. **Rectification of a Cartesian Oval. Genocchi's Result.**

The last form was used by Mr. Roberts in a proof of Prof. Angelo Genocchi's Theorem, that the arc of a Cartesian oval can be expressed in terms of three elliptic arcs.

Thus, for this oval, viz. $l_1 r_1 + l_2 r_2 = c l_3$,

we have $$\frac{\rho_1}{l_1} = \frac{\rho_2}{l_2} = \frac{Nc}{cl_3} = \frac{N}{l_3} = \nu, \text{ say,}$$

and $$r_1^2 = N^2 + \rho_2^2 - 2N\rho_2 \cos\theta_1 = \nu^2(l_2^2 - 2l_2 l_3 \cos\theta_1 + l_3^2),$$
$$r_2^2 = N^2 + \rho_1^2 - 2N\rho_1 \cos\theta_2 = \nu^2(l_3^2 - 2l_3 l_1 \cos\theta_2 + l_1^2),$$
$$c^2 = \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos\theta_3 = \nu^2(l_1^2 + 2l_1 l_2 \cos\theta_3 + l_2^2).$$

Hence
$$\frac{s}{c} = \int \frac{l_1}{l_2^2 - l_1^2}\frac{r_1}{\nu}\,d\theta_1 + \int \frac{l_2}{l_2^2 - l_1^2}\frac{r_2}{\nu}\,d\theta_2 + \int \frac{l_3}{l_2^2 - l_1^2}\frac{c}{\nu}\,d\theta_3,$$

and $$(l_2^2 - l_1^2)\frac{s}{c} = l_1 \int \sqrt{l_2^2 - 2l_2 l_3 \cos\theta_1 + l_3^2}\,d\theta_1$$
$$+ l_2 \int \sqrt{l_3^2 - 2l_3 l_1 \cos\theta_2 + l_1^2}\,d\theta_2$$
$$+ l_3 \int \sqrt{l_1^2 + 2l_1 l_2 \cos\theta_3 + l_2^2}\,d\theta_3.$$

And these are the integrations required in the rectification of ellipses. This is Genocchi's result.

For a full description of the elements of these ellipses and for many other important properties of the Cartesian Ovals, the student should consult Professor Williamson's *Differential Calculus*, pp. 375–382, and *Integral Calculus*, pp. 239–243.

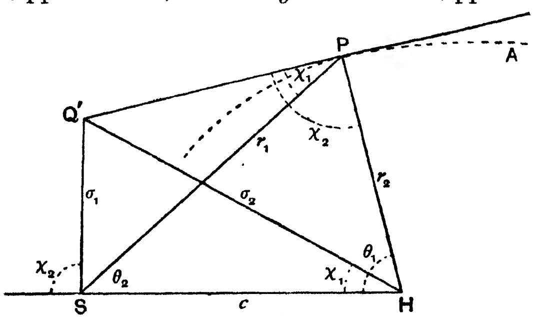


Fig. 162.

616. In a similar manner, if the *tangent* to the curve cut the circumcircle of the triangle SPH at a point Q' whose

bi-polar coordinates are σ_1, σ_2, and T be the length of the tangent PQ', which makes angles χ_1, χ_2 with r_1 and r_2, we have

$$\frac{dr_1}{ds} = -\cos\chi_1 = -\frac{c^2+\sigma_2^2-\sigma_1^2}{2c\sigma_2}$$

$$\frac{dr_2}{ds} = -\cos\chi_2 = \frac{c^2+\sigma_1^2-\sigma_2^2}{2c\sigma_1};$$

$$\therefore\ \sigma_2\frac{dr_1}{ds}+\sigma_1\frac{dr_2}{ds}=\frac{\sigma_1^2-\sigma_2^2}{c} \quad \text{and} \quad \sigma_1\frac{dr_2}{ds}-\sigma_2\frac{dr_1}{ds}=c;$$

$$\left.\begin{aligned}\therefore\ cs &= \int\sigma_1 dr_2 - \int\sigma_2 dr_1\\ \text{and}\qquad \frac{s}{c} &= \int\frac{\sigma_2}{\sigma_1^2-\sigma_2^2}dr_1+\int\frac{\sigma_1}{\sigma_1^2-\sigma_2^2}dr_2.\end{aligned}\right\}$$

617. **A General Theorem.**

Let there be two given curves

$$r_1=f_1(\theta), \quad r_2=f_2(\theta),$$

and let OP_2P_1 be a radius vector from the origin cutting these curves at P_2 and P_1.

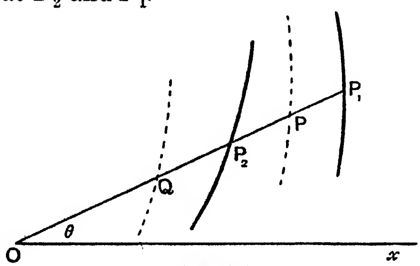


Fig. 163.

Let a point P be taken on OP_2P_1 so that

$$OP=\lambda_1 OP_1+\lambda_2 OP_2,$$

i.e. $$r=\lambda_1 r_1+\lambda_2 r_2$$

and $$\dot r=\lambda_1\dot r_1+\lambda_2\dot r_2,$$

λ_1, λ_2 being constants and dots denoting differentiation with regard to θ.

Hence

$$r^2+\dot r^2=\lambda_1^2(r_1^2+\dot r_1^2)+\lambda_2^2(r_2^2+\dot r_2^2)+2\lambda_1\lambda_2(r_1r_2+\dot r_1\dot r_2). \quad\ldots\ldots(1)$$

Let s_1, s_2, s_P be corresponding arcs of the three curves.

Now $$(r_1r_2+\dot r_1\dot r_2)^2+(r_1\dot r_1-r_2\dot r_2)^2=(r_1^2+\dot r_2^2)(r_2^2+\dot r_1^2)$$

and $$(r_1r_2+\dot r_1\dot r_2)^2+(r_1\dot r_2-r_2\dot r_1)^2=(r_1^2+\dot r_1^2)(r_2^2+\dot r_2^2).$$

Hence there are two cases of simplication, viz.

(A) when $r_1\dot{r}_1 - r_2\dot{r}_2 = 0$; (B) when $r_1\dot{r}_2 - r_2\dot{r}_1 = 0$.

Case (A) arises when the given curves are so related that

$$r_1^2 - r_2^2 = \text{const.} = a^2.$$

Case (B) arises when $\qquad \dfrac{\dot{r}_1}{r_1} = \dfrac{\dot{r}_2}{r_2}$,

i.e. $\dfrac{r_1}{r_2}$ = constant and the original curves similar and similarly situated with regard to O.

In case (A)

$$(r_1r_2 + \dot{r}_1\dot{r}_2)^2 = (r_2^2 + a^2 + \dot{r}_2^2)(r_1^2 - a^2 + \dot{r}_1^2)$$
$$= (\dot{s}_1^2 - a^2)(\dot{s}_2^2 + a^2)$$

and $\qquad \dot{s}_P^2 = \lambda_1^2\dot{s}_1^2 + \lambda_2^2\dot{s}_2^2 + 2\lambda_1\lambda_2\sqrt{(\dot{s}_1^2 - a^2)(\dot{s}_2^2 + a^2)}.$

If we take $\qquad \lambda_1 = \lambda_2 = \lambda$, say,

$$\dot{s}_P^2 = \lambda^2[\dot{s}_1^2 - a^2 + \dot{s}_2^2 + a^2 + 2\sqrt{\dot{s}_1^2 - a^2}\sqrt{\dot{s}_2^2 + a^2}]$$

and $\qquad \dot{s}_P = \lambda[\sqrt{\dot{s}_2^2 + a^2} + \sqrt{\dot{s}_1^2 - a^2}].$

If another point Q be taken on the same radius vector such that $\qquad \lambda_1 = -\lambda_2 = \lambda$, say,

then $\qquad \dot{s}_Q = \lambda[\sqrt{\dot{s}_2^2 + a^2} - \sqrt{\dot{s}_1^2 - a^2}].$

The radicals are placed in this order because

$$\dot{s}_2^2 + a^2 > \dot{s}_1^2 - a^2,$$

as may be seen as follows:

$$\dot{s}_2^2 + a^2 - (\dot{s}_1^2 - a^2) = (r_2^2 + \dot{r}_2^2) - (r_1^2 + \dot{r}_1^2) + 2a^2$$
$$= \dot{r}_2^2 - \dot{r}_1^2 + a^2$$
$$= \frac{r_1^2}{r_2^2}\dot{r}_1^2 - \dot{r}_1^2 + a^2$$
$$= \frac{a^2}{r_2^2}\dot{r}_1^2 + a^2 = a^2\left(1 + \frac{\dot{r}_1^2}{r_2^2}\right),$$

and is positive.

If we take

$$\lambda = \tfrac{1}{2}, \quad i.e.\ r_P = \frac{r_1 + r_2}{2} \quad \text{and} \quad r_Q = \frac{r_1 - r_2}{2}$$

then the P-curve is the locus of the mid-points of P_1P_2, and the Q-curve is such that $OQ = P_2P = PP_1$ and,

$$r_P \cdot r_Q = \frac{r_1 + r_2}{2} \cdot \frac{r_1 - r_2}{2} = \frac{a^2}{4},$$

so that the P and Q loci are inverse to each other.

For such derived loci we therefore have

$$s_P+s_Q=\int\sqrt{\dot{s}_2{}^2+a^2}\,d\theta,$$

$$s_P-s_Q=\int\sqrt{\dot{s}_1{}^2-a^2}\,d\theta,$$

and when these integrals can be found, s_P and s_Q can be found.

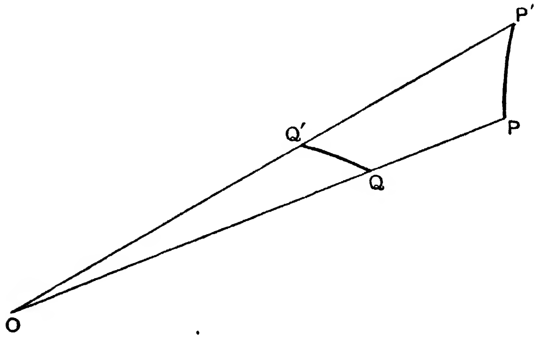


Fig. 164.

Again, the P and Q loci being inverse to each other, the constant of inversion being $\dfrac{a}{2}$,

$$ds_Q=\left(\frac{a}{2}\right)^2\frac{ds_P}{r_P{}^2}=\frac{r_1{}^2-r_2{}^2}{(r_1+r_2)^2}\,ds_P=\frac{r_1-r_2}{r_1+r_2}\,ds_P\,;$$

$$\therefore\ ds_P+ds_Q=\frac{2r_1}{r_1+r_2}\,ds_P=\frac{2r_1}{r_1-r_2}\,ds_Q,$$

$$ds_P-ds_Q=\frac{2r_2}{r_1+r_2}\,ds_P=\frac{2r_2}{r_1-r_2}\,ds_Q\,;$$

$$\therefore\ s_P=\tfrac{1}{2}\int\left(1+\frac{r_2}{r_1}\right)\sqrt{\dot{s}_2{}^2+a^2}\,d\theta=\tfrac{1}{2}\int\left(\frac{r_1}{r_2}+1\right)\sqrt{\dot{s}_1{}^2-a^2}\,d\theta,$$

$$s_Q=\tfrac{1}{2}\int\left(1-\frac{r_2}{r_1}\right)\sqrt{\dot{s}_2{}^2+a^2}\,d\theta=\tfrac{1}{2}\int\left(\frac{r_1}{r_2}-1\right)\sqrt{\dot{s}_1{}^2-a^2}\,d\theta.$$

618. In Case (B), $\quad r_1r_2+\dot{r}_1\dot{r}_2=\dot{s}_1\dot{s}_2\,;$

whence $\quad \dot{s}_P=\lambda_1\dot{s}_1+\lambda_2\dot{s}_2$

and $\quad s_P=\lambda_1s_1+\lambda_2s_2\,;$

but as the curves are then similar this is an obvious fact, and this part of the investigation does not render any new information.

619. **A Useful Case.**

In Case (A), it may happen that the derived curves are different branches of the same curve locus,

$$r^2-bF(\theta)r+\frac{a^2}{4}=0, \text{ say},$$

whose roots are $\qquad r_P, r_Q$ and $r_Pr_Q=\dfrac{a^2}{4}$,

and therefore $\qquad r_1^2-r_2^2=4r_Pr_Q=a^2.$

In this case the two branches of the curve are

$$r=\frac{bF(\theta)\pm\sqrt{b^2\{F(\theta)\}^2-a^2}}{2},$$

which are inverse to each other with regard to the pole, the constant of inversion being $\dfrac{a}{2}$.

And the "given" curves from which this curve is derived are

$$r_1=bF(\theta),$$
$$r_2=\sqrt{b^2\{F(\theta)\}^2-a^2}.$$

And if $\dot{s}_1$ and $\dot{s}_2$ be the differential coefficients of the arcs of these curves, the arcs of the derived P and Q curves are given by

$$2s_P=\int\sqrt{\dot{s}_2^2+a^2}\,d\theta+\int\sqrt{\dot{s}_1^2-a^2}\,d\theta,$$

$$2s_Q=\int\sqrt{\dot{s}_2^2+a^2}\,d\theta-\int\sqrt{\dot{s}_1^2-a^2}\,d\theta.$$

620. Ex. 1. Consider the rectification of the curve

$$4(x^2+y^2)(x-a)+a^2x=0.$$

Putting this into Polars,

$$r^2-ar\sec\theta+\frac{a^2}{4}=0,$$

$$r=\frac{a\sec\theta\pm a\tan\theta}{2}.$$

The original curves from which this is derived are obviously

$$r_1=a\sec\theta$$

and $\qquad r_2=a\tan\theta,$

the first being a straight line and incidentally an asymptote of the curve we wish to rectify.

The P and Q curves are branches of the same curve and inverse to each other. If N be the node on this curve (see Fig. 165) and A the point where the asymptote $x=2a$ cuts the x-axis, the several arcs are $AP_1=s_1$; $OP_2=s_2$, $NP=s_P$, $NQ=s_Q$.

Now
$$\dot{s}_1 = a\sec^2\theta, \quad \dot{s}_2^2 = a^2(\tan^2\theta + \sec^4\theta),$$
$$\dot{s}_1^2 - a^2 = a^2\tan^2\theta(\sec^2\theta + 1),$$
$$\dot{s}_2^2 + a^2 = a^2\sec^2\theta(\sec^2\theta + 1);$$
$$\therefore \sqrt{\dot{s}_1^2 - a^2} = a\frac{\sin\theta}{\cos^2\theta}\sqrt{1+\cos^2\theta},$$
$$\sqrt{\dot{s}_2^2 + a^2} = a\sec^2\theta\sqrt{1+\cos^2\theta}.$$

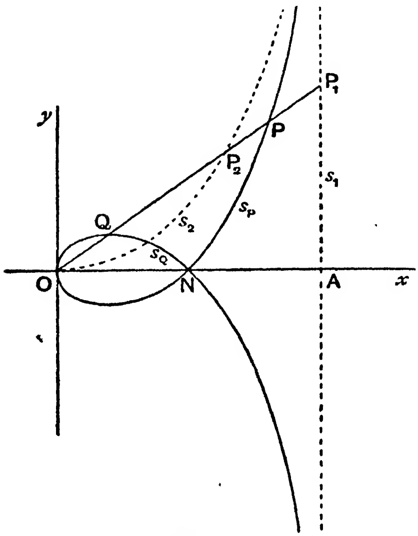


Fig. 165.

Now $\displaystyle\int \frac{\sin\theta}{\cos^2\theta}\sqrt{1+\cos^2\theta}\,d\theta$
$$= \sec\theta\sqrt{1+\cos^2\theta} + \int \frac{\sin\theta}{\sqrt{1+\cos^2\theta}}\,d\theta$$
$$= \sec\theta\sqrt{1+\cos^2\theta} - \sinh^{-1}(\cos\theta)$$
$$= \sqrt{\sec^2\theta + 1} - \sinh^{-1}(\cos\theta),$$
and $\displaystyle\int \sec^2\theta\sqrt{1+\cos^2\theta}\,d\theta = \tan\theta\sqrt{1+\cos^2\theta} + \int\frac{1-\cos^2\theta}{\sqrt{1+\cos^2\theta}}\,d\theta$
$$= \tan\theta\sqrt{1+\cos^2\theta} + \int\left(\frac{2}{\sqrt{1+\cos^2\theta}} - \sqrt{1+\cos^2\theta}\right)d\theta$$
$$= \tan\theta\sqrt{1+\cos^2\theta} + \sqrt{2}\int\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} - \sqrt{2}\int\sqrt{1-\tfrac{1}{2}\sin^2\theta}\,d\theta$$
$$= \sin\theta\sqrt{\sec^2\theta+1} + \sqrt{2}\,F\left(\theta, \frac{1}{\sqrt{2}}\right) - \sqrt{2}\,E\left(\theta, \frac{1}{\sqrt{2}}\right).$$

Hence
$$\text{arc } NP + \text{arc } NQ = a\left[\sin\theta\sqrt{\sec^2\theta+1} + \sqrt{2}\,F\left(\theta, \frac{1}{\sqrt{2}}\right) - \sqrt{2}\,E\left(\theta, \frac{1}{\sqrt{2}}\right)\right],$$
$$\text{arc } NP - \text{arc } NQ = a\left[\sqrt{\sec^2\theta+1} - \sinh^{-1}\cos\theta\right].$$

Thus arc NP and arc NQ are found by addition and subtraction. It is to be noted in this case, that although each separate arc NP, NQ requires for its expression the elliptic integrals of the first and second kinds, their difference is free from these functions, and expressible in terms of trigonometric and logarithmic functions.

Ex. 2. As a further example, consider the "derived" curves to be the branches of the Cartesian oval

$$r^2-(A+B\cos\theta)r+\frac{a^2}{4}=0.$$

The roots being r_P and r_Q, we have

$$r_1=r_P+r_Q=A+B\cos\theta,$$
$$r_2=r_P-r_Q=\sqrt{(A+B\cos\theta)^2-a^2},$$

and these are the "original" curves from which the Cartesian ovals are derived, the first being a Limaçon.

$$\dot{s}_1^2=r_1^2+\dot{r}_1^2=(A+B\cos\theta)^2+B^2\sin^2\theta$$
$$=A^2+2AB\cos\theta+B^2,$$
$$s_P-s_Q=\int\sqrt{A^2+B^2-a^2+2AB\cos\theta}\,d\theta. \quad \text{(See Art. 573.)}$$

Hence the difference between corresponding portions of the inner and outer loops of the curve

$$r^2-(A+B\cos\theta)r+\frac{a^2}{4}=0$$

can be expressed as the corresponding arc of a certain ellipse.

[This polar equation to the Cartesian oval is an ordinary conversion to polars, retaining one of the poles as origin, of $lr+mr'=n$, writing $r^2+c^2-2rc\cos\theta$ for r'^2 and performing the rationalization.]

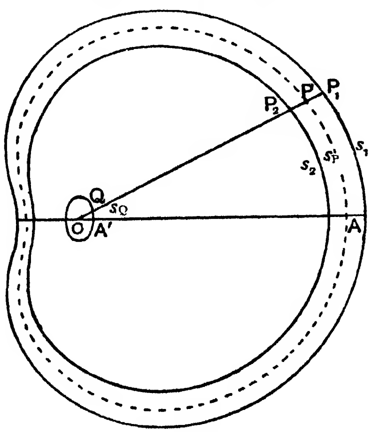


Fig. 166.

We may remind the student that any arc of this curve has already been expressed in terms of three elliptic arcs (Art. 615).

The arcs $s_P = AP$, $s_Q = A'Q$ to which the integration refers are shown in the figure.

We may construct the ovals as follows. Having drawn the limaçon $r_1 = A + B\cos\theta$ as explained in Art. 424, *Diff. Calc.*, take any radius vector OP_1, and on OP_1 for diameter construct a circle. Take centre P_1 and radius a and draw a second circle cutting the first at R. Then with centre O and radius OR draw a circle cutting OP_1 at P_2.

Then
$$OP_1 = A + B\cos\theta,$$
$$OP_2 = \sqrt{(A+B\cos\theta)^2 - a^2}.$$

Bisect P_1P_2 at P and make $OQ = PP_1$; then the points P and Q are points on the Cartesian oval.

MISCELLANEOUS PROBLEMS.

1. Prove that the three equations
$$x = c\log\sec\psi, \quad y = c(\tan\psi - \psi), \quad s = c(\sec\psi - 1),$$
represent one and the same curve. [I. C. S., 1893.]

2. Find the area of the curve
$$r_1 r_2 = b^2,$$
considering all cases which may arise.

3. Prove that the value of the integral
$$\int p\,xy\left(\frac{x}{a^3} + \frac{y}{b^3}\right)^2 ds,$$
taken round the ellipse $x^2/a^2 + y^2/b^2 = 1$, is $\frac{\pi}{2}$, p denoting the central perpendicular on the tangent at (x, y) and ds an element of arc.
[I. C. S., 1912.]

4. If the point x, y lies on the curve
$$y^2 = x^2 + 2px + q,$$
prove that
$$\frac{dx}{y} = \frac{dy}{x+p} = \frac{dx+dy}{x+y+p},$$
and hence obtain the integral of $\dfrac{dx}{\sqrt{x^2+2px+q}}$.

If, however, the point (x, y) lie on the circle $x^2 + y^2 = a^2$, show that the corresponding relation is
$$\frac{dx}{y} = -\frac{dy}{x} = \pm\frac{ds}{a},$$
where s is the length of the arc measured to the point (x, y). Deduce the known formula for the integral of $\dfrac{dx}{\sqrt{a^2-x^2}}$.
[I. C. S., 1908.]

5. Show that if δ stands for $\dfrac{d}{dx}$,

$$\delta^{-n}(uv)=v\,\delta^{-n}u-n\frac{dv}{dx}\delta^{-n-1}u+\frac{n(n+1)}{1\,.\,2}\frac{d^2v}{dx^2}\delta^{-n-2}u-\dots.$$

6. If $\dfrac{1}{R}\dfrac{dR}{dx}=\dfrac{fx+g}{ax^2+bx+c}$,

and if b^2-4ac be positive and the roots of $ax^2+bx+c=0$ be λ and μ, prove that $R=(x-\lambda)^p(x-\mu)^q$, where

$$\frac{p}{q}=\frac{f}{2a}\pm\frac{2ag-bf}{2a^2(\lambda-\mu)}.$$

And if $a=-1$, $b=0$, $c=1$,

$$R=(1-x^2)^{-\frac{f}{2}}\left(\frac{1+x}{1-x}\right)^{\frac{g}{2}}.$$

If b^2-4ac be negative,

$$R=(ax^2+bx+c)^{\frac{f}{2a}}e^{\frac{2ag-bf}{2\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}}.$$

If $b^2-4ac=0$,

$$R=\left(\frac{2ax+b}{2a}\right)^{\frac{f}{a}}e^{-\frac{2ag-bf}{a(2ax+b)}}.$$

[E. J. ROUTH, *Proc. L.M.S.*, vol. xvi., p. 250.]

7. Show that

$$\iiint\dots\int(x^2-1)^{-k-1}\left(\frac{x-1}{x+1}\right)^l dx\,dx\dots dx,$$

there being $2k+1$ integrations, $2k+1$ being an integer, though k may be a fraction, is equal to

$$\frac{1}{2^{2k+1}M}(x^2-1)^k\left(\frac{x-1}{x+1}\right)^l,$$

where $M=(k+l)(k+l-1)\dots$ to $2k+1$ factors.

[Cf. ROUTH, *Proc. L.M.S.*, vol. xvi., p. 249.]

8. ABC is a triangle with the corner A fixed and with sides AC, CB respectively $\sqrt{n}$ and $\sqrt{n+1}$, given lengths.

The side AB $(=r)$ makes an angle $\theta=nA-(n+1)B$ with a fixed straight line AX.

Show (1) that the path of B is rectifiable by the formula

$$s=\sqrt{n}\int_0^A\frac{d\theta}{\sqrt{1-\dfrac{n}{n+1}\sin^2\theta}}=\sqrt{n}\,\mathrm{am}^{-1}A,\quad \text{mod.}\sqrt{\frac{n}{n+1}}.$$

(2) When $n=1$ the rectification is the same as that of a Bernoulli's Lemniscate.

(3) The inclination of the normal to the radius vector is $A+B$.

(4) The area of the triangle is equal to the area of a sector of the curve starting from the axis AX.

[M. SERRET'S PROBLEM, *Calc. Int.*, p. 269.]

9. C is a point of maximum curvature on the Limaçon

$$r = a\cos\theta + b, \quad b > a;$$

A and A' are the two vertices Prove that the difference between the arcs AC, $A'C$ is $4a$. [ST. JOHN'S, 1891.]

10. If $y = x^3 - 3a^2x$, prove that

$$\frac{dx}{\sqrt{x^2 - 4a^2}} = \frac{dy}{3\sqrt{y^2 - 4a^6}},$$

and by integration express x explicitly in terms of y.

[OXFORD I. P., 1916.]

Apply this method to solve the cubic

$$x^3 - 3x^2 - 45x - 473 = 0.$$

11. Prove that

$$\int_0^{\frac{\pi}{2}} e^{-\sin 2x}\cos x\,dx = e^{-1}\left\{1 + \frac{1}{3.1!} + \frac{1}{5.2!} + \frac{1}{7.3!} + \dots\right\}.$$

[OXFORD I. P., 1916.]

12. Prove that if n be an odd positive integer greater than 3,

$$n\int_0^{\frac{\pi}{6}} \sin^n x\,dx = (2 - \sqrt{3})\frac{n-1}{n-2}\cdot\frac{n-3}{n-4}\dots\frac{4}{3}$$

$$-\frac{\sqrt{3}}{8}\left[\frac{1}{2^{n-3}} + \frac{n-1}{n-2}\,\frac{1}{2^{n-5}} + \dots + \frac{n-1}{n-2}\cdot\frac{n-3}{n-4}\dots\frac{4}{3}\right].$$

[OXFORD I. P., 1916.]

13. The parameters t_1, t_2 of two points A, B of the unicursal curve

$$x/(1-t^2) = y/(t-t^3) = a/(1+t^2)$$

are equal to $\tan\alpha$, $\tan\beta$, where

$$-\tfrac{1}{2}\pi < \alpha < -\tfrac{1}{4}\pi, \quad \tfrac{1}{4}\pi < \beta < \tfrac{1}{2}\pi.$$

Prove that the area of the curvilinear triangle AOB, where O is the double point, is

$$a^2\left[2 - \frac{\pi}{2} + \beta - \alpha - \sec\beta\sec\alpha\sin(\beta-\alpha) + \frac{1}{2}\tan\beta\tan\alpha\sin 2(\beta-\alpha)\right].$$

[OXFORD I. P., 1916.]

14. If n be a positive integer, show that

$$\int_0^{\pi} x\sin^2 nx\operatorname{cosec}^2 x\,dx = \tfrac{1}{2}n\pi^2.$$ [OXFORD I. P., 1912.]

15. By assuming

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx \text{ to be of the form } \frac{a+bx+cx^2+dx^3+ex^4}{\sqrt{1+x^4}},$$

obtain the integration by differentiation and equating coefficients, also obtain the result directly by putting $x^4 = z$.

16. Given a rational integral relation between x and y of the form

$$y^n + A_1 y^{n-1} + A_2 y^{n-2} + \ldots + A_n = 0 = F(x, y), \text{ say,}$$

where $A_1, A_2, \ldots A_n$ are rational functions of x, prove that when $\int y\,dx$ can be expressed algebraically in terms of x, then

$$\int y\,dx = B_0 + B_1 y + B_2 y^2 + B_3 y^3 + \ldots + B_{n-1} y^{n-1},$$

where $B_0, B_1, B_2 \ldots$ are rational functions of x. [ABEL.]

17. Assuming X to be a rational function of x, and $y^m = X$, and that $\int y\,dx$ is integrable in algebraic form and expressible as

$$\int y\,dx = P_0 + P_1 y + P_2 y^2 + \ldots + P_{m-1} y^{m-1},$$

where $P_0, P_1, \ldots P_{m-1}$ are rational functions of x, show that

$$P_0 = P_2 = P_3 = \ldots = P_{m-1} = 0,$$

that is that the integration must contain one term only, and that $\frac{1}{y}\int y\,dx$ is a rational algebraic function of x. [LIOUVILLE.]

18. If M and T be two rational polynomials, then, provided $\int \frac{M}{\sqrt[m]{T}}\,dx$ can be integrated in algebraic form at all, the form of the integral is $\frac{\theta}{\sqrt[m]{T}}$, where θ is a function of x.

Show also that

(1) θ is a rational function of x.

(2) That $MT = T\frac{d\theta}{dx} - \frac{1}{m}\theta\frac{dT}{dx}$.

(3) That θ is an integral polynomial expression and not of such form as $\frac{U}{V}$, where U and V are complete polynomials, *i.e.* not such that V contains x.

(4) That the degree of the polynomial θ is greater by unity than the degree of M.

Use these facts to show that $\int \frac{x^2\,dx}{\sqrt{1+x^4}}$ is not expressible algebraically.

[BERTRAND, *C. I.*, p. 94.]

19. P, Q, R being any rational algebraic polynomials, and assuming that when $\int \frac{P}{Q}\frac{dx}{\sqrt{R}}$ is integrable by means of the ordinary elementary functions, the integral must be of the form

$$\int \frac{P}{Q}\frac{dx}{\sqrt{R}} = \eta + \frac{\theta}{\sqrt{R}} + A\log(\alpha_1 + \beta_1\sqrt{R}) + B\log(\alpha_2 + \beta_2\sqrt{R}) + \dots,$$

where η, θ, α_1, β_1, α_2, etc., are rational functions of x, a result established by Abel,* show that when the integration can be reduced to one term, the general type of the result is either algebraic of form $\theta/\sqrt{R}$ or may be written as

$$\int \frac{P}{Q}\frac{dx}{\sqrt{R}} = A\tanh^{-1}\frac{\beta\sqrt{R}}{\alpha}.$$

In the latter case show that

$$(1)\quad \alpha^2 - \beta^2 R = Q.$$

$$(2)\quad \alpha\beta R\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha} + \frac{1}{2}\frac{R'}{R}\right) = \frac{P}{A},$$

$$(3)\quad 2\alpha' Q - Q'\alpha = \frac{2P}{A}\beta,$$

where accents denote differentiation with regard to x.

20. Show that

$$\int \frac{x^6(11x^2+37x+28)}{x^{14}-x^3-4x^2-5x-2}\frac{dx}{\sqrt{x+2}} = -2\tanh^{-1}\frac{(x+1)\sqrt{x+2}}{x^7}.$$

21. Prove that

$$\int \frac{x\,dx}{\sqrt{(x^2+2x-5)(x^2+4x-8)}} = \frac{1}{2}\tanh^{-1}\frac{x+4}{x+5}\sqrt{\frac{x^2+4x-8}{x^2+2x-5}}.$$

22. Prove that

$$(1)\quad \int \frac{5x^2+3x+1}{2x+1}\frac{dx}{\sqrt{x^4+2x^2+2x+1}} = \tanh^{-1}\frac{x\sqrt{x^4+2x^2+2x+1}}{x^3+x+1}.$$

$$(2)\quad \int \frac{3\tan^2\theta+2}{2\tan^4\theta+5\tan^2\theta+4}\sec\theta\,d\theta = \tan^{-1}\frac{\sin\theta}{1+\cos^2\theta}.$$

$$(3)\quad \int \frac{\sinh 2x\,dx}{\sqrt{\cosh^2 x+1}\sqrt{\cosh^2 x+2}} = 2\tanh^{-1}\sqrt{\frac{\cosh^2 x+1}{\cosh^2 x+2}}.$$

* *Œuvres.* See Bertrand, *Calc Intég.*, chap. v.

23. Show that

(i) $$\frac{2n+1}{2}a^2\int\sqrt{\frac{a^{2n-1}+x^{2n-1}}{a^{2n+1}+x^{2n+1}}}\frac{dx}{a^2-x^2}-\frac{2n-1}{2}\int\sqrt{\frac{a^{2n+1}+x^{2n+1}}{a^{2n-1}+x^{2n-1}}}\frac{dx}{a^2-x^2}$$
$$=\tanh^{-1}x\sqrt{\frac{x^{2n-1}+a^{2n-1}}{x^{2n+1}+a^{2n+1}}}.$$

(ii) $$\frac{2n+1}{2}\int\sqrt{\frac{1+\sin^{2n-1}\theta}{1+\sin^{2n+1}\theta}}\sec\theta\,d\theta$$
$$-\frac{2n-1}{2}\int\sqrt{\frac{1+\sin^{2n+1}\theta}{1+\sin^{2n-1}\theta}}\sec\theta\,d\theta=\tanh^{-1}\left(\sin\theta\sqrt{\frac{1+\sin^{2n-1}\theta}{1+\sin^{2n+1}\theta}}\right).$$

24. Integrate the following:

(i) $\displaystyle\int\frac{(2x+1)\,dx}{\sqrt{x^4+2x^3-3x^2-4x+3}}$. (ii) $\displaystyle\int\frac{x\,dx}{\sqrt{x^4+2x^3-3x^2-ax+a}}$.

[ABEL.]

(iii) $\displaystyle\int\frac{5x^2+15x+12}{5x^2+15x+9}\frac{dx}{\sqrt{(x+1)(x+2)}}$.

(iv) $\displaystyle\int\frac{(1+8x)\,dx}{\sqrt{1+6x+4x^2}\sqrt{1-2x+4x^2}}$.

(v) $\displaystyle\int\frac{(2x+a)\,dx}{\sqrt{x^4+2ax^3+3a^2x^2+2a^3x-a^4}}$.

(vi) $\displaystyle\int\frac{3x^4-2x^3+1}{2x^3-x^2+1}\frac{dx}{\sqrt{x^4+1}}$. (vii) $\displaystyle\int\frac{3x+1}{1-x-2x^2-x^3}\frac{dx}{\sqrt{x}}$.

(viii) $\displaystyle\int\frac{3x^4+1}{1-x^2-x^6}\frac{dx}{\sqrt{x^4+1}}$. (ix) $\displaystyle\int\frac{x^2-(a+b)x-ab}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}}\,dx$

(x) $\displaystyle\int\frac{1+x-x^2}{1-x+x^2+x^3}\frac{dx}{\sqrt{x^2-1}}$. (xi) $\displaystyle\int\frac{1+x^4}{1+x^2-x^4}\frac{dx}{\sqrt{x^4-1}}$.

(xii) $\displaystyle\int\frac{1+3x^4+2x^5}{1+2x-x^6}\frac{dx}{\sqrt{1+x^4}}$.

25. Show that the whole perimeter and area of a single loop of the curve $r=2a\cos n\theta$ $(n>1)$ are respectively equal to the whole perimeter and area of the ellipse $x^2+n^2y^2=a^2$. [OXF. I. P., 1911.]

26. If an element ds of a curve lie at distance r from the origin, and subtends an angle $d\theta$ there, it is known that unit electric current flowing along ds produces a magnetic force at the origin at right angles to the plane of the curve proportional to $\dfrac{d\theta}{r}$.

Show that if unit current flows through a thin endless wire of given length in the form of an ellipse, the magnetic force due to the current at the centre of the ellipse is inversely proportional to the area of the ellipse. [OXFORD II. P., 1913.

27. A current of electricity is flowing round a fine wire $ABCD \dots KA$ bent into a plane polygon. O is any point within the polygon, and perpendiculars OP, OQ, OR, ... are drawn to the sides KA, AB, BC, etc., respectively, and again perpendiculars whose lengths are α, β, γ, ... from O upon the sides PQ, QR, RS, ... of the inscribed polygon $PQRS \dots$. Show that the magnetic force on unit particle situated at O is

$$\iota \Sigma \frac{\sin A}{\alpha},$$

where ι is the current strength.

28. Show that the perimeter of an ellipse of axes $2a$, $2b$ and small eccentricity e is approximately equal to the perimeter of a circle of diameter $a+b$, with an error which is only about 0·0025 per cent. when e is as great as 0·2. [MATH. TRIP. PART II., 1913.]

CHAPTER XVIII.

RECTIFICATION (III.). MISCELLANEOUS THEOREMS.

621. **Arc of an Inverse Curve.**

Let s and s' be the corresponding arcs of a curve and of its inverse with regard to a fixed point O, the constant of inversion being k.

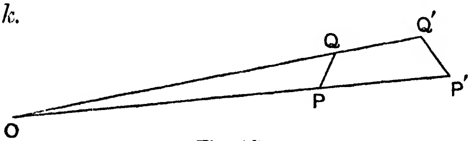


Fig. 167.

Then if P, Q be points on the curve and P', Q' the inverse points, we have

$$P'Q' = k^2\frac{PQ}{OP\,.\,OQ}.\quad \text{(See } \textit{Diff. Calc.}\text{, p. 174.)}$$

And ultimately, when Q and Q' are made to travel along their respective paths to ultimate coincidence with P and P',

$$ds' = k^2\frac{ds}{r^2},$$

i.e. $$s' = k^2\int\frac{ds}{r^2}, \quad \text{..........................(1)}$$

giving the arc of the inverse in terms of elements of the original curve.

622. **Modifications for Various Coordinate Systems.**

This formula may be modified as required for different systems of coordinates, and with the usual notation, we have for polars, the inversion being with regard to the pole,

$$s' = k^2\int\frac{\sqrt{dr^2 + r^2 d\theta^2}}{r^2} = k^2\int\sqrt{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}\,d\theta. \quad \text{.........(2)}$$

$$= k^2\int\frac{d\theta}{p} \quad \text{..........................(3)}$$

Again, we may write

$$s' = k^2\int \frac{1}{r^2}\frac{ds}{d\psi}\,d\psi = k^2\int \frac{\rho}{r^2}\,d\psi, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

i.e. as a formula suitable for tangential polars,

$$s' = k^2\int \frac{p+\dfrac{d^2p}{d\psi^2}}{p^2+\left(\dfrac{dp}{d\psi}\right)^2}\,d\psi, \quad \ldots\ldots\ldots\ldots\ldots(5)$$

or for pedal equations,

$$s' = k^2\int \frac{1}{r^2}\frac{ds}{dr}\,dr = k^2\int \frac{dr}{r^2\cos\phi} = k^2\int \frac{dr}{r\sqrt{r^2-p^2}}, \quad \ldots\ldots(6)$$

and for Cartesians,

$$s' = \int \frac{\sqrt{dx^2+dy^2}}{x^2+y^2} = k^2\int \frac{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{x^2+y^2}\,dx, \quad \ldots\ldots\ldots\ldots(7)$$

the inversion being with regard to the origin;

or $$s' = k^2\int \frac{\sqrt{dx^2+dy^2}}{(x-a)^2+(y-b)^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

if the inversion is with regard to the point (a, b).

623. **Illustrative Examples.**

1. Consider the arc of the inverse of the parabola

$$p^2 = ar \quad \left(\text{or } \frac{2a}{r} = 1+\cos\theta\right)$$

with regard to the focus; *i.e.* a cardioide.

Here

$$s' = k^2\int \frac{dr}{r\sqrt{r^2-p^2}} = k^2\int \frac{dr}{r\sqrt{r^2-ar}} = -k^2\int \frac{du}{\sqrt{1-au}}, \quad \text{if } r=\frac{1}{u},$$

$$= \frac{2k^2}{a}\sqrt{1-au} = \frac{2k^2}{a}\sin\frac{\theta}{2}.$$

2. Rectification of the inverse with regard to the centre of the first negative pedal of an ellipse with regard to the centre.

The ellipse being $x^2/a^2+y^2/b^2=1$, the first negative pedal is the envelope of

$$x\cos\psi + y\sin\psi = p, \quad \text{where } \frac{1}{p^2} = \frac{\cos^2\psi}{a^2}+\frac{\sin^2\psi}{b^2}.$$

Hence the tangential polar equation is

$$p = \frac{ab}{\sqrt{a^2\sin^2\psi + b^2\cos^2\psi}}.$$

Differentiating we have

$$\frac{dp}{d\psi} = -ab\,\frac{(a^2-b^2)\sin\psi\cos\psi}{(a^2\sin^2\psi+b^2\cos^2\psi)^{\frac{3}{2}}},$$

$$\frac{d^2p}{d\psi^2} = -ab(a^2-b^2)\frac{b^2\cos^4\psi - a^2\sin^4\psi - 2(a^2-b^2)\sin^2\psi\cos^2\psi}{(a^2\sin^2\psi+b^2\cos^2\psi)^{\frac{5}{2}}};$$

whence

$$p+\frac{d^2p}{d\psi^2} = \quad ab\,\frac{a^2(2a^2-b^2)\sin^2\psi+b^2(2b^2-a^2)\cos^2\psi}{(a^2\sin^2\psi+b^2\cos^2\psi)^{\frac{5}{2}}}$$

and

$$p^2+\left(\frac{dp}{d\psi}\right)^2 = \quad a^2b^2\,\frac{a^4\sin^2\psi+b^4\cos^2\psi}{(a^2\sin^2\psi+b^2\cos^2\psi)^3}.$$

Hence

$$s'\cdot\frac{ab}{k^2} = \int\frac{a^2(2a^2-b^2)\sin^2\psi+b^2(2b^2-a^2)\cos^2\psi}{a^4\sin^2\psi+b^4\cos^2\psi}\sqrt{a^2\sin^2\psi+b^2\cos^2\psi}\,d\psi$$

$$=\int\left(2-\frac{a^2b^2}{a^4\sin^2\psi+b^4\cos^2\psi}\right)\cdot\sqrt{a^2\sin^2\psi+b^2\cos^2\psi}\,d\psi$$

$$=\int\Bigg[2\sqrt{a^2\sin^2\psi+b^2\cos^2\psi}$$

$$-\frac{a^2b^2}{a^2+b^2}\frac{(a^4\sin^2\psi+b^4\cos^2\psi+a^2b^2)}{(a^4\sin^2\psi+b^4\cos^2\psi)\sqrt{a^2\sin^2\psi+b^2\cos^2\psi}}\Bigg]d\psi$$

Hence if e be the eccentricity of the ellipse, and the integration be taken from ψ to $\frac{\pi}{2}$ and if χ be the complement of ψ, we have

$$s'=\frac{k^2}{b}\left[2E(\chi, e)-\frac{1-e^2}{2-e^2}F(\chi, e)-\frac{(1-e^2)^2}{2-e^2}\Pi(\chi, e, e^4-2e^2)\right].$$

This curve therefore requires all three kinds of the Legendrian integrals for its rectification.

Note for the first negative central pedal of an ellipse that we have incidentally

(1) $$\rho=ab\,\frac{a^2(2a^2-b^2)\sin^2\psi+b^2(2b^2-a^2)\cos^2\psi}{(a^2\sin^2\psi+b^2\cos^2\psi)^{\frac{5}{2}}};$$

(2) $$r^2=a^2b^2\,\frac{a^4\sin^2\psi+b^4\cos^2\psi}{(a^2\sin^2\psi+b^2\cos^2\psi)^3};$$

(3) $$\int p^2\,d\theta=\int\frac{p^3\rho}{r^2}d\psi=3ab\tan^{-1}\left(\frac{a}{b}\tan\psi\right)-(a^2+b^2)\tan^{-1}\left(\frac{a^2}{b^2}\tan\psi\right);$$

(4) $$\int_\psi^{\frac{\pi}{2}}p\,d\psi=b\int_\psi^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-e^2\cos^2\psi}}=b\int_0^\chi\frac{d\chi}{\sqrt{1-e^2\sin^2\chi}}=bF(\chi, e).$$

3. Central inversion of epi- or hypo-cycloids.

Here $p = A \sin B\psi$, where $\left.\begin{array}{l} A = a + 2b, \\ B = \dfrac{a}{a+2b}. \end{array}\right\}$ See *Diff. Calc.*, Art. 410

$$s' = k^2 \int_0^\psi \frac{p + \dfrac{d^2p}{d\psi^2}}{p^2 + \left(\dfrac{dp}{d\psi}\right)^2} d\psi,$$

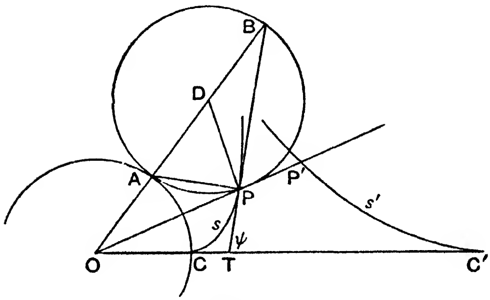


Fig. 168.

$$s' = k^2 \int_0^\psi \frac{A(1-B^2)\sin B\psi \, d\psi}{A^2\{1-(1-B^2)\cos^2 B\psi\}}, \quad \text{or} \quad -k^2 \int_0^\psi \frac{A(B^2-1)\sin B\psi \, d\psi}{A^2\{1+(B^2-1)\cos^2 B\psi\}}$$

$$= \frac{k^2}{AB}\sqrt{1-B^2}\left[\tanh^{-1}(\sqrt{1-B^2}\cos B\psi)\right]_\psi^0 \left.\right\} \text{ for the inverses of epi-cycloids,}$$

or $$= \frac{k^2}{AB}\sqrt{B^2-1}\left[\tan^{-1}(\sqrt{B^2-1}\cos B\psi)\right]_0^\psi \left.\right\} \text{ for the inverses of hypo-cycloids.}$$

E.g. in the case of the inverse of the cardioide with regard to the centre of the fixed circle $a = b$, $A = 3a$, $B = \frac{1}{3}$,

$$s' = \frac{2k^2\sqrt{2}}{3a}\left[\tanh^{-1}\left(\frac{2\sqrt{2}}{3}\cos\frac{\psi}{3}\right)\right]_\psi^0.$$

In the case of the inverse of the three-cusped hypocycloid

$$b = -\tfrac{1}{3}a, \quad A = \frac{a}{3}, \quad B = 3,$$

$$s' = \frac{2k^2\sqrt{2}}{a}\left[\tan^{-1}(2\sqrt{2}\cos 3\psi)\right]_0^\psi.$$

Note that these inverses are such that their arcs are expressible logarithmically if derived from epicycloids, or by means of circular functions if derived from hypocycloids.

4. Inverse of the parabola $y^2 = 4ax$ with regard to the point $x = -3a$, $y = 0$.

The general problem for any point on the axis is discussed by Mr. R. A. Roberts, in the *Proceedings of the London Mathematical Society*, vol. xviii., p. 202.

Taking am^2, $2am$ as the current coordinates of a point P on the curve $y^2=4ax$, an element of arc is given by

$$ds=\sqrt{dx^2+dy^2}=2a\sqrt{1+m^2}\,dm.$$

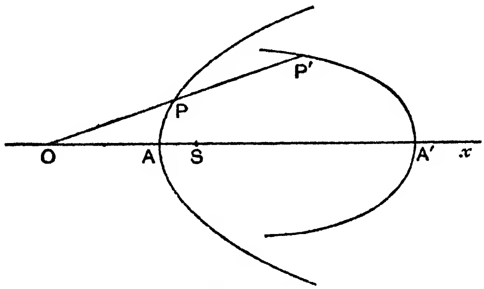


Fig. 169.

Also $$OP^2=(am^2+3a)^2+4a^2m^2=a^2m^4+10a^2m^2+9a^2$$
$$=a^2(m^2+1)(m^2+9),$$

and the element ds' of the inverse is $ds'=k^2\dfrac{ds}{OP^2}$;

$$\therefore\ s'=k^2.\,2a\int_0^m\frac{\sqrt{1+m^2}\,dm}{a^2(m^2+1)(m^2+9)}$$

$$=\frac{2k^2}{a}\int_0^m\frac{dm}{(m^2+9)\sqrt{m^2+1}}\qquad(\text{Let } m=\tan\phi.)$$

$$=\frac{2k^2}{a}\int_0^\phi\frac{\cos\phi\,d\phi}{\sin^2\phi+9\cos^2\phi}$$

$$=\frac{2k^2}{a}\int_0^\phi\frac{d\sin\phi}{9-8\sin^2\phi}=\frac{k^2}{4a}\int\frac{d\sin\phi}{\frac{9}{8}-\sin^2\phi}$$

$$=\frac{k^2}{6a\sqrt{2}}\log\frac{3+2\sqrt{2}\sin\phi}{3-2\sqrt{2}\sin\phi}.$$

Example. Mr. Roberts shows in the article above cited that for points between $-\infty$ and $-3a$ on the x-axis the arc of the inverse curve can be expressed as a pure logarithm. For points from $-3a$ to a such arcs are partly logarithmic, partly inverse circular. For points from a to $+\infty$ the arcs are inverse circular expressions. Examine the truth of this.

624. John Bernoulli's Theorem.

Let a number of points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, etc., be moving in a plane, and let ds_1, ds_2, ds_3, etc., be elements of the paths described. Let us impose upon their motion the condition that they are all moving at every instant in parallel directions

in the same sense. Let ψ be the angle the tangents to their respective paths make with the x-axis.

Suppose heavy particles of masses m_1, m_2, etc., to be placed at P_1, P_2, etc., and let $\bar{x}$, $\bar{y}$ be their centroid.

Then
$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \quad \bar{y}=\frac{\Sigma my}{\Sigma m},$$

$$d\bar{x}=\frac{\Sigma m\,dx}{\Sigma m}=\frac{\Sigma m\,ds}{\Sigma m}\cos\psi,$$

$$d\bar{y}=\frac{\Sigma m\,dy}{\Sigma m}=\frac{\Sigma m\,ds}{\Sigma m}\sin\psi.$$

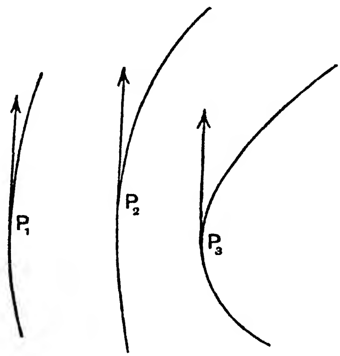


Fig. 170.

Hence $\dfrac{d\bar{x}}{\cos\psi}=\dfrac{d\bar{y}}{\sin\psi}$, and therefore the motion of the centroid is always parallel to the motion of the several particles; moreover, if $d\bar{s}$ be the corresponding element of the path of the centroid,
$$d\bar{s}=\frac{\Sigma m\,ds}{\Sigma m}$$

and
$$\bar{s}=\frac{\Sigma ms}{\Sigma m}.$$

625. This result is ascribed by Mr. R. A. Roberts, in the paper before cited, as due to John Bernoulli, the intention being to give a method for the generation of new rectifiable curves from any system of curves whose rectification has already been effected.

It is to be remarked that the same theorem obviously holds for any system of particles moving in the manner prescribed upon twisted or tortuous curves in space.

Again, several of the points may be moving on different branches of the same curve.

It appears from Bernoulli's result that as $m_1, m_2, m_3, \ldots$ can be arranged at will, we can from any set of rectifiable curves generate an infinite number of other curves which are rectifiable in the same manner and in terms of the same functions.

Thus, for instance, taking any set of catenaries with parallel directrices whose typical equation is $s=a+c\tan\psi$;
or any set of equal equiangular spirals, type $s=a+be^{m(\psi+\alpha)}$;
or any set of circles, type $s=a+b\psi$;
or any set of involutes of circles, type $s=a+b(\psi+a)^2$;
or any set of similar epi- or hypocycloids, type

$$s=a+b\sin(n\psi+a);$$

or any set of semi-cubical parabolas with parallel axes, type

$$s=a+b\sec^3\psi;$$

or, in fact, any of the cases in which $\dfrac{\Sigma ms}{\Sigma m}$ reduces to an expression of the same form, the locus of the centroid

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \quad \bar{y}=\frac{\Sigma my}{\Sigma m},$$

is another curve of the same kind, and the length of any portion of its arc is to be found from the formula

$$\bar{s}=\frac{\Sigma ms}{\Sigma m}.$$

And further, when curves of different nature are taken as the original curves, though the derived locus be not of the same nature as that of any one of the original curves, yet it is still rectifiable in terms of the same functions as those in terms of which the original curves are rectifiable.

626. **Extension of Bernoulli's Theorem.**

When the forward-drawn tangents at the several points are not all in the same sense, we may still apply the theorem, but with the precaution of reckoning all those elementary arcs which are traversed in the same sense as positive, and the remaining ones as negative.

Thus, if $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be at opposite extremities of a diameter of an ellipse, or centric oval, and if $\cos\psi$, $\sin\psi$ be the direction ratios of the tangent at P_1, $-\cos\psi$, $-\sin\psi$ will be the direction ratios of the forward drawn tangent at P_2, and

$$d\bar{x}=\frac{m_1\,dx_1+m_2\,dx_2}{m_1+m_2}=\frac{m_1(ds_1\cos\psi)+m_2(-ds_2\cos\psi)}{m_1+m_2}$$

$$=\frac{m_1\,ds_1-m_2\,ds_2}{m_1+m_2}\cos\psi,$$

and $$d\bar{y}=\frac{m_1\,ds_1-m_2\,ds_2}{m_1+m_2}\sin\psi,$$

and $$d\bar{s}=\frac{m_1\,ds_1-m_2\,ds_2}{m_1+m_2},\quad\text{giving}\quad \bar{s}=\frac{m_1s_1-m_2s_2}{m_1+m_2}.$$

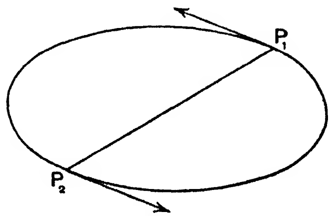


Fig. 171.

Moreover, for an ellipse, or centric oval, obviously $ds_1=ds_2$ and $s_1=s_2$, and if we make $m_1=m_2$, $\bar{s}=0$, as it should be, since all diameters are bisected at the centre, and the centroid locus degenerates into a point.

In the case when one of the curves degenerates to a point and one other point describes a given curve, Bernoulli's Theorem states that the similar and similarly situated centroid-locus is such that corresponding arcs on this locus and on the original curve are proportional, which is *a priori* obvious.

627. **Ovoid with One Axis of Symmetry.**

Let us consider the case of any ovoid with one axis of symmetry, and discuss the locus of the mid-points of chords which are such that the tangents at their extremities are parallel. Let P_1P_2 be such a chord and G its mid-point. If we take the direction ratios at P_1 as $\cos\psi$, $\sin\psi$, then at P_2, where the forward-drawn tangent is parallel, but in the opposite direction, they must be taken as $-\cos\psi$, $-\sin\psi$. If

it be a question of applying the theorem to the locus of the mid-point G of the chord P_1P_2, we have

$$d\sigma = \frac{ds_1 - ds_2}{2},$$

where ds_1, ds_2, $d\sigma$ are the elementary arcs traced by P_1, P_2, G respectively, and as the inclination of all three tangents to the x-axis is the same,

$$\rho = \frac{\rho_1 - \rho_2}{2},$$

where ρ_1, ρ_2, ρ are the corresponding radii of curvature.

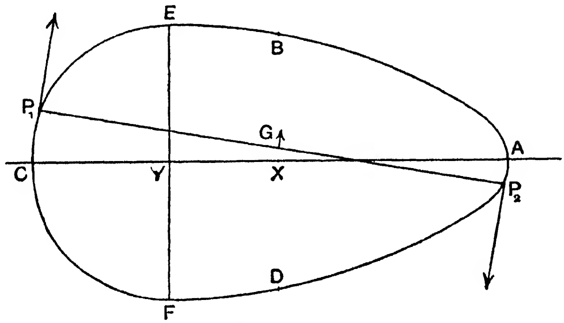


Fig. 172.

Now, in integrating to find σ for the whole length of the path of G, considerable care is necessary, for when the points P_1, P_2 pass through positions at which the radii of curvature become equal, $ds_1 - ds_2$ in general changes sign. So that in estimating $\int d\sigma$ for the whole G-locus, for some parts we must take $\sigma = \int \frac{ds_1 - ds_2}{2}$ and for others $\int \frac{ds_2 - ds_1}{2}$; *i.e.* we must take care that the difference of the elementary arcs at the ends of the chord is reckoned positively.

Hence we shall write the result

$$\sigma = \int \frac{ds_1 \sim ds_2}{2}.$$

In such an ovoid there will in general be points A, B, C, D, of which the first and third are the extremities of the axis of symmetry, where the radii of curvature are respectively

minimum, maximum, minimum, maximum;

and there may be a pair of points, one between D and A and one between B and C, at which the tangents are parallel, and such that the radii of curvature at those points are equal; and the same is true of the portions AB, CD of the ovoid. In such case, on the G-locus there is therefore a point at which $\rho = 0$, with a change of sign of ρ. Hence there is at such a point a singularity on the G-locus, in general a cusp at which the tangent is parallel to the tangents at the corresponding points on the ovoid.

628. **Geometrical Examination.**

Let us examine more closely, in a geometrical manner, what is in general happening at such a point.

Let P_0P_1, P_1P_2, P_2P_3, P_3P_4, P_4P_5, ... be elements of the ovoid, with equal increments $d\psi$ in the angle of contingence,

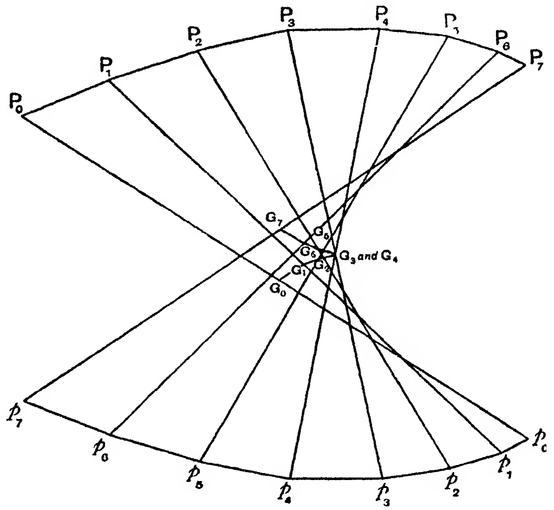


Fig. 173.

and drawn in the neighbourhood of a point on the ovoid, which has the peculiarity under consideration, viz. that the radius of curvature at that point is equal to that at the opposite extremity of the chord.

And let p_0p_1, p_1p_2, p_2p_3, p_3p_4, p_4p_5, ... be the opposite parallel elements, the angles between consecutive pairs of either system being therefore $d\psi$, and let $P_3P_4 = p_3p_4$.

Let $G_0, G_1, G_2, G_3, G_4, \ldots$ be the mid-points of the chords $P_0p_0, P_1p_1, P_2p_2, P_3p_3, \ldots$ respectively; then it will be obvious that

$$\begin{aligned} G_0G_1 &= \tfrac{1}{2}(P_0P_1 - p_0p_1), \\ G_1G_2 &= \tfrac{1}{2}(P_1P_2 - p_1p_2), \\ G_2G_3 &= \tfrac{1}{2}(P_2P_3 - p_2p_3), \\ G_3G_4 &= \tfrac{1}{2}(P_3P_4 - p_3p_4) = 0, \\ G_4G_5 &= \tfrac{1}{2}(p_4p_5 - P_4P_5), \\ G_5G_6 &= \tfrac{1}{2}(p_5p_6 - P_5P_6), \\ &\text{etc.} \end{aligned}$$

The points G_3, G_4 coincide, the element G_4G_5 makes an angle $2d\psi$ with the element G_2G_3, the direction of the tangent to

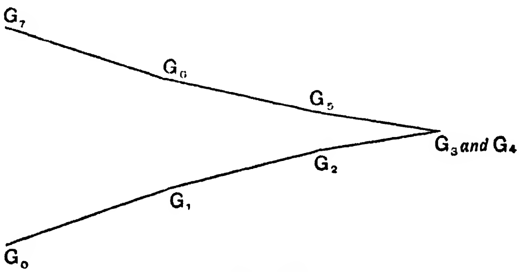


Fig. 174.

the path having turned through an angle $\pi + 2d\psi$. Ultimately then we have at G_3 two coincident tangents to the G-locus, *i.e.* there is a cusp on the G-locus at such a point, and this cusp lies upon the envelope of the chord, for G_3 is the point of intersection of two consecutive positions of the chord.

629. Again, at the points E, F on the double ordinate at the widest part of the ovoid the radii of curvature are obviously equal, and at the mid-point Y of EF there will be a cusp on the G-locus, whilst at X, the mid-point of the axis of symmetry AC, the tangent to the G-locus will be perpendicular to AC.

Let IJ be that chord of the ovoid for which the tangents at I and J are parallel and for which the radii of curvature at the ends are equal, and whose mid-point is situated at the cusp L of the G-locus, and let $I'J'$ be the corresponding chord through the cusp M, symmetrically situated with regard to the axis of symmetry.

Then, integrating along corresponding arcs,

$$\text{arc } MXL = \frac{\text{arc } I'CI - \text{arc } J'AJ}{2},$$

$$\text{arc } LY \quad = \frac{\text{arc } JDF - \text{arc } IE}{2},$$

$$\text{arc } YM \quad = \frac{\text{arc } EBJ' - \text{arc } FI'}{2}.$$

Thus the whole perimeter of the tricuspidal G-locus

$$= \tfrac{1}{2}(\text{arc } I'CI - \text{arc } IE + \text{arc } EJ' - \text{arc } J'J + \text{arc } JF - \text{arc } FI'),$$

i.e. in short, half the difference of the two sums of alternate arcs of the original ovoid, the points of division being those

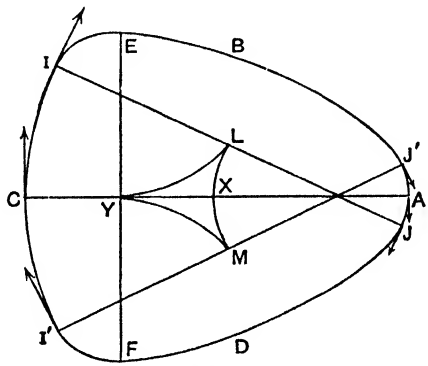


Fig. 175.

at which, whilst the opposite tangents are parallel, the radii of curvature are equal.

630. Of course, in the case of any closed oval symmetrical about two perpendicular axes, such as an ellipse, the diameters are all bisected at the intersection of the axes of symmetry, and the tricusp is evanescent, the radii of curvature at all opposite points being equal and the tangents parallel.

631. Note (i) that if lines be drawn through the points G parallel to the tangents at the extremities of the chords through G, then the points G are the points of contact of such lines with their envelope;

(ii) that the cuspidal tangents to the G-locus are parallel to those parallel tangents to the ovoid at whose points of contact the opposite radii of curvature are equal;

(iii) if R be a point on such a chord P_1P_2 as has been described, and dividing it in the ratio $m_2 : m_1$, then the theorem

$$\sigma=\frac{m_1s_1-m_2s_2}{m_1+m_2}$$

is true for the whole perimeter s of the ovoid,

i.e.
$$\sigma=\frac{m_1-m_2}{m_1+m_2}s$$

(for in integrating round the curve $s_1=s_2=s$), provided that R does not lie intermediate between a certain pair of points R_1, R_2 on the chord, for which $m_1\rho_1-m_2\rho_2$ can vanish, *i.e.* if λ and λ^{-1} be the greatest and least values of the ratio ρ_1/ρ_2 attained as P_1 travels round the perimeter of the ovoid, the points R_1, R_2 are the positions of R for which $m_1=\lambda m_2$ and $m_2=\lambda m_1$ respectively. Thus, for all points R on the chord or the chord produced which do not lie between R_1 and R_2, the perimeter of the R-locus is

$$\sigma=\frac{m_1-m_2}{m_1+m_2}s.$$

But for points between R_1 and R_2 thus defined, precautions similar to those described for the mid-point must be taken.

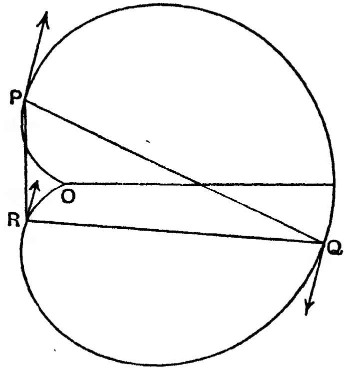


Fig. 176.

632. An Instructive Problem.

Let us discuss the locus of the centroid of the triangle PQR when these points lie upon a cardioide and are such that the tangents at P, Q, R are always parallel.

The equation of a normal to the curve $r=a(1+\cos\theta)$ at the point $\theta=2\alpha$ is

$$(3t-t^3)x-(1-3t^2)y=\frac{a}{2}\{(3t-t^3)+t(1+t^2)\},$$

where $t\equiv\tan\alpha$ (*Diff. Calc.*, p. 158).

The three normals will be parallel at points such that

$$\frac{3t-t^3}{1-3t^2}=k, \text{ say,} \quad i.e. \ \tan 3\alpha=k.$$

Let $$\tan 3\chi=\tan 3\alpha.$$

Then $$3\chi=n\pi+3\alpha,$$

$$\chi=\alpha, \quad \alpha+\frac{\pi}{3}, \quad \alpha+\frac{2\pi}{3}.$$

Hence 2α, $2\alpha+\dfrac{2\pi}{3}$, $2\alpha+\dfrac{4\pi}{3}$ are points at which the normals, and therefore also the tangents, are parallel.

Let these be called 2α, 2β, 2γ.

If (x_1, y_1), (x_2, y_2), (x_3, y_3) be the coordinates of P, Q, R,

$$x_1=2a\cos^2\frac{\theta}{2}\cos\theta=2a\cos^2\alpha\cos 2\alpha=\frac{a}{2}(1+2\cos 2\alpha+\cos 4\alpha), \text{ etc.},$$

$$y_1=2a\cos^2\frac{\theta}{2}\sin\theta=2a\cos^2\alpha\sin 2\alpha=\frac{a}{2}(2\sin 2\alpha+\sin 4\alpha), \text{ etc.};$$

$$\therefore\ 3\bar{x}=\Sigma x_1=\frac{3a}{2}, \quad 3\bar{y}=\Sigma y_1=0.$$

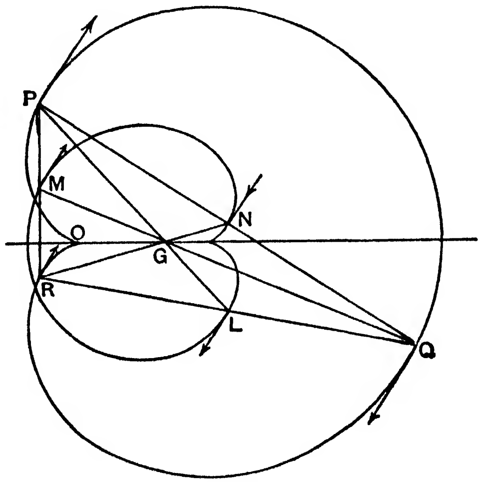


Fig. 177.

(i) $\therefore\ \bar{x}=\dfrac{a}{2}$, $\bar{y}=0$, and the centroid is therefore at a fixed point G on the axis.

(ii) Let PG, QG, RG cut the sides of the triangle at L, M, N. Then, since $GP=2GL$, etc., the points L, M, N, *i.e.* the mid-points of the sides lie on another cardioide of half the linear dimensions of the former.

(iii) The tangents at L, M, N to this cardioide are parallel to the tangents to the original cardioide at P, Q, R.

(iv) The triangle PQR might have been described as one in which each of the sides subtends an angle 120° at the pole O.

(v) All other points which divide the sides, or the medians, in a constant ratio, or any points connected with the triangle PQR by the formulae

$$\xi=\frac{\Sigma lx}{\Sigma l}, \quad \eta=\frac{\Sigma ly}{\Sigma l},$$

where l, m, n are either numerical or not dependent upon the magnitude, shape and position of the triangle, also trace cardioides; and lines through such points parallel to the tangents at P, Q, R, envelope cardioides.

633. Areals and Trilinears.

It has already been explained that such systems are not well adapted for metrical purposes (Art. 460).

We can, however, readily obtain suitable formulae for such cases if necessary.

Denoting the trilinear coordinates of any two points by $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, the triangle of reference being some given triangle ABC of sides a, b, c, and area Δ, the distance between these points is

$$r^2=-\frac{abc}{4\Delta^2}[a(\beta_1-\beta_2)(\gamma_1-\gamma_2)+b(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)$$
$$+c(\alpha_1-\alpha_2)(\beta_1-\beta_2)]$$

or $$\frac{abc}{4\Delta^2}[a\cos A(\alpha_1-\alpha_2)^2+b\cos B(\beta_1-\beta_2)^2+c\cos C(\gamma_1-\gamma_2)^2]$$

(Ferrers' *Trilinears*, p. 6).

Accordingly, the length of an elementary arc ds between two points (α, β, γ), $(\alpha+d\alpha, \beta+d\beta, \gamma+d\gamma)$ may be written either as

$$ds^2=-\frac{abc}{4\Delta^2}(a\,d\beta\,d\gamma+b\,d\gamma\,d\alpha+c\,d\alpha\,d\beta)$$

or as $$ds^2=\frac{abc}{4\Delta^2}(a\cos A\,d\alpha^2+b\cos B\,d\beta^2+c\cos C\,d\gamma^2),$$

where $$a\alpha+b\beta+c\gamma=2\Delta,$$

and therefore $$a\,d\alpha+b\,d\beta+c\,d\gamma=0.$$

The corresponding expressions in Areals will obviously be

$$ds^2 = -(a^2\,dy\,dz + b^2\,dz\,dx + c^2\,dx\,dy)$$

or

$$ds^2 = bc\cos A\,dx^2 + ca\cos B\,dy^2 + ab\cos C\,dz^2,$$

with the identical relations

$$x+y+z=1, \quad dx+dy+dz=0.$$

The Areal results are a little the simpler.

634. **Unicursal Curves.**

In the case of a curve being unicursal, *i.e.* such that the coordinates of a point upon it can be expressed as rational functions of some parameter t, then if we have taken areal coordinates x, y, z, so that their sum is unity, we may write

$$\frac{x}{f_1(t)} = \frac{y}{f_2(t)} = \frac{z}{f_3(t)} = \frac{1}{f(t)},$$

where

$$f(t) = f_1(t) + f_2(t) + f_3(t).$$

Let these functions be made homogeneous and of the same degree, say the n^{th}, by the insertion of a proper power of another letter τ, where $\tau = 1$.

Then

$$\frac{dx}{dt} = \frac{f(t)f_1'(t) - f'(t)f_1(t)}{\{f(t)\}^2}.$$

Now, by Euler's Theorem,

$$\begin{vmatrix} f(t), & f_1(t) \\ f'(t), & f_1'(t) \end{vmatrix} = \frac{1}{n}\begin{vmatrix} t\dfrac{\partial f}{\partial t} + \tau\dfrac{\partial f}{\partial \tau}, & t\dfrac{\partial f_1}{\partial t} + \tau\dfrac{\partial f_1}{\partial \tau} \\ \dfrac{\partial f}{\partial t}, & \dfrac{\partial f_1}{\partial t} \end{vmatrix}$$

$$= \frac{1}{n}\begin{vmatrix} \dfrac{\partial f}{\partial \tau}, & \dfrac{\partial f_1}{\partial \tau} \\ \dfrac{\partial f}{\partial t}, & \dfrac{\partial f_1}{\partial t} \end{vmatrix}$$

$$= \frac{1}{n} J_1,$$

where J_1 is the Jacobian of f_1 and f with regard to t and τ, *i.e.*

$$= \frac{1}{n}\frac{\partial(f_1, f)}{\partial(t, \tau)},$$

and τ is to be put $=1$ after the differentiations are performed.

Thus

$$dx = \frac{1}{n}\frac{J_1}{f^2}\,dt.$$

Similarly
$$dy=\frac{1}{n}\frac{J_2}{f^2}dt,$$
$$dz=\frac{1}{n}\frac{J_3}{f^2}dt,$$

where J_2 and J_3 are respectively

$$\frac{\partial(f_2, f)}{\partial(t, \tau)},\quad \frac{\partial(f_3, f)}{\partial(t, \tau)}.$$

Thus the areal formulae for rectification in the case of a unicursal curve become

$$s=\frac{1}{n}\int\sqrt{-\frac{1}{f^4}\left[a^2J_2J_3+b^2J_3J_1+c^2J_1J_2\right]}\,dt$$

or
$$s=\frac{1}{n}\int\sqrt{\frac{1}{f^4}\left[bc\cos AJ_1{}^2+ca\cos BJ_2{}^2+ab\cos CJ_3{}^2\right]}\,dt.$$

These simplify a little further in the case where it is possible to take the reference triangle equilateral.

635. Ex. 1. For example, if it be required to apply this method to rectify a circle referred to a pair of tangents inclined at 60° and the chord of contact, the equation is

$$x^2=yz,$$

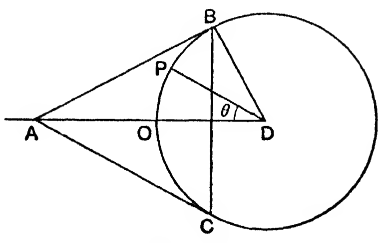


Fig. 178.

and we may put
$$\frac{x}{t}=\frac{y}{1}=\frac{z}{t^2}=\frac{1}{1+t+t^2},$$

$$\frac{dx}{dt}=\frac{1-t^2}{(1+t+t^2)^2},\quad \frac{dy}{dt}=-\frac{1+2t}{(1+t+t^2)^2},\quad \frac{dz}{dt}=\frac{2t+t^2}{(1+t+t^2)^2},$$

$$ds^2=+\frac{a^2}{2}(dx^2+dy^2+dz^2)=a^2\frac{dt^2}{1+t+t^2};$$

$$\therefore\ ds=-a\frac{dt}{1+t+t^2}.$$

We take the negative sign, because we measure the arc from O, where $t=1$, the nearest point to A, and as the current point P moves from O towards B (Fig. 178), t decreases, *i.e.* s increases as t decreases, *i.e.*

$$s=\left[-\frac{2a}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}}\right]_1^t$$

$$=\frac{2a}{\sqrt{3}}\left(\tan^{-1}\sqrt{3}-\tan^{-1}\frac{2t+1}{\sqrt{3}}\right)$$

$$=\frac{2a}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\frac{1-t}{1+t}\right).$$

[Clearly the radius $=\frac{a}{\sqrt{3}}$; hence we can determine the geometrical meaning of the parameter t, viz. $t=\frac{1-\sqrt{3}\tan\frac{1}{2}O\hat{D}P}{1+\sqrt{3}\tan\frac{1}{2}O\hat{D}P}$.]

Ex. 2. Take as triangle of reference any pair of tangents to a parabola and the chord of contact. The equation of the curve then is

$$x^2=4yz,$$

and we may write
$$\frac{x}{2t}=\frac{y}{1}=\frac{z}{t^2}=\frac{1}{(1+t)^2};$$

$$\therefore\ \frac{dx}{dt}=2\frac{1-t}{(1+t)^3},\quad \frac{dy}{dt}=-\frac{2}{(1+t)^3},\quad \frac{dz}{dt}=\frac{2t}{(1+t)^3};$$

$$\therefore\ ds^2=\frac{4}{(1+t)^6}[bc\cos A(1-t)^2+ca\cos B+ab\cos Ct^2]dt^2$$

$$=\frac{4}{(1+t)^6}(c^2-2bc\cos At+b^2t^2)dt^2,$$

$$s=2\int\frac{\sqrt{(bt-c\cos A)^2+c^2\sin^2 A}}{(1+t)^3}dt.$$

Put $bt-c\cos A=c\sin A\cdot\tan\theta$; $\therefore\ b\,dt=c\sin A\sec^2\theta\,d\theta$;

$$\therefore\ s=2b^2c^2\sin^2 A\int\frac{d\theta}{[(b+c\cos A)\cos\theta+c\sin A\sin\theta]^3}$$

$$=8\Delta^2\int\frac{d\theta}{(p\cos\theta+q\sin\theta)^3},\quad \text{where } p=b+c\cos A,\ q=c\sin A,$$

$$=\frac{8\Delta^2}{(p^2+q^2)^{\frac{3}{2}}}\int\sec^3\left(\theta-\tan^{-1}\frac{q}{p}\right)d\theta$$

$$=\frac{4\Delta^2}{(b^2+2bc\cos A+c^2)^{\frac{3}{2}}}\left[\tan\left(\theta-\tan^{-1}\frac{q}{p}\right)\sec\left(\theta-\tan^{-1}\frac{q}{p}\right)+\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}-\frac{1}{2}\tan^{-1}\frac{q}{p}\right)\right],$$

where
$$\frac{q}{p}=\frac{c\sin A}{b+c\cos A}\quad\text{and}\quad\tan\theta=\frac{bt-c\cos A}{c\sin A},$$

which, when taken between limits t_1, t_2, determines the length of the intercepted arc in terms of t_1, t_2 and the elements of the triangle of reference.

636. Connexion between Quadrature and Rectification.

It is perhaps of historical rather than mathematical importance to point out the connexion between the problems of rectification and of quadrature.

If $y=f(x)$ be the Cartesian equation of the curve to be considered, we shall suppose a new curve to be constructed from

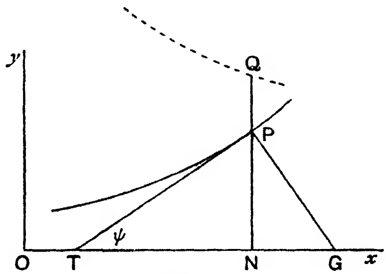


Fig. 179.

it, taking the same abscissa and an ordinate $\eta = a \sec\psi$, where ψ is the slope of the tangent to the original curve and a is any constant. Then

$$ds = dx \sec\psi = \frac{\eta}{a}\,dx;$$

$$\therefore \quad as = \int \eta\,dx.$$

Hence the rectification of the first curve may be regarded as the quadrature of the second.

Sec ψ may be interpreted in various ways to facilitate the drawing of the graph of the new curve; for example,

$$\sec\psi = \frac{\text{Tangent}}{\text{Subtangent}} \quad \text{or} \quad = \frac{\text{Normal}}{\text{Ordinate}}, \text{ etc.}$$

Accordingly, if the ordinates of the original curve be all increased to a length η so that

$$\eta : a = \frac{\text{Tangent}}{\text{Subtangent}} \quad \text{or} \quad \frac{\text{Normal}}{\text{Ordinate}},$$

a new curve will be found for which the area bounded by the new curve, the x-axis and the terminal ordinates is equal to a rectangle, one side of which is a and the other side is the corresponding arc of the given curve. Also a, being at our choice, may be taken as unit length.

637. Ex. If the ordinate of the semicubical parabola $ay^2=x^3$ be produced to a length η so that $\eta=a\dfrac{\text{Normal}}{\text{Ordinate}}$, show that the path of the new point thus found is the parabola

$$4\eta^2=4a^2+9ax.$$

Find the area of a portion of this parabola bounded by two given ordinates, and deduce the result of Ex. 1, Art. 516, for the length of the corresponding arc of the semicubical parabola.

Van Huraet's rectification of the semicubical parabola referred to in Art. 516 was effected thus. (Williamson, *Int. Calc.*, p. 249.)

638. **On a Class of Rectifiable Curves.**

If
$$\left.\begin{aligned}\frac{dx}{dt}&=F(t)\cos f(t)\\ \text{and}\quad \frac{dy}{dt}&=F(t)\sin f(t,)\end{aligned}\right\}\text{ we have }\frac{ds}{dt}=F(t).$$

Hence in the curve

$$\left.\begin{aligned}x&=\int F(t)\cos f(t)\,dt,\\ y&=\int F(t)\sin f(t)\,dt,\end{aligned}\right\}\text{ we have } s=\int F(t)\,dt.$$

The functions $F(t)$ and $f(t)$ being at our choice a large number of rectifiable curves arise.

In constructing a rectifiable curve, a common method is to make $f(t)=n\tan^{-1}t$ and make use of the formulae

$$\cos(n\tan^{-1}t)=\frac{1}{(1+t^2)^{\frac{n}{2}}}(1-{}^nC_2t^2+{}^nC_4t^4-\ldots),$$

$$\sin(n\tan^{-1}t)=\frac{1}{(1+t^2)^{\frac{n}{2}}}({}^nC_1t-{}^nC_3t^3+{}^nC_5t^5-\ldots),$$

and either to choose an even value for n, or to take $(1+t^2)^{\frac{n}{2}}$ as one of the factors of $F(t)$, if n be odd, to facilitate integration.

639. Ex. 1. Thus, taking

$$\left.\begin{aligned}\frac{dx}{dt}&=2t,\\ \frac{dy}{dt}&=1-t^2\end{aligned}\right\}\text{ here } n=2 \text{ and } F(t)=1+t^2,$$

we have
$$\left.\begin{aligned}x&=t^2,\\ y&=t-\frac{t^3}{3},\end{aligned}\right\}$$

and
$$\frac{ds}{dt}=1+t^2;\quad\text{whence } s=t+\frac{t^3}{3}.$$

The curve in question is then

$$y^2 = x\left(1-\frac{x}{3}\right)^2, \text{ a cubic,}$$

and we have in this curve

$$s^2 = x\left(1+\frac{x}{3}\right)^2$$

or

$$s^2 - y^2 = \tfrac{4}{3}x^2.$$

Ex. 2. Let us take

$$\left.\begin{aligned}\frac{dx}{dt} &= a\left(\frac{1}{t} - t\right),\\ \frac{dy}{dt} &= 2a.\end{aligned}\right\}$$

Then

$$\frac{ds}{dt} = a\left(\frac{1}{t} + t\right),$$

$$x = a\left(\log t - \frac{t^2}{2}\right), \quad y = 2at, \quad s = a\left(\log t + \frac{t^2}{2}\right);$$

$$\therefore \; ax + \frac{y^2}{8} = a^2 \log \frac{y}{2a}$$

is the Cartesian equation of the curve.

Also

$$s - x = \frac{y^2}{4a},$$

and the intrinsic equation is

$$s = \frac{a}{2}\tan^2\frac{\psi}{2} + a\log\tan\frac{\psi}{2}.$$

Ex. 3. Take

$$\frac{dx}{dt} = \frac{2a}{\sqrt{t}}(1-t^2)$$

and

$$\frac{dy}{dt} = \frac{2a}{\sqrt{t}} \cdot 2t.$$

Then

$$\left.\begin{aligned}x &= 4a\left(t^{\frac{1}{2}} - \frac{t^{\frac{5}{2}}}{5}\right),\\ y &= \tfrac{8}{3}at^{\frac{3}{2}},\end{aligned}\right\}$$

and

$$\frac{ds}{dt} = \frac{2a}{\sqrt{t}}(1+t^2),$$

$$s = 4a\left(t^{\frac{1}{2}} + \frac{t^{\frac{5}{2}}}{5}\right).$$

Hence $s^2 - x^2 = \frac{8}{3}y^2$, and the intrinsic equation is

$$s = 4a\sqrt{\tan\frac{\psi}{2}}\left(1 + \tfrac{1}{5}\tan^2\frac{\psi}{2}\right).$$

Ex. 4. In the curve for which

$$\frac{dx}{dt} = 1 - 6t^2 + t^4,$$

$$\frac{dy}{dt} = 4t(1-t^2),$$

we have $$\frac{ds}{dt}=1+2t^2+t^4,$$

i.e. $$s=t+\frac{2t^3}{3}+\frac{t^5}{5},$$

where $$x=t-2t^3+\frac{t^5}{5},$$
$$y=2t^2-t^4,$$
$$\tan\psi=\frac{dy}{dx}=\tan 4\theta, \quad \text{if } t\equiv\tan\theta;$$
$$\therefore\ \theta=\frac{\psi}{4},$$

and the intrinsic equation is
$$s=\tan\frac{\psi}{4}\left(1+\frac{2}{3}\tan^2\frac{\psi}{4}+\frac{1}{5}\tan^4\frac{\psi}{4}\right),$$
the Cartesian equation being the t-eliminant from the values of x and y.

Several examples of this class of curve will be found in Wolstenholme's *Problems* (No. 1800 onwards).

640. Since $(m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2$ we may construct a curve such that
$$x=\int\phi(t)[f_1^2(t)-f_2^2(t)]\,dt,$$
$$y=2\int\phi(t)f_1(t)f_2(t)\,dt,$$
and then we shall obviously have
$$s=\int\phi(t)[f_1^2(t)+f_2^2(t)]\,dt,$$
where $\phi(t)$, $f_1(t)$, $f_2(t)$ are all at our choice. This artifice amounts to a form of the last method.

641. Ex. Let $$\frac{dx}{dt}=at(t^{2p}-t^{2q}),$$
$$\frac{dy}{dt}=2at\,.\,t^pt^q.$$

Then $$\frac{ds}{dt}=at(t^{2p}+t^{2q}).$$

Hence, for the curve $$\left.\begin{aligned}x&=a\left(\frac{t^{2p+2}}{2p+2}-\frac{t^{2q+2}}{2q+2}\right),\\ y&=2a\frac{t^{p+q+2}}{p+q+2},\end{aligned}\right\}$$

we have $$s=a\left(\frac{t^{2p+2}}{2p+2}+\frac{t^{2q+2}}{2q+2}\right),$$

i.e. $$s^2-x^2=\frac{(p+q+2)^2}{(p+1)(q+1)}\frac{y^2}{4}.$$

642. A Theorem by Mr. R. A. Roberts.

An important transformation may be used in some cases to derive one rectifiable curve from another, as follows:

Put
$$\left.\begin{aligned} x+\iota y&=u,\\ x-\iota y&=v,\end{aligned}\right\}\text{ where } \iota=\sqrt{-1}.$$

Then clearly
$$\begin{aligned} ds^2=dx^2+dy^2&=(dx+\iota\,dy)(dx-\iota\,dy)\\ &=du\,dv.\end{aligned}$$

In cases where the equation of the original curve takes the form
$$\phi(u)\,\phi(v)=\text{const., say unity,}$$
if another curve be derived from this one by taking
$$u'=\int[\phi(u)]^n\,du,$$
$$v'=\int[\phi(v)]^n\,dv,$$
it is plain that
$$du'\,dv'=[\phi(u)]^n\,[\phi(v)]^n\,du\,dv=du\,dv,$$
and therefore $\quad ds'^2=ds^2 \quad$ and $\quad ds'=ds,$

and corresponding arcs will be equal.

The theorem is given by Mr. R. A. Roberts [*Proc. L.M.S.*, vol. xviii.].

643. Precautions.

Some circumspection is necessary in the inference to be made as to the whole perimeter of the derived curve. For instance when the point $P(x, y)$ of the curve, supposed closed, traces out the complete path $\phi(u)\,\phi(v)=1$, the corresponding point P' on the derived curve may not trace out the *whole* of the derived curve, or it may trace the derived curve several times. This point must be examined in all cases of application of the theorem.

644. In illustration it will be instructive to consider the most elementary case, viz. that in which the primary curve is the circle $x^2+y^2=a^2$.

With the proposed transformation, viz. $x+\iota y=u$, $x-\iota y=v$, we have
$$uv=a^2.$$

Taking the derived curve as
$$u'=\int\frac{u^2}{a^2}\,du,\quad v'=\int\frac{v^2}{a^2}\,dv,$$
we get $ds'=ds$, and corresponding arcs are equal.

Now $u'=\dfrac{u^3}{3a^2}$ gives $\qquad x'+\iota y'=\dfrac{1}{3a^2}(x+\iota y)^3.$

Therefore $$3a^2x' = x^3 - 3xy^2, \quad\ldots\ldots(1)$$
$$3a^2y' = 3x^2y - y^3, \quad\ldots\ldots(2)$$

And upon squaring and adding,
$$9a^4(x'^2+y'^2) = (x^2+y^2)^3 = a^6.$$

Hence the corresponding locus is the circle
$$x'^2+y'^2 = \frac{a^2}{9},$$
viz. one of radius $\frac{a}{3}$.

The whole perimeters are obviously not equal.

But noticing that if we put $\frac{y'}{x'} = \tan\theta'$ and $\frac{y}{x} = \tan\theta$, we get
$$\tan\theta' = \tan 3\theta, \quad \text{or} \quad \theta' = 3\theta,$$
and it appears that the derived circle is traced out at *three times* the angular rate of the primary circle, and whilst the point $P(x, y)$ traces the whole of the primary circle, the derived point $P'(x', y')$ traces the derived circle thrice, and the circumference of the first, viz. $2\pi a$, is thrice the circumference of the second, *i.e.* $3 \times \left(\frac{2\pi a}{3}\right)$.

645. As an illustration of the derivation of a new rectifiable curve by this method, take as primary curve the lemniscate
$$r^2 = a^2 \cos 2\theta,$$
i.e. $$(x^2+y^2)^2 = a^2(x^2-y^2),$$
i.e. $$u^2v^2 = \frac{a^2}{2}(u^2+v^2),$$
or $$\left(u^2 - \frac{a^2}{2}\right)\left(v^2 - \frac{a^2}{2}\right) = \frac{a^4}{4}.$$

Let us derive a new curve from this by putting
$$u' = \frac{2}{a^2}\int\left(u^2 - \frac{a^2}{2}\right)du,$$
$$v' = \frac{2}{a^2}\int\left(v^2 - \frac{a^2}{2}\right)dv,$$
and therefore $$du'\,dv' = \frac{4}{a^4}\left(u^2 - \frac{a^2}{2}\right)\left(v^2 - \frac{a^2}{2}\right)du\,dv = du\,dv;$$
whence $ds' = ds$, and corresponding arcs are equal.

Now $$u' = \frac{2}{a^2}\left(\frac{u^3}{3} - \frac{a^2}{2}u\right), \quad v' = \frac{2}{a^2}\left(\frac{v^3}{3} - \frac{a^2}{2}v\right),$$
i.e. $$\frac{a^2}{2}(x'+\iota y') = \frac{(x+\iota y)^3}{3} - \frac{a^2}{2}(x+\iota y),$$
$$\left.\begin{aligned} 3a^2x' &= 2(x^3-3xy^2) - 3a^2x, \\ 3a^2y' &= 2(3x^2y-y^3) - 3a^2y, \end{aligned}\right\} \text{ where } (x^2+y^2)^2 = a^2(x^2-y^2),$$

which may be written as

$$\left.\begin{aligned}\frac{3x'}{a}&=\sqrt{\cos 2\theta}\,[\cos 5\theta-2\cos\theta],\\ \frac{3y'}{a}&=\sqrt{\cos 2\theta}\,[\sin 5\theta-2\sin\theta],\end{aligned}\right\}\ \theta \text{ being an arbitrary parameter.}$$

Hence as arcs of a lemniscate can be expressed as elliptic integrals of the first kind, the same is true of this derived curve.

The elimination of u and v from the equations

$$u'=\frac{2}{3}\frac{u^3}{a^2}-u,\quad v'=\frac{2}{3}\frac{v^3}{a^2}-v,\quad \left(u^2-\frac{a^2}{2}\right)\left(v^2-\frac{a^2}{2}\right)=\frac{a^4}{4}$$

in this example may be performed as follows :

Let $\qquad u^2=\frac{a^2}{2}(1+t^2),\quad v^2=\frac{a^2}{2}\left(1+\frac{1}{t^2}\right).$

Then $\qquad 3u'=u(t^2-2),\quad 3v'=v\left(\frac{1}{t^2}-2\right);$

$$\therefore\ 9u'^2=\frac{a^2}{2}(1+t^2)(t^2-2)^2,\quad 9v'^2=\frac{a^2}{2}\left(1+\frac{1}{t^2}\right)\left(\frac{1}{t^2}-2\right)^2;$$

$$\therefore\ \frac{18u'^2}{a^2}-4=t^6-3t^4,\quad \frac{18v'^2}{a^2}-4=\frac{1}{t^6}-\frac{3}{t^4}$$

$$=A,\text{ say},\qquad\qquad =B,\text{ say}.$$

Then $\qquad t^2+\frac{1}{t^2}=\frac{10-AB}{3}=p,$ say;

$$\therefore\ t^4+\frac{1}{t^4}=p^2-2,\qquad t^6+\frac{1}{t^6}=p^3-3p;$$

$$\therefore\ A+B=(p^3-3p)-3(p^2-2);$$

$$\therefore\ A+B-5=(p+1)(p^2-4p+1),$$

$$27(A+B-5)=(13-AB)(A^2B^2-8AB-11);$$

$\therefore\ A^3B^3-21A^2B^2+93AB+27(A+B)+8=0$ is the locus required, where

$$A+B=\frac{36}{a^2}(x'^2-y'^2)-8,$$

$$AB=\frac{324}{a^4}(x'^2+y'^2)^2-\frac{144}{a^2}(x'^2-y'^2)+16.$$

The desired curve is therefore one of the 12th degree, and its arcs are of the same length as corresponding arcs on Bernoulli's lemniscate.

646. Serret's Mode of Derivation of Rectifiable Curves.

M. Serret (*Calcul Intégral*, p. 252) indicates a process by means of which algebraic curves can be produced which are rectifiable in terms of arcs of a circle, *i.e.* without the aid of the elliptic functions. Let $\iota\equiv\sqrt{-1}$.

Taking ι and $-\iota$, a and α, b and β, c and γ, etc., to be k pairs of conjugate constant complex quantities, C any real

constant quantity, and ω a real constant angle, and m, n, p, q, etc., positive integers, and putting

$$t=(z-a)^{n+1}(z-b)^{p+1}(z-c)^{q+1}\ldots,$$
$$T=(z-\alpha)^{n+1}(z-\beta)^{p+1}(z-\gamma)^{q+1}\ldots,$$

he states that the proposed problem is answered by the formula

$$x+\iota y=Ce^{\iota\omega}\int\frac{t}{T}\frac{(z-\iota)^m}{(z+\iota)^{m+2}}\,dz, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

provided the $k-1$ pairs of constants (a, α), (b, β), etc., be chosen so as to make the result of integration algebraic. As there are k repeated factors in the denominator of the integrand, this will entail the satisfying of $k-1$ independent conditions (Art. 149), for the degree of the denominator is greater by 2 than the degree of the numerator.

To see the truth of M. Serret's assertion, we observe that

$$dx+\iota\,dy=Ce^{\iota\omega}\,\frac{t}{T}\,\frac{(z-\iota)^m}{(z+\iota)^{m+2}}\,dz;$$

$$\therefore\ dx-\iota\,dy=Ce^{-\iota\omega}\,\frac{T}{t}\,\frac{(z+\iota)^m}{(z-\iota)^{m+2}}\,dz.$$

Hence $$ds^2=dx^2+dy^2=C^2\frac{dz^2}{(1+z^2)^2}$$

and $$ds=C\,\frac{dz}{1+z^2},$$

giving $$s=C\tan^{-1}z. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

647. M. Serret discusses a slightly different form in Liouville's *Journal*, vol. x.,* viz.

$$x+\iota y=Ce^{\iota\omega}\int\frac{(z-a)^m(z+a)^n}{(z-\alpha)^{m+1}(z+\alpha)^{n+1}}\,dz. \quad \ldots\ldots(3)$$

Here $$dx+\iota\,dy=Ce^{\iota\omega}\frac{(z-a)^m(z+a)^n}{(z-\alpha)^{m+1}(z+\alpha)^{n+1}}\,dz,$$

$$dx-\iota\,dy=Ce^{-\iota\omega}\,\frac{(z-\alpha)^m(z+\alpha)^n}{(z-a)^{m+1}(z+a)^{n+1}}\,dz;$$

whence $$ds^2=dx^2+dy^2=C^2\,\frac{dz^2}{(z^2-a^2)(z^2-\alpha^2)}$$

and $$s=C\int\frac{dz}{\sqrt{(z^2-a^2)(z^2-\alpha^2)}},$$

a form readily made to depend upon an elliptic integral.

*See also *Lond. Math. Soc. Proc.*, vol. xviii.; Mr. R. A. Roberts; and Cayley, *Ell. Funct.*, Art. 448 (where the $Ce^{\iota\omega}$ is omitted).

In the equation (3), the denominator is still in degree higher by 2 than the degree of the numerator, and there are two repeated factors in the denominator; hence one condition only is necessary that the resulting rectifiable curve should be purely algebraic (Art. 149). The integral (3) is not in all cases obtainable, but if one of the indices, say m, be a positive integer and if the equation of condition be satisfied, the integration can be effected in terms of z, involving complex constants. Then, equating real and imaginary parts, x and y can be found, and when z has been eliminated the Cartesian form of the equation of the derived curve will result.

648. **The Equation of Condition.**

The form of the conditional equation is very remarkable, viz. taking
$$\xi = \frac{(a+\alpha)^2}{4a\alpha},$$
it is
$$\frac{1}{\xi^{n-m}}\left(\frac{d}{d\xi}\right)^m \xi^n(\xi-1)^m = 0.$$

This is discussed at length by Cayley, chap. xv., *Ell. Funct.*, to which we must refer the advanced student for the work.

MISCELLANEOUS PROBLEMS.

1. Show that any point on the Lemniscate $r^2 = a^2\cos 2\theta$ may be represented by
$$x = a\frac{z+z^3}{1+z^4}, \quad y = a\frac{z-z^3}{1+z^4},$$
and hence obtain the rectification of the curve. [SERRET.]

Show that the integral obtained for s reduces to the standard Legendrian form by the further substitution
$$\cos\phi = \frac{z\sqrt{2}}{\sqrt{1+z^4}}.$$
[CAYLEY, *Ell. Functions*, Art. 63.]

2. By the transformation $\dfrac{z-\iota}{z+\iota} = \dfrac{a-\iota}{a+\iota}u$,

show that the equation
$$x+\iota y = Ce^{\iota\omega}\int\frac{(z-a)^{n+1}(z-\iota)^m}{(z-\alpha)^{n+1}(z+\iota)^{m+2}}\,dz$$
takes the form $\quad x+\iota y = A\displaystyle\int\frac{u^m(u-1)^{n+1}}{(u-\xi)^{n+1}}\,du,$

where $\quad \xi = \dfrac{(a+\iota)(\alpha-\iota)}{(a-\iota)(\alpha+\iota)}, \quad A = \dfrac{C}{2\iota}e^{\iota\omega}\left(\dfrac{a+\iota}{\alpha+\iota}\right)^{n+1}\left(\dfrac{a-\iota}{a+\iota}\right)^{m+1}$

Hence show that the condition that $x+\iota y$ should be purely algebraic is
$$\frac{d^n}{d\xi^n}\xi^m(\xi-1)^{n+1}=0,$$
a and α being supposed conjugate, and m, n positive integers.

Discuss the roots of this equation. [SERRET, *Calc. Intég.*, p. 254.]

3. In Bernoulli's Lemniscate $r^2=2a^2\cos 2\theta$,
show that if $\qquad x+\iota y=u \quad$ and $\quad x-\iota y=v$,
the equation of the curve may be written
$$(u^2-a^2)(v^2-a^2)=a^4.$$

Further, expressing u^2 and v^2 as $a^2(1+t^2)$ and $a^2\left(1+\frac{1}{t^2}\right)$ respectively, show that the tangent of the angle which the tangent at any point makes with the x-axis is
$$\iota\frac{1+t^3}{1-t^3}.$$

Hence, putting the coordinates of two points at which the tangents are parallel, as $\omega\mu$, $\omega^2\mu$ where $\omega^3=1$, show that the locus of the mid-points of chords joining such points is
$$[16u^2v^2-8a^2(u^2+v^2)+3a^4]^2$$
$$=4a^4[16\{u^4+v^4-u^2v^2\}-12a^2(u^2+v^2)+9a^4],$$
i.e. a curve of the eighth degree.

[R. A. ROBERTS, *Proc. L.M. Soc.*, vol. xviii.]

4. Obtain an integral for the rectification of the inverse of the parabola $y^2=4ax$, with regard to a point on the axis whose coordinates are $(h, 0)$.

If $h=-3a$, show that
$$s=\frac{1}{6a\sqrt{2}}\log\frac{3+2\sqrt{2}\sin\omega}{3-2\sqrt{2}\sin\omega},$$
where $a\tan^2\omega$, $2a\tan\omega$ are taken as the current coordinates of a point on the parabola, and the arc of the inverse is measured from the point corresponding to the vertex of the parabola.

[MR. ROBERTS, *loc. cit.*]

Show that the semiperimeter is bisected at the point $\omega=\sin^{-1}\frac{3}{4}$.

5. Show that the tangents to the parabola $y^2=4a(x+a)$ at the points
$$\{a\sinh^2(u\pm v)-a,\ 2a\sinh(u\pm v)\},$$
where u is variable but v is a constant, intersect on a confocal parabola; and that if T be a point on this second parabola, and TP_1, TP_2 the tangents to the first, then
$$TP_1+TP_2-\text{arc } P_1P_2=a(\sinh 2v-2v),$$
and is constant. [OXFORD I. P., 1911.]

6. Show that $\iint \frac{dA}{r}$, taken over the area cut from a parabola of latus rectum $4a$ by an ordinate distant c from the vertex $(c<a)$, where r denotes the distance from the focus, is equal to

$$4\sqrt{ac} - 2(a-c)\log\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}.$$ [OXFORD I. P., 1911.]

7. Show that $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 15\theta}{\sin\theta}\, d\theta = \frac{\pi}{4} + \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}.$

8. If

$$u = e^{-\int\phi\,dx}\int e^{\int\phi\,dx}(c_0 + c_1x + c_2x^2 + \ldots + c_nx^n)f\,dx + Ce^{-\int\phi\,dx},$$

where $c_0, c_1, c_2, \ldots, c_n, C$ are $(n+2)$ *arbitrary* constants, and

$$\frac{\phi}{f} = a_0 + a_1x + a_2x^2 + \ldots + a_mx^m,$$

where $a_0, a_1, \ldots, a_m$ are $(m+1)$ *given* constants, show that if m be not greater than n, $\frac{du}{dx}$, obtained by the direct differentiation of u with regard to x, contains only $(n+1)$ *arbitrary* constants.

[MATH. TRIPOS, 1878.]

9. If $f(m, n) = \int_0^\infty x^m(\cosh x)^{-n}dx$, where m and n are positive integers, each greater than 2, prove that

$$(n-1)(n-2)f(m, n) = (n-2)^2 f(m, n-2) - m(m-1)f(m-2, n-2).$$

[OXFORD I. P., 1914.]

10. Given that a and c are positive, show that the limit when $m \to \infty$ and $n \to \infty$ of

$$\frac{1}{n}\left[\frac{1}{a^r} + \frac{1}{\left(a+\frac{c}{n}\right)^r} + \frac{1}{\left(a+\frac{2c}{n}\right)^r} + \frac{1}{\left(a+\frac{3c}{n}\right)^r} + \ldots + \frac{1}{\left(a+mn\frac{c}{n}\right)^r}\right]$$

is finite when $r>1$; and find this limit. [OXF. I. P., 1914.]

11. The increase dS in a man's satisfaction S by an increased expenditure dx on a certain commodity, is expressed by the law $dS = \frac{\lambda}{x-a}dx$. Similar laws, viz.

$$dS = \frac{\mu}{y-b}dy, \quad dS = \frac{\nu}{z-c}dz,$$

hold for two other commodities, where $\lambda, \mu, \nu, a, b, c$ are all positive. Find how the man should expend a given sum $E\ (>a+b+c)$ so that his total satisfaction is greatest. [OXFORD I. P., 1914.]

Show that the maximum satisfaction is measured by

$$S = \log \frac{\lambda^\lambda \mu^\mu \nu^\nu (E - a - b - c)^{\lambda+\mu+\nu}}{(\lambda+\mu+\nu)^{\lambda+\mu+\nu}}.$$

12. Evaluate

$$\int_{e^u}^{\infty} \frac{x-1}{x^2 - 2x\cos v + 1} \frac{dx}{\sqrt{x^2 - 2x\cosh u + 1}}.$$

[Oxford II. P., 1914.]

13. Show that the tangent to the curve

$$3a^2(y - x) + x^3 = 0,$$

at the point whose abscissa is h, cuts the curve again at the point whose abscissa is $-2h$, and that the area included between the curve and the tangent is $9h^4/4a^2$. [Oxf. I. P., 1918.]

14. If $f_1(x)$ and $f_2(x)$ are both polynomials in x, show that the integral of $f_1(x)/f_2(x)$ with respect to x can always be written in the form

$$\phi_1(x)/\phi_2(x) + \log \phi_3(x)/\phi_4(x),$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 also denote polynomials, not necessarily real.

Find the general form of the integral with respect to x of

$$f_1(x + \sqrt{x^2 - 1})/f_2(x - \sqrt{x^2 - 1}).$$ [Oxf. I. P., 1918.]

15. Show that the area bounded by the curve

$$x = \frac{3at^2}{1+t^3}, \quad y = \frac{3at}{1+t^3},$$

its real asymptote $x + y + a = 0$, and by two lines at right angles to this asymptote through the points $t = -a$, $t = 0$ of the curve, is

$$\frac{3a^2}{4}\left\{1 + \frac{u^4 - 1}{(u^2 + u + 1)^2}\right\},$$

and find the whole area between the curve and its real asymptote.

[Oxf. I. P., 1917.]

16. If $\phi(z)$ be a rational function of z without singularities in the range $0 \leqslant z \leqslant 1$, prove that

$$\int_0^{\frac{\pi}{2}} \phi(\sin 2x)\cos^2 x \cos 2x\,dx = \int_0^{\frac{\pi}{2}} \phi(\sin 2x)\cos^2 x\cos^2 2x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \phi(\sin 2x)\cos^4 x \cos 2x\,dx.$$

[Oxford I. P., 1907.]

17. Integrate (i) $\displaystyle\int \frac{x(x-a)^{b-1}\,dx}{(x-b)\{(x-b)^{2b} - (x-a)^{2a}\}^{\frac{1}{2}}}$,

(ii) $\displaystyle\int \frac{x^{q-1}\{px^{p+q} - qa^{p+q}\}\,dx}{(x^{p+q} + a^{p+q})^2 + x^{2q}a^{2q}}$.

18. In the curve $\dfrac{15x}{a}=\left(\dfrac{2y}{a}\right)^{\frac{3}{4}}-\left(\dfrac{2y}{a}\right)^{\frac{5}{4}}$, show that $s^2=x^2+\frac{16}{15}y^2$, s being measured from the origin.

Show that the curve is a quintic of which the y-axis is an axis of symmetry, and that the area of the loop $=\left(\dfrac{8}{189}\right)\left(\dfrac{5}{3}\right)^{\frac{7}{2}}a^2$.

19. If 2ϕ be the eccentric angle of the point r, θ on the ellipse $c=r(1-e\cos\theta)$, prove that

$$\{(1+e)^2-4e\sin^2\phi\}\left(\frac{d\theta}{d\phi}\right)^2=4(1-e^2\cos^2\theta).$$

Use the fact that

$$\int_0^\pi F(\cos^2\theta)\,d\theta=2\int_0^{\frac{\pi}{2}}F(\cos^2\theta)\,d\theta$$

and the above to obtain a value of α, such that

$$\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{(1+e)^2-4e\sin^2\phi}}=2\int_0^{\alpha}\frac{d\phi}{\sqrt{(1+e)^2-4e\sin^2\phi}}.$$

[OXFORD I. P., 1917.]

20. A uniform rod of mass M has its extremities at the points x_1, y_1; x_2, y_2. Show that the product of inertia of the rod with respect to the axes is given by

$$M\int_0^1\{tx_2+(1-t)x_1\}\{ty_2+(1-t)y_1\}\,dt.$$

Hence show that the product of inertia of the rod is the same as that of three particles of masses

$$\frac{M}{6},\quad\frac{M}{6},\quad\frac{2M}{3},$$

placed at the extremities and the middle point of the rod respectively.

[OXFORD I. P., 1913.]

21. Show that the coordinates of any point on the curve whose intrinsic equation is

$$s=a(\sec^n\psi-1),$$

where n is an odd integer greater than unity, can be expressed rationally in terms of $\tan\psi$, and show that when $x=0$ the curve is a cubic with a cusp. [OXF. I. P., 1911.]

22. Show how to evaluate the integral $\int f(x,y)\,dx$, where

$$y^2=ax^2+2bx+c$$

and $f(x,y)$ is a rational function of x and y.

Prove that

$$\text{(i)} \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \tfrac{1}{4}\pi,$$

$$\text{(ii)} \int_0^a \frac{a\,dx}{(x+\sqrt{a^2-x^2})^2} = \frac{1}{\sqrt{2}} \log(1+\sqrt{2}),$$

the positive sign being taken for the radical in each of the subjects of integration. [MATH. TRIP., PART. II., 1913.]

23. Show by means of the transformation $y = \dfrac{(x^2+1)^{\frac{1}{2}}}{x+1}$ that

$$\int_0^\infty \frac{dx}{(x+1)(x^2+1)^{\frac{1}{2}}} = 2\int_{1/\sqrt{2}}^1 \frac{dy}{(2y^2-1)^{\frac{1}{2}}} = \sqrt{2}\log(\sqrt{2}+1),$$

and verify the result in an independent manner.

[MATH. TRIP., PART II., 1914.]

24. Integrate $\displaystyle\int \frac{\sin x}{\sin(x-a)}\,dx.$ [MATH. TRIP., PART II., 1914.]

25. Evaluate

$$\int \frac{x+2}{(x+1)^2(x^2+4)}, \quad \int \frac{dx}{(x^2+1)^4}, \quad \int \frac{dx}{(5-3\cos x)^2},$$

and the corresponding definite integrals taken between the limits $(0, \infty)$, $(0, \infty)$ and $(0, \pi)$ respectively. [MATH. TRIP., PART II., 1914.]

26. Show that

$$\text{(i)} \int \frac{\sin 4x}{\sin 6x}\,dx = \frac{\sqrt{3}}{6} \tanh^{-1}\left(\frac{\sin 2x}{\cos\frac{\pi}{6}}\right).$$

$$\text{(ii)} \int \frac{\sin 3x}{\sin 5x}\,dx = \frac{1}{5}\left[\sin\frac{\pi}{5}\log\frac{\sin\left(\frac{2\pi}{5}+x\right)}{\sin\left(\frac{2\pi}{5}-x\right)} + \sin\frac{2\pi}{5}\log\frac{\sin\left(\frac{\pi}{5}+x\right)}{\sin\left(\frac{\pi}{5}-x\right)}\right].$$

27. Prove that

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1+\sin^2\theta)(2+\sin^2\theta)} = \frac{\pi}{\sqrt{3}}\sin\frac{\pi}{12}.$$

28. Prove that

$$\int \frac{2\cos\theta+\sin\theta}{(1+\sin\theta\cos\theta)^{\frac{3}{2}}}\,d\theta = \frac{2\sin\theta}{(1+\sin\theta\cos\theta)^{\frac{1}{2}}}.$$

CHAPTER XIX.

MOVING CURVES.

Quadrature and Rectification of Loci of Carried Points and Envelopes of Carried Lines.

649. "**Instantaneous Centre.**"

It is a very well-known geometrical theorem that if two triangles ABC, abc are equal in all respects and lie in the same plane, the one can be superposed upon the other by a rotation about some point in the plane.

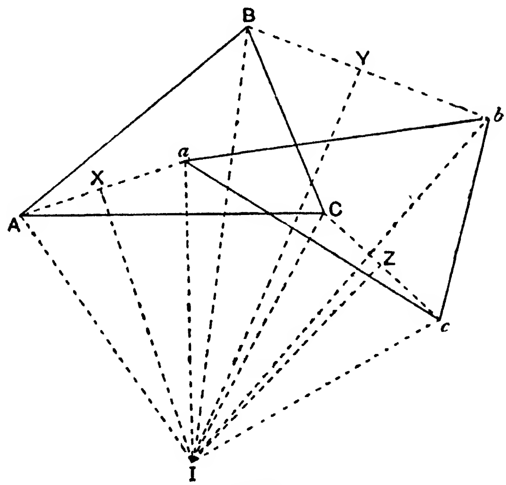


Fig. 180.

Let XI, YI, the perpendicular bisectors of Aa, Bb, meet at I. Join IA, Ia; IB, Ib; IC, Ic; and join I to the mid-point Z of Cc.

Then IA, AB, BI being respectively equal to Ia, ab, bI, the triangles IAB, Iab are congruent, and angle $I\hat{B}A = I\hat{b}a$. Hence $I\hat{B}C = I\hat{b}c$, and having also IB, BC respectively equal to Ib, bc, the triangles IBC, Ibc are congruent, and $IC = Ic$; whence IZ bisects Cc perpendicularly, so that the perpendicular bisectors of Aa, Bb, Cc are concurrent. Moreover angle $A\hat{I}B$ being equal to $a\hat{I}b$, and $B\hat{I}C$ being equal to $b\hat{I}c$, it is clear that

$$A\hat{I}a = B\hat{I}b = C\hat{I}c,$$

and therefore a rotation through the angle $A\hat{I}a$ about the point I in the proper direction will accomplish the superposition of the one triangle upon the other.

If Aa, Bb are parallel, I is at ∞ in the plane, and the motion is one of translation without rotation.

Two of the three points A, B, C may be regarded as fixing the position of the lamina upon which the triangle is drawn, and the third point may be regarded as *any* point carried by the lamina.

Thus a displacement of a lamina of any shape in its own plane may be regarded as brought about by a rotation about a point in its plane, and any consistent motion of two points attached to the plane lamina will define the motion of the lamina in its own plane.

650. If the equal angles AIa, BIb be infinitesimal, Aa, Bb may be regarded ultimately as the direction of the tangents to the paths of A and B, and I is called the *instantaneous centre.* The position of this point is immediately discovered when the direction of motion of the two points A and B are known, by drawing through A and B perpendiculars to the direction of motion of these points; these perpendiculars meet in the "instantaneous centre of rotation" I. If I be joined to any other point P of the moving lamina, the tangent to the path of P is at right angles to PI, and PI is the normal to the path.

651. For instance, if a hoop of any shape be in motion in a plane, and the direction of motion of two points of the hoop be known, say, PT, QT, then I is at the intersection of perpendiculars through P and Q to PT, QT respectively, and the motion of any other point of the hoop, R, is at

right angles to IR. Hence at any instant the directions of instantaneous motion of all particles on the hoop envelop the first negative pedal of the hoop with regard to the instantaneous centre. When the hoop is

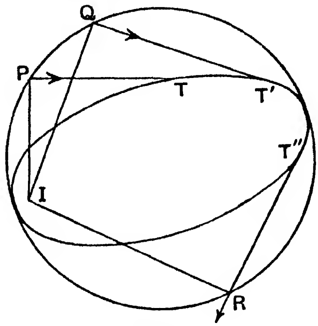


Fig. 181.

circular, this will be an ellipse if I falls within the hoop, a hyperbola if I falls without the hoop, and a point if I falls upon the hoop.

652. The instantaneous centre itself is not in general a fixed point. If it has a path upon the fixed plane, it has another path relatively to the moving lamina.

When a circular hoop rolls along the ground in a vertical plane, the point of contact is the instantaneous centre, for at any instant the point

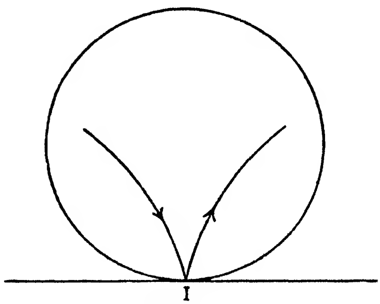


Fig. 182.

of the hoop in contact with the ground is not moving *along* the ground, for by supposition there is no slipping, and it has just ceased to approach the ground, and is on the point of beginning to leave the ground, and therefore for the instant it has no motion *at right angles* to the ground. The path of the instantaneous centre on the fixed plane is evidently the line on which the hoop rolls. The path on the plane of the hoop is the hoop itself.

653. Exactly the same is true when any curve traced upon a lamina is made to roll without sliding upon a fixed curve. The point of contact is the instantaneous centre. The two I-loci are respectively the fixed curve and the moving curve themselves.

654. When a rod AB, of given length, slips down between two perpendicular axes Oy, Ox, the instantaneous centre I is at the intersection of the perpendiculars AI, BI to Oy and Ox, and its locus on the fixed plane

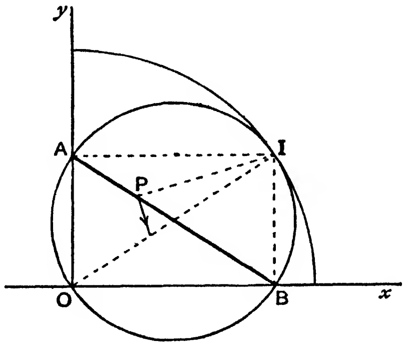


Fig. 183.

is a circle with centre at O and radius equal to the rod. The path relative to the rod is a circle of radius half the rod, described on the rod for diameter.

Any point P attached to the rod describes an ellipse, of which IP is the normal and a perpendicular through P to IP is the tangent.

655. **General Motion of a Lamina reduced to a Case of Rolling.**

Let us define the manner of motion of the lamina to be such that its angular velocity at every instant is some given quantity; $I_1, I_2, I_3, I_4, I_5 \ldots$, being the corresponding successive positions at equal intervals of time dt of the instantaneous centre on the fixed plane upon which the lamina moves.

Let $d\psi_1, d\psi_2, d\psi_3, \ldots$ be the infinitesimal angles turned through in successive rotations about $I_1, I_2, I_3, \ldots$. Then

(a) Let there be a rotation $d\psi_2$ about I_2.

Then a line on the moving lamina, which was originally coincident with I_2I_1, will be brought by rotation about I_2 into the position $I_2\iota_1$.

(*b*) Now let rotation commence about I_3 through $d\psi_3$.

Then the line $I_3 I_2 \iota_1$ on the moving lamina is brought into the position $I_3 \iota_2 \iota_1'$.

(*c*) Let rotation now commence about I_4 through $d\psi_4$.

Then the line $I_4 I_3 \iota_2 \iota_1'$ is brought into the position $I_4 \iota_3 \iota_2' \iota_1''$.

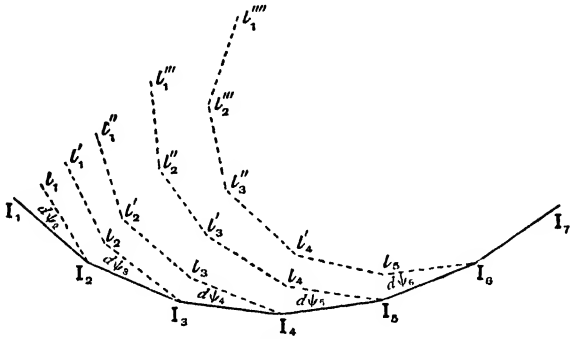


Fig. 184.

(*d*) Let rotation now commence about I_5 through $d\psi_5$.

Then the line $I_5 I_4 \iota_3 \iota_2' \iota_1''$ is brought to the position $I_5 \iota_4 \iota_3' \iota_2'' \iota_1'''$, and so on.

Hence it is clear that when the intervals of time are infinitesimally small, and the chords $I_1 I_2$, $I_2 I_3$, etc., indefinitely diminished, the motion of the lamina may be constructed by the rolling of the curve locus of the instantaneous centres relative to the lamina, viz. $I_5 \iota_4 \iota_3' \iota_2'' \iota_1'''$ upon the curve locus of the instantaneous centres upon the fixed plane, viz. $I_5 I_4 I_3 I_2 I_1$.

Hence the general motion of a lamina in its own plane may be constructed by the rolling of one curve upon another. It therefore becomes important to study the motion of points and lines attached to curves which roll.

656. The Two Loci of the Instantaneous Centre.

The locus of *I* both on the lamina itself and on the fixed plane upon which the lamina moves becomes important. Each may be readily found.

Let *OX*, *OY* be fixed rectangular axes upon the fixed plane.

Let *O′x*, *O′y* be rectangular axes attached to the moving lamina.

Let ξ, η be the coordinates of O' relatively to OX, OY; x, y the coordinates of any point P on the lamina relatively to $O'x$, $O'y$.

Let θ be the inclination of $O'x$ to OX.

The motion of the lamina will then be fully defined by the three coordinates ξ, η, θ, and their differential coefficients with regard to time, where ξ and η are definite known functions of θ.

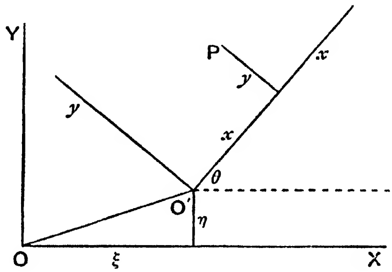


Fig. 185.

The coordinates of P relatively to OX, OY will be

$$\left.\begin{aligned} X &= \xi + x\cos\theta - y\sin\theta, \\ Y &= \eta + x\sin\theta + y\cos\theta. \end{aligned}\right\} \quad \text{.................. (1)}$$

Differentiating,

$$dX = d\xi + (dx - y\,d\theta)\cos\theta - (dy + x\,d\theta)\sin\theta,$$

$$dY = d\eta + (dx - y\,d\theta)\sin\theta + (dy + x\,d\theta)\cos\theta.$$

To find the position of I about which the lamina is turning at any instant, we must remember that

(a) it is for the moment stationary in space,

(b) it is for the moment stationary in the lamina.

Hence for this point

$$dX = dY = 0 \quad \text{and} \quad dx = dy = 0.$$

Therefore $$\left.\begin{aligned} d\xi - y\,d\theta\cos\theta - x\,d\theta\sin\theta &= 0, \\ d\eta - y\,d\theta\sin\theta + x\,d\theta\cos\theta &= 0, \end{aligned}\right\} \text{ at such a point,}$$

and ξ, η being known functions of θ, x and y are found from

$$\left.\begin{aligned} x &= \frac{d\xi}{d\theta}\sin\theta - \frac{d\eta}{d\theta}\cos\theta, \\ y &= \frac{d\xi}{d\theta}\cos\theta + \frac{d\eta}{d\theta}\sin\theta, \end{aligned}\right\} \quad \text{.................... (2)}$$

and the θ-eliminant from these equations gives the locus of I on the lamina.

Next, substituting in equation (1),

$$\left.\begin{aligned} X&=\xi-\frac{d\eta}{d\theta},\\ Y&=\eta+\frac{d\xi}{d\theta},\end{aligned}\right\} \qquad \text{.........................(3)}$$

and the θ-eliminant from these equations gives the I-locus on the fixed plane.

657. Ex. 1. Taking the case of a rod AB $(=2a)$ sliding between two straight lines OX, OY at right angles, making an angle θ with the

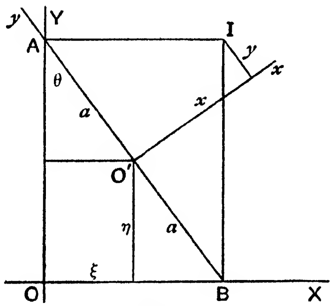


Fig. 186.

latter, and taking the centre of the rod O' as origin for the moving axes and the rod itself as the y-axis,

$$\xi=a\sin\theta, \quad \eta=a\cos\theta\,;$$

$$\therefore \left.\begin{aligned} x&=a\cos\theta\sin\theta+a\sin\theta\cos\theta=a\sin 2\theta,\\ y&=a\cos\theta\cos\theta-a\sin\theta\sin\theta=a\cos 2\theta,\end{aligned}\right\} \text{ from equations (2)};$$

$$\left.\begin{aligned} X&=a\sin\theta+a\sin\theta=2a\sin\theta,\\ Y&=a\cos\theta+a\cos\theta=2a\cos\theta,\end{aligned}\right\} \text{ from equations (3)},$$

and the locus of I on the lamina is

$$x^2+y^2=a^2,$$

and on the fixed plane $\quad x^2+y^2=4a^2$;

as is geometrically obvious (see Art. 654); as indeed are also all the equations established, the point I being at the intersection of the perpendiculars at B and A to OX, OY respectively.

All carried points which lie on the circle with AB for diameter describe two cusped hypo-cycloids, *i.e.* straight lines, and all points attached to the line itself describe ellipses (see Besant, *Conic Sections*, Art. 245).

Ex. 2. Taking the case of an involute of a circle of radius a, sliding between two perpendicular lines OX, OY, let the radius of the circle through the cusp make an angle θ with the line OX. Then

$$\xi = a\left(\frac{\pi}{2}+\theta\right), \quad \eta = a\theta;$$

$$\therefore \left.\begin{array}{l} x = a(\sin\theta - \cos\theta), \\ y = a(\cos\theta + \sin\theta), \end{array}\right\} \text{ from equations (2);}$$

$$\left.\begin{array}{l} X = a\left(\frac{\pi}{2}+\theta\right) - a, \\ Y = a\theta + a, \end{array}\right\} \text{ from equations (3).}$$

Hence the locus of I on the lamina is $x^2+y^2=2a^2$, *i.e.* a circle; the locus of I on the fixed plane is $Y-X=2a-\frac{\pi a}{2}$, *i.e.* a straight line. These loci are shown in Fig. 187.

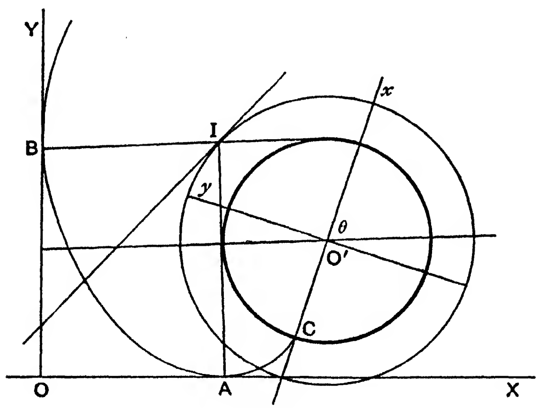


Fig. 187.

The first of the loci is geometrically obvious, as the tangents from I to the generating circle of the involute are at right angles.

The motion is that of the rolling of a circle of radius $a\sqrt{2}$ upon a straight line which makes an angle $\frac{\pi}{4}$ with the axes OX, OY and an intercept $\left(2-\frac{\pi}{2}\right)a$ on the Y-axis. The locus of the starting-point C of the involute is plainly a trochoid, and the locus of the centre of the generating circle a straight line. Points on the circular I-locus describe cycloids, all other attached points describe trochoids.

The student will find this example done (in a different way) in Besant's *Roulettes and Glisettes*, p. 37. The object here is to illustrate the use of the general formulae of the preceding article.

Ex. 3. Consider a case of motion of apparently different nature.

Let a lamina PQR rotating at a constant angular velocity ω be moving so that an attached point C describes a straight line with uniform velocity v.

Take the path of C as the axis of X, and ξ, η the coordinates of the centre, and θ the angle turned through in time t, and suppose that initially ξ and θ both vanish. Let accents denote differentiations with regard to θ

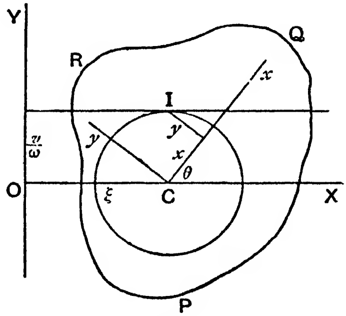


Fig. 188.

Then, O being the starting point for the point C,

$$\xi = vt, \quad \theta = \omega t, \quad \xi = \frac{v}{\omega}\theta, \quad \eta = 0, \quad \xi' = \frac{v}{\omega}, \quad \eta' = 0.$$

The equations of Art. 656 give

$$X = \frac{v}{\omega}\theta, \quad Y = \frac{v}{\omega}; \quad x = \frac{v}{\omega}\sin\theta, \quad y = \frac{v}{\omega}\cos\theta;$$

$\therefore$ the I-loci are a straight line, $Y = \frac{v}{\omega}$, on the fixed plane, and

$$x^2 + y^2 = \frac{v^2}{\omega^2},$$

i.e. a circle whose centre is C on the lamina.

The motion is therefore that of a circle rolling on a fixed straight line, All carried points describe cycloids or trochoids.

658. In the same way, if the point C be made to describe a circle of radius a with angular velocity ω, whilst the lamina rotates with an angular velocity ω', we have, taking rectangular axes through the centre of the fixed circle, and rectangular moving axes through the point C attached to the lamina, and supposing η and θ to vanish together,

$$\xi = a\cos\omega t, \quad \eta = a\sin\omega t, \quad \theta = \omega' t;$$

$$X = \xi - \frac{d\eta}{d\theta} = a\frac{\omega' - \omega}{\omega'}\cos\frac{\omega\theta}{\omega'}, \quad Y = \eta + \frac{d\xi}{d\theta} = a\frac{\omega' - \omega}{\omega'}\sin\frac{\omega\theta}{\omega'};$$

$$x = -\frac{a\omega}{\omega'}\cos\frac{\omega' - \omega}{\omega'}\theta, \qquad y = \frac{a\omega}{\omega'}\sin\frac{\omega' - \omega}{\omega'}\theta,$$

and the motion is that of a circle of radius $\frac{a\omega}{\omega'}$ rolling upon a fixed circle of radius $a\frac{\omega'-\omega}{\omega'}$, and therefore all carried points on the lamina trace epi- or hypo-cycloids or epi- or hypo-trochoids.

659. Ex. Suppose that a point O' of a lamina PQR travel upon an equiangular spiral, with pole O, fixed upon a plane over which the lamina slides. Suppose that the lamina rotates at $\frac{1}{n}$th of the rate of the radius vector OO'. It is required to reduce this motion to one of rolling.

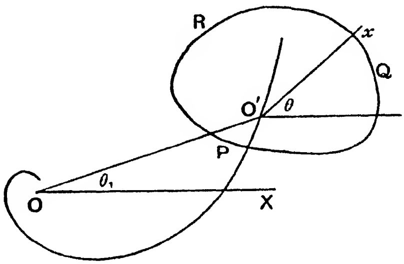


Fig. 189.

Let OO' make an angle θ_1 with the initial line, and let a line $O'x$ fixed on the rotating lamina make an angle θ with the axis OX fixed in space. Suppose OX be taken such that θ_1, θ vanish together. Then $\theta_1 = n\theta$.

If ξ, η be the coordinates of O', we have

$$\xi = ae^{\theta_1 \cot\alpha}\cos\theta_1, \quad \eta = ae^{\theta_1\cot\alpha}\sin\theta_1, \quad \frac{d\theta_1}{d\theta} = n,$$

with the usual notation as to the spiral.

Then
$$\xi' \equiv \frac{d\xi}{d\theta} = nae^{\theta_1\cot\alpha}(\cot\alpha\cos\theta_1 - \sin\theta_1),$$

$$\eta' \equiv \frac{d\eta}{d\theta} = nae^{\theta_1\cot\alpha}(\cot\alpha\sin\theta_1 + \cos\theta_1);$$

$\therefore$ by Art. 656,

$$X = \xi - \eta' = ae^{\theta_1\cot\alpha}[(1-n)\cos\theta_1 - n\cot\alpha\sin\theta_1],$$

$$Y = \eta + \xi' = ae^{\theta_1\cot\alpha}[(1-n)\sin\theta_1 + n\cot\alpha\cos\theta_1].$$

Putting $1-n = k\cos\beta$, $n\cot\alpha = k\sin\beta$,

$$\left.\begin{aligned} X &= ae^{\theta_1\cot\alpha}k\cos(\theta_1+\beta), \\ Y &= ae^{\theta_1\cot\alpha}k\sin(\theta_1+\beta), \end{aligned}\right\}$$

i.e. the locus of X, Y is $R = kae^{(\Theta-\beta)\cot\alpha}$, where R, Θ are current coordinates, and

$$k = \sqrt{(1-n)^2 + n^2\cot^2\alpha}, \quad \tan\beta = \frac{n}{1-n}\cot\alpha,$$

i.e. the fixed I-locus is an equal equiangular spiral.

Again, $x=\xi'\sin\theta-\eta'\cos\theta,\quad y=\xi'\cos\theta+\eta'\sin\theta,$

and
$$x^2+y^2=\xi'^2+\eta'^2=\frac{n^2a^2e^{2\theta_1\cot\alpha}}{\sin^2\alpha},$$

and if R_1, Θ_1 be the polar coordinates of a point on the I-locus upon the lamina,

$$R_1\cos\Theta_1=\xi'\sin\theta-\eta'\cos\theta,$$
$$R_1\sin\Theta_1=\xi'\cos\theta+\eta'\sin\theta,$$

$$\tan\Theta_1=\frac{\cot\theta+\frac{\eta'}{\xi'}}{1-\frac{\eta'}{\xi'}\cot\theta}=\frac{\cot\theta+\tan(\theta_1+\alpha)}{1-\cot\theta\tan(\theta_1+\alpha)}$$
$$=\tan\left(\frac{\pi}{2}-\theta+\theta_1+\alpha\right)$$
$$=\tan\left(\frac{\pi}{2}+\alpha+\overline{n-1}\,\theta\right),$$
$$\Theta_1=\frac{\pi}{2}+\alpha+\frac{n-1}{n}\theta_1;$$

$\therefore$ the polar equation of the (x, y) locus is

$$R_1=\frac{na}{\sin\alpha}\,e^{\frac{n\cot\alpha}{n-1}\left(\Theta_1-\alpha-\frac{\pi}{2}\right)},$$

i.e. another equiangular spiral, but of different angle, which is replaced by the straight line

$$\Theta_1=\frac{\pi}{2}+\alpha,\quad \text{when } n=1.$$

The motion is therefore that of one equiangular spiral with angle α, rolling upon another of different angle, or when $n=1$, upon a straight line. The case when $n=1$ is that in which the lamina rotates at the same rate as the radius vector of the original spiral.

660. The Curvatures of the two Loci. Analytical Consideration.

It will be found in later articles that we frequently have to find the difference of the curvatures of these two I-loci. And for convenience of drawing it is customary, as in Arts. 665, 667, and in *Diff. Calc.*, Ch. XX., to consider the concavities of the fixed and rolling curves as being in opposite directions. That is, the expressions $\frac{1}{\rho_1}+\frac{1}{\rho_2}$ which occur in theorems on Roulettes and Glisettes are the algebraic differences of curvatures as measured in the same direction.

For the present we consider the concavities in the *same* direction. Both the I-loci have been found in the form

$x=F(\theta)$, $y=f(\theta)$, and therefore the curvatures can readily be obtained from the formula

$$\frac{1}{\rho}=\frac{f'(\theta)F''(\theta)-f''(\theta)F'(\theta)}{[\{F'(\theta)\}^2+\{f'(\theta)\}^2]^{\frac{3}{2}}}.$$

Representing by accents differentiations with regard to θ, we have

(a) For the I-locus on the fixed plane,

$$X=\xi-\eta',\quad Y=\eta+\xi',$$
$$X'=\xi'-\eta'',\quad Y'=\eta'+\xi'',$$
$$X''=\xi''-\eta''',\quad Y''=\eta''+\xi''';$$
$$\therefore\ X'^2+Y'^2=(\xi''+\eta')^2+(\eta''-\xi')^2,$$

and $\quad X'Y''-X''Y'=(\xi'-\eta'')(\eta''+\xi''')-(\eta'+\xi'')(\xi''-\eta''');$

and if ρ_1 be the radius of curvature of this fixed I-locus,

$$\frac{1}{\rho_1}=\frac{(\xi'-\eta'')(\eta''+\xi''')-(\eta'+\xi'')(\xi''-\eta''')}{[(\xi''+\eta')^2+(\eta''-\xi')^2]^{\frac{3}{2}}}.$$

(b) For the locus of I on the moving lamina,

$$x=\xi'\sin\theta-\eta'\cos\theta,\quad y=\xi'\cos\theta+\eta'\sin\theta,$$
$$\left.\begin{aligned}x'&=(\xi''+\eta')\sin\theta-(\eta''-\xi')\cos\theta,\\ y'&=(\xi''+\eta')\cos\theta+(\eta''-\xi')\sin\theta,\end{aligned}\right\}$$
$$\left.\begin{aligned}x''&=(\xi'''+2\eta''-\xi')\sin\theta-(\eta'''-2\xi''-\eta')\cos\theta,\\ y''&=(\xi'''+2\eta''-\xi')\cos\theta+(\eta'''-2\xi''-\eta')\sin\theta,\end{aligned}\right\}$$

and $$x'^2+y'^2=(\xi''+\eta')^2+(\eta''-\xi')^2,$$

$$x'y''-y'x''=(\xi''+\eta')(\eta'''-2\xi''-\eta')-(\eta''-\xi')(\xi'''+2\eta''-\xi').$$

And if ρ_2 be the radius of curvature of the I-locus on the moving lamina estimated in the same direction as I,

$$\frac{1}{\rho_2}=\frac{(\xi''+\eta')(\eta'''-2\xi''-\eta')-(\eta''-\xi')(\xi'''+2\eta''-\xi')}{[(\xi''+\eta')^2+(\eta''-\xi')^2]^{\frac{3}{2}}}.$$

Hence $$\frac{1}{\rho_2}\sim\frac{1}{\rho_1}=\frac{\xi''^2+\eta''^2+\xi'^2+\eta'^2-2\xi'\eta''+2\xi''\eta'}{[(\xi''+\eta')^2+(\eta''-\xi')^2]^{\frac{3}{2}}}$$

$$=\frac{1}{[(\xi''+\eta')^2+(\eta''-\xi')^2]^{\frac{1}{2}}}$$

$$=\frac{1}{(X'^2+Y'^2)^{\frac{1}{2}}},$$

which gives the difference of the curvatures sought.

Finally, $x'^2+y'^2=(\xi''+\eta')^2+(\eta''-\xi')^2=X'^2+Y'^2$,
and therefore if ds be the elementary arc of either curve,

$$\frac{ds}{d\theta}=\sqrt{(\xi''+\eta')^2+(\eta''-\xi')^2} \quad\text{and}\quad s=\int\sqrt{(\xi''+\eta')^2+(\eta''-\xi')^2}\,d\theta,$$

whence
$$\frac{1}{\rho_2}\sim\frac{1}{\rho_1}=\frac{1}{\frac{ds}{d\theta}}.$$

661. Geometrical Consideration.

This last result may be seen at once geometrically; for $\frac{ds}{\rho_2}$ and $\frac{ds}{\rho_1}$ are the angles turned through by ρ_2 and ρ_1, and their difference is the angle turned through by the moving lamina, *i.e.*

$$\frac{ds}{\rho_2}\sim\frac{ds}{\rho_1}=d\theta. \quad \text{(See Fig. 190.)}$$

662. (1) Thus, in the case of the sliding rod of Art. 657, Ex. 1, we have

$$\begin{aligned}\xi&=a\sin\theta, & \eta&=a\cos\theta,\\ \xi'&=a\cos\theta, & \eta'&=-a\sin\theta,\\ \xi''&=-a\sin\theta, & \eta''&=-a\cos\theta,\end{aligned}$$

and
$$\frac{1}{\rho_2}-\frac{1}{\rho_1}=\frac{1}{\sqrt{4a^2\sin^2\theta+4a^2\cos^2\theta}}=\frac{1}{2a}$$

which agrees with the previous result for which $\rho_1=2a$, $\rho_2=a$.

(2) In the case of the sliding involute (Art. 657, Ex. 2),

$$\xi=a\left(\frac{\pi}{2}+\theta\right), \qquad \eta=a\theta,$$

$$\xi'=a, \quad \eta'=a, \quad \xi''=\eta''=0,$$

and
$$\frac{1}{\rho_2}-\frac{1}{\rho_1}=\frac{1}{\sqrt{a^2+a^2}}=\frac{1}{a\sqrt{2}},$$

which agrees with the previous result, for which $\rho_1=\infty$, $\rho_2=a\sqrt{2}$; and

$$s=\int\sqrt{(\xi''+\eta')^2+(\eta''-\xi')^2}\,d\theta$$

gives $2a\theta$ in case (1) above, and $a\theta\sqrt{2}$ in case (2).

663. Besant's Equations for the Fixed I-locus for sliding curves.

When the motion of the lamina is defined by two curves attached to the lamina making sliding contact with fixed perpendicular axes OX, OY, the equations

$$X=\xi-\eta', \quad Y=\eta+\xi'$$

give $\quad X'=\xi'-\eta''=Y-\eta-\eta''$
and $\quad Y''=\eta'+\xi''=\xi-X+\xi''$ $\Big\}$ respectively,

and show that $\quad X'-Y=-(\eta+\eta'')=-\rho_1,$
$\quad Y'+X=\quad \xi+\xi''=\quad \rho_2,$ $\Big\}$ by Legendre's formula,

where ρ_1 and ρ_2 are the radii of curvature of the sliding curves at the points of contact with the straight lines OX, OY.

These equations are obtained by geometrical considerations by Mr. Besant (*Roulettes and Glisettes*, Art. 51), and are the equations he uses for the determination of the I-locus on the fixed plane in such cases of sliding contact. They require the integration of two simultaneous differential equations for the determination of the locus.

When the intrinsic equations of the two curves are known, viz. $s=f_1(\psi)$, $s=f_2(\psi)$, Mr. Besant's equations are very convenient, and the fixed I-locus can be deduced by solving the simultaneous equations

$$\frac{dX}{d\psi}-Y=-f_1'(\psi), \quad \frac{dY}{d\psi}+X=f_2'\left(\psi+\frac{\pi}{2}\right),$$

the constant being determined by the starting conditions.

664. "**Roulettes and Glisettes.**"

The path of a point carried by a curve which rolls upon another curve is called the Roulette of the point. (See *Diff. Calc.*, Art. 561.)

The path of a point carried by a lamina which moves so that a curve drawn upon it slides in such a manner as to touch two given fixed curves is called a Glisette.

The latter name is due to Mr. W. H. Besant.

The terms Roulette and Glisette have been extended to include the case of the envelope of a carried curve.

A very full and interesting account of the principal properties of Roulettes and Glisettes was given by Mr. Besant in his *Tract on Roulettes and Glisettes* (1870).

665. **The Curvature of the Roulettes described by a Carried Point, and as the Envelope of a Carried Curve** are worked out in Articles 564 and 565 respectively of the *Diff. Calc.* The student who has not access to Mr. Besant's tract, should revise

these articles before reading the articles which follow, which are mainly concerned with quadrature and rectification.

The formula established for the radius of curvature of the envelope of a curve carried by another curve which rolls without sliding upon a fixed curve is shown to be

$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

Here ρ_1, ρ_2 are the radii of curvature of the fixed and rolling curves respectively, ρ that of the carried curve, R that of its envelope, whilst r is the normal distance of the point of contact of the carried curve with its envelope from the point of contact of the rolling and fixed curves, and ϕ is the angle r makes with the common normal of the latter.

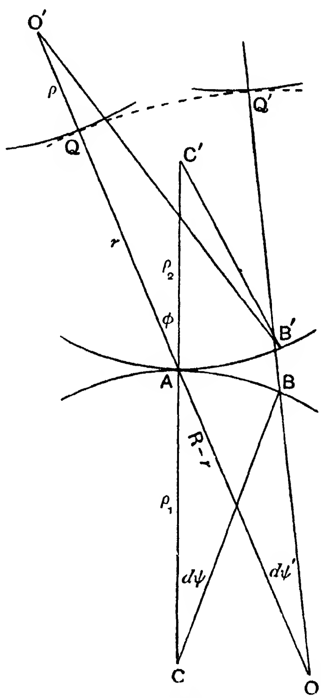


Fig. 190.

If all these several quantities can be expressed in terms of ψ', the angle which r makes with any fixed line, then $\int R\,d\psi'$ gives the length of an arc of the envelope, *i.e.*

$$\text{Arc}=\int\frac{r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\rho}{r+\rho}\cos\phi}{\frac{1}{\rho_1}+\frac{1}{\rho_2}-\frac{\cos\phi}{r+\rho}}\,d\psi'.$$

This is the **general result.** It includes the roulette of a carried point, viz. when $\rho=0$, or of a carried straight line (when $\rho=\infty$), or the case when the fixed curve is a straight line ($\rho_1=\infty$), or when the rolling curve is a circle ($\rho_2=a$), or when the rolling curve is a straight line ($\rho_2=\infty$), or any combination of such cases.

The standard figure is that shown above and described in *Diff. Calc.*, Art. 565. If the concavity of any of the curves be in the opposite direction, the formula will require modification

by the change of sign in the particular radius of curvature or particular radii of curvature involved.

It must be remembered from *Diff. Calc.*, Art. 565, that the angle between

two consecutive positions of ρ_1 is $\dfrac{ds}{\rho_1}$,

.... ρ_2 is $\dfrac{ds}{\rho_2}$,

.... r is $\dfrac{ds\cos\phi}{R-r}$,

.... ρ is $\dfrac{ds\cos\phi}{r+\rho}$.

Thus
$$d\psi'=\frac{ds\cos\phi}{R-r};$$

$$\therefore \textbf{ arc of envelope}=\int\left[r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\rho}{r+\rho}\cos\phi\right]ds.$$

666. Again, the **area swept out by** r is plainly

$$\int\tfrac{1}{2}R^2 d\psi'-\int\tfrac{1}{2}(R-r)^2 d\psi'=\frac{1}{2}\int(2Rr-r^2)\,.\,\frac{ds\cos\phi}{R-r}$$
$$=\frac{1}{2}\int r\,\frac{2R-r}{R-r}\cos\phi\,ds,$$

and since
$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2},$$

$$\frac{2R-r}{R-r}\cos\phi=\left(2+\frac{r}{R-r}\right)\cos\phi=2\cos\phi+r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}-\frac{\cos\phi}{r+\rho}\right)$$
$$=r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{2\rho+r}{\rho+r}\cos\phi;$$

$\therefore$ r sweeps out an area
$$\frac{1}{2}\int\left[r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+r\frac{2\rho+r}{\rho+r}\cos\phi\right]ds.$$

667. When the carried curve *reduces to a point*, *i.e.* $\rho=0$, $\cos\phi=\dfrac{d\theta}{ds}$, where $d\theta$ is the angle between consecutive radii vectores of the rolling curve.

Hence, for a carried point,

$$\textbf{Arc of roulette}=\int r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds,$$

and
$$\textbf{Area swept out by } r=\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds+\frac{1}{2}\int r^2 d\theta.$$

Hence the area swept out by r exceeds the corresponding portion of the sectorial area of the rolling curve,

$$\text{viz. } \frac{1}{2}\int r^2 d\theta, \quad \text{by} \quad \frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds.$$

And if the rolling curve be a straight line, $\rho_2=\infty$, and these expressions reduce further to

$$\text{Arc}=\int\frac{r}{\rho_1}\,ds \quad \text{and} \quad \text{Area swept}=\frac{1}{2}\int\frac{r^2}{\rho_1}\,ds+\frac{1}{2}\int r^2\,d\theta$$

respectively.

668. **Important Cases.**

The most important case, perhaps, is when a curve which carries a point or a straight line rolls upon a *fixed straight line.*

In this case $\rho_1=\infty$.

If also the roulette be that of a carried point, $\rho=0$,

$$R=r+\rho_2\cos\phi\,\frac{r}{r-\rho_2\cos\phi}=\frac{r^2}{r-\rho_2\cos\phi}.$$

If the roulette be that enveloped by a carried straight line, $\rho=\infty$, and

$$R=r+\rho_2\cos\phi.$$

In these cases ϕ is the angle which the normal to the roulette makes with a fixed line, and in accordance with the usual custom in dealing with intrinsic equations may be written ψ.

Hence the intrinsic equations of the roulette in the two cases will be respectively

$$s=\int\left(r+\rho_2\cos\psi\,\frac{r}{r-\rho_2\cos\psi}\right)d\psi, \text{ for a carried point,}$$

$$s=\int(r+\rho_2\cos\psi)\,d\psi, \qquad \text{for a carried straight line.}$$

669. It is to be further noted that if the concavity of any of the curves concerned be turned in the opposite direction to that in which they are represented in Fig. 190, the general formula for R will need modification by the corresponding change of sign of the particular radii of curvature involved with a corresponding modification in all the deduced results. To avoid error it is therefore desirable to examine each case on

its own merits, rather than to deduce the formulae required from the general result

$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

Moreover, special cases have their own special geometrical peculiarities. Hence, in succeeding articles, we adopt this course though it necessitates some repetition. This will also have the advantage of exhibiting a somewhat different treatment.

670. Ex. 1. A circular wheel rolls in a vertical plane along a straight line. To find the intrinsic equation of the envelope of a given diameter.

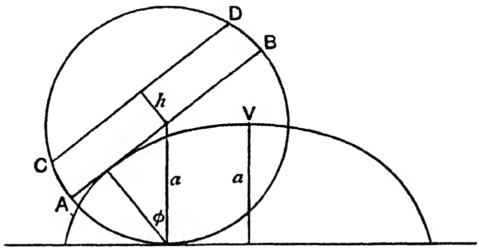


Fig. 191.

Here ρ_2 = the radius of the wheel $=a$, say;

$$r=a\cos\phi;$$

$$\therefore\ R=r+\rho_2\cos\phi=2a\cos\phi;$$

$$\therefore\ s=2a\sin\phi;$$

i.e. the envelope is a cycloid with an axis of length a, s being measured from the vertex of the cycloidal envelope.

For a parallel chord at a distance h from this diameter, we have

$$r=h+a\cos\phi$$

and $$s=h\phi+2a\sin\phi,$$

viz. a parallel to a cycloid. Moreover, the cycloid which is the envelope of the diameter of the rolling circle, is itself an involute of another cycloid. Hence the parallels to the cycloid are involutes of a cycloid. This then is the result for any carried line.

Ex. 2. Let the rolling curve be $r^n=a^n\cos n\theta$, and suppose the initial position be that in which the vertex of a foil of the curve is in contact with the line.

First, let us find the roulette of the pole.

We have $$p=\frac{r^{n+1}}{a^n},\qquad \rho_2=\frac{r\,dr}{dp}=\frac{1}{n+1}\frac{a^n}{r^{n-1}}.$$

Let P be the point of contact, O the pole, A its initial position, ψ the angle turned through by the tangent at O to the roulette, $x'Cx$ the fixed line.

Then $\tan OPx' = \frac{r\,d\theta}{dr} = -\cot n\theta$; $\therefore\ OPx' = \frac{\pi}{2} + n\theta$;

$$\therefore\ \phi = n\theta,\ \rho_2 \cos\phi = \frac{1}{n+1} r \quad\text{and}\quad \phi = \psi.$$

$$R = \frac{r^2}{r - \rho_2 \cos\phi} = \frac{n+1}{n} r;$$

$$\therefore\ \frac{ds}{d\psi} = \frac{n+1}{n} a \cos^{\frac{1}{n}} \psi,$$

and $s = \frac{n+1}{n} a \int \cos^{\frac{1}{n}} \psi\, d\psi$ is the intrinsic equation sought.

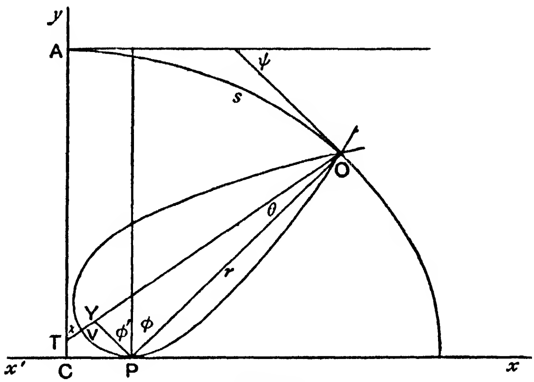


Fig. 192.

If $n=1$, we have the case of a rolling circle of diameter a, and the intrinsic equation of the cycloid traced is $s = 2a \sin\psi$.

In the general case, if we refer the curve to tangential polar coordinates, we can perform one integration. For taking A as pole.

$$\frac{d^2p}{d\psi^2} + p = \frac{ds}{d\psi} = \frac{n+1}{n} a \cos^{\frac{1}{n}} \psi.$$

Multiplying by $\sin\psi$,

$$\sin\psi \frac{d^2p}{d\psi^2} + p \sin\psi = \frac{n+1}{n} a \cos^{\frac{1}{n}} \psi \sin\psi,$$

and integrating, $\sin\psi \frac{dp}{d\psi} - p\cos\psi = -a \cos^{1+\frac{1}{n}} \psi + a,$

for p and $\frac{dp}{d\psi}$ vanish if $\psi = 0$ and p be measured from the vertex A;

$$\therefore\ \frac{dp}{d\psi} - p\cot\psi = a \operatorname{cosec}\psi \left(1 - \cos^{1+\frac{1}{n}} \psi\right).$$

Again, multiplying by $\cos\psi$ and integrating

$$\cos\psi\frac{dp}{d\psi}+p\sin\psi=\frac{n+1}{n}a\int_0^\psi \cos^{1+\frac{1}{n}}\psi\,d\psi,$$

or
$$\frac{dp}{d\psi}+p\tan\psi=\frac{n+1}{n}a\sec\psi\int_0^\psi \cos^{1+\frac{1}{n}}\psi\,d\psi.$$

Eliminating $\dfrac{dp}{d\psi}$, we obtain

$$p(\tan\psi+\cot\psi)=\frac{n+1}{n}a\sec\psi\int_0^\psi \cos^{1+\frac{1}{n}}\psi-a\operatorname{cosec}\psi(1-\cos^{1+\frac{1}{n}}\psi),$$

or
$$p=\frac{n+1}{n}a\sin\psi\int_0^\psi \cos^{\frac{n+1}{n}}\psi\,d\psi-a\cos\psi\left(1-\cos^{\frac{n+1}{n}}\psi\right),$$

as the tangential-polar equation of the roulette, the origin being at the vertex of the roulette.

To find the roulette enveloped by the axis of the rolling curve, we have $R=r'+\rho_2\cos\phi'$, where ϕ' is the angle between a parallel to CA and the perpendicular upon the axis of the curve, and r' is the perpendicular from P upon the axis.

Then
$$\phi'=\frac{\pi}{2}-\theta-\phi=\frac{\pi}{2}-(n+1)\theta=\frac{\pi}{2}-\chi,$$

where χ is the angle the axis of the rolling curve makes with the line CA, and
$$r'=r\sin\theta=a\cos^{\frac{1}{n}}n\theta\sin\theta;$$

$$\therefore\ R=r\sin\theta+\frac{1}{n+1}\frac{r}{\cos n\theta}\sin(n+1)\theta$$

$$=a\cos^{\frac{1}{n}}n\theta\left[\sin\theta+\frac{\sin\overline{n+1}\,\theta}{(n+1)\cos n\theta}\right],$$

$$\frac{ds}{d\chi}=a\cos^{\frac{1}{n}}\frac{n}{n+1}\chi\left[\sin\frac{\chi}{n+1}+\frac{\sin\chi}{(n+1)\cos\frac{n}{n+1}\chi}\right],$$

and the intrinsic equation of the envelope of the axis is therefore

$$s=a\int_0^\chi\left\{\sin\frac{\chi}{n+1}+\frac{\sin\chi}{(n+1)\cos\frac{n}{n+1}\chi}\right\}\cos^{\frac{1}{n}}\frac{n\chi}{n+1}\,d\chi.$$

Ex. Special Case of the Epi- and Hypo-cycloids.

Here $\rho_1=a$, the radius of the fixed circle; $\rho_2=b$, the radius of the rolling circle; $\rho=0$.

$$(\pi-2\phi)b=a\theta,\quad \psi=\theta+\frac{\pi}{2}-\phi;$$

$$\therefore\ \phi=\frac{\pi}{2}-\frac{a}{a+2b}\psi\quad\text{and}\quad r=2b\cos\phi;$$

and
$$\frac{\cos\phi}{R-r}=\frac{1}{a}+\frac{1}{b}-\frac{\cos\phi}{r}=\frac{1}{a}+\frac{1}{2b}=\frac{a+2b}{2ab};$$

$$\therefore \quad R = r\left(1+\frac{a}{a+2b}\right) = 2\frac{a+b}{a+2b}\cdot 2b\cos\phi,$$

$$\frac{ds}{d\psi} = -4b\frac{a+b}{a+2b}\sin\frac{a}{a+2b}\psi$$

(s measured from the vertex increases as ψ diminishes);

$$s = \frac{4b}{a}(a+b)\cos\frac{a}{a+2b}\psi,$$

s being measured from the vertex (Art. 412, *Diff. Calc.*).

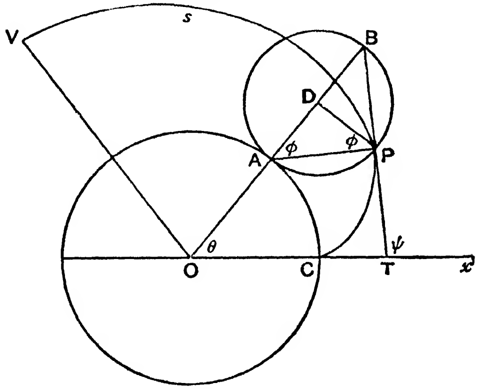


Fig. 193.

671. When $\rho_1 = \infty$, the formula for the roulette of a carried point,

$$\text{viz.} \quad R = \frac{r^2}{r - \rho_2\cos\phi},$$

is expressible otherwise.

For with the usual notation, taking the carried point as the pole of the rolling curve,

$$\rho_2 = \frac{r\,dr}{dp} \quad \text{and} \quad \cos\phi = \frac{p}{r}.$$

Hence
$$\frac{1}{R} = \frac{r - p\dfrac{dr}{dp}}{r^2} = \frac{d}{dp}\left(\frac{p}{r}\right),$$

which gives a convenient measure for R in this case.

672. **General Theorems with regard to Rolling on a Fixed Straight Line. Roulette of a Carried Point. Theorems of Jacob Steiner and W. H. Besant.**

Let APB be any curve rolling along a straight line xz, P being the point of contact, P' the adjacent point on the curve which will come into contact with the line at Q. Let O be a

carried point and O' the point at which it arrives when the rolling of the curve has carried P' to Q.

Let OY, OY' be the perpendiculars from O upon the contiguous tangents at P and P'. Let $OO'=d\sigma$, the elementary arc traced by O as the point of contact travels from P to Q. Let $O'O$ cut xz at R. Then OY plays a double part.

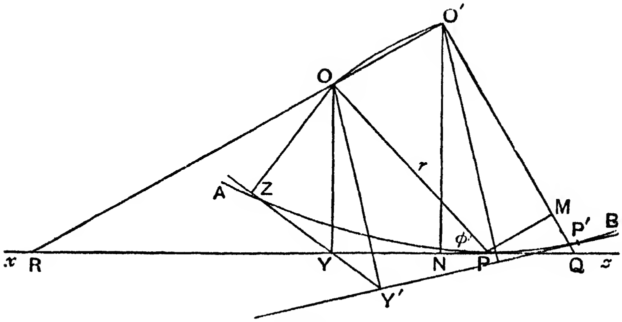


Fig. 194.

(1) It is the ordinate of the point O of the roulette of O.

(2) It is the radius vector of the pedal of the rolling curve with regard to O.

Let the elementary arc of the pedal curve, viz. YY', be called ds_p.

Then $$\frac{dy}{d\sigma}=Lt\sin z\hat{R}O=Lt\cos R\hat{P}O,$$

for OP is the normal to the roulette (Art. 562, *Diff. Calc.*)

$$=Lt\cos O\hat{Y}'Y=\frac{dOY}{ds_p}=\frac{dy}{ds_p}.$$

That is, in the limit, $\qquad d\sigma=ds_p.$ ....(1)

Hence corresponding arcs of the roulette of O and of the pedal of the rolling curve with regard to O are equal.

This theorem is due to JACOB STEINER (1796-1863).*

673. Again, if OZ be the perpendicular from O on YY', we have ultimately

$$y\frac{dx}{d\sigma}=Lt\, y\cos z\hat{R}O=Lt\, y\sin R\hat{P}O=Lt\, y\sin O\hat{Y}'Y$$
$$=Lt\, y\sin O\hat{Y}Z=OZ;$$
$$\therefore\ y\,dx=OZ\,d\sigma=OZ\,ds_p,$$

i.e. the element $OYNO'$ is ultimately double the element OYY'.

*Cajori's *History of Mathematics*, p. 295.

Hence integrating, the area swept out by the ordinate of the roulette during any portion of the rolling is double the corresponding sectorial area of the pedal curve.

This theorem appears to be due to the late W. H. BESANT (Art. 26, *Roulettes and Glisettes*).

674. We consider next the **area swept out by the normal** OP **to the roulette.**

Draw PM perpendicular to $O'Q$. Let ϕ be the angle OP makes with the tangent.

We have $PM=\delta s \sin\phi$, δs being the element PP' or PQ of the rolling curve. Let $OP=r$, $POP'=\delta\theta$ and $YOY'=\delta\psi$.

Then to the first order,

$$\begin{aligned}\text{Quadrilateral } OPQO' &= \tfrac{1}{2}OP\,.\,OO'+\tfrac{1}{2}O'Q\,.\,PM\\ &=\tfrac{1}{2}r(\delta\sigma+\delta s\sin\phi)\\ &=\tfrac{1}{2}r(YY'+r\,\delta\theta)\\ &=\tfrac{1}{2}r(r\sin\delta\psi+r\,\delta\theta)\end{aligned}$$

(for $OYY'P$ being ultimately cyclic, $YY'=\text{diam.}\times\sin YOY'$)

$$=\tfrac{1}{2}r^2\delta\psi+\tfrac{1}{2}r^2\delta\theta.$$

$\therefore$ area swept out by normal in any portion of the rolling

$$=\text{corresponding sectorial area of curve}+\frac{1}{2}\int r^2 d\psi,$$

the limits for ψ being its initial and final values.

675. If the curve be a closed oval, every point of whose perimeter comes into contact with the line in one revolution, and if we suppose the rolling to start with OP at right angles to the line, so that the limits for ψ may be specified as 0 to 2π, we have for a complete revolution

$$\begin{aligned}\text{Area swept by normal}&=\text{area of rolling curve}+\frac{1}{2}\int_0^{2\pi} r^2 d\psi\\ &=\text{area of curve}+\frac{1}{2}\int_0^{2\pi}\frac{r^2}{\rho}\,ds.\end{aligned}$$

But by Art. 426

$$2\text{ area of pedal}=\text{area of curve}+\frac{1}{2}\int_0^{2\pi}\frac{r^2}{\rho}\,ds;$$

$$\therefore \left.\begin{array}{l}\textbf{area swept out by normal}\\ \textbf{in a complete revolution}\end{array}\right\}=2\text{ area of pedal.}$$

This theorem is also due to **Steiner.***

* See Bertrand, *Calc. Intég.*, p 362 and Besant, *Roulettes and Glisettes*, p. 19.

676. It is worth noting also that

$$\textbf{Area of oval} = 2 \text{ area of pedal} - \frac{1}{2}\int_0^{2\pi} r^2\,d\psi$$
$$= \int_0^{2\pi} p^2\,d\psi - \frac{1}{2}\int_0^{2\pi} r^2\,d\psi$$
$$= \frac{1}{2}\int_0^{2\pi} (2p^2 - r^2)\,d\psi.$$

(Besant, *R. and G.*, p. 19.)

677. **Illustrative Examples.**

1. When an ellipse rolls upon a straight line, any arc of the roulette of the focus is equal to the corresponding portion of the circumference of the circle which is the first positive pedal of the ellipse with regard to the focus, *i.e.* the auxiliary circle.

The roulette of the centre is of the same length as the corresponding arc of its central pedal, viz. $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

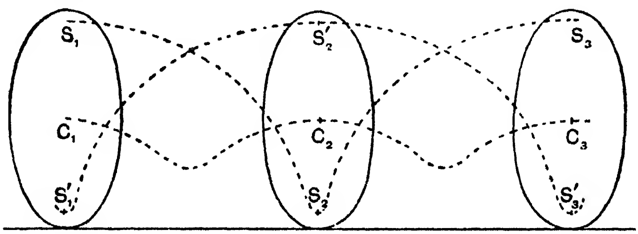


Fig. 195.

And in both cases the areas swept by the ordinate are double of the corresponding sectorial area of the pedal. In a complete revolution these areas are $2\pi a^2$ for the area swept by the ordinate of the focus in a complete revolution of the ellipse and $\pi(a^2+b^2)$ for the roulette of the centre. These paths are illustrated in the accompanying diagram.

2. The arc of the roulette of a point rigidly connected *with a circle rolling* on a straight line (*i.e. a Trochoid*) is equal to the corresponding portion of the limaçon which is the first positive pedal of the circle with regard to the point. And when the point is on the circumference of the rolling circle, we see that the arc of a cycloid is of the same length as the corresponding arc of a cardioide.

3. If *a rectangular hyperbola rolls along a straight line,* any arc of the roulette of the centre is equal to the corresponding arc of the lemniscate which is the pedal of the hyperbola with regard to the centre, and is therefore expressible as an elliptic integral (Art. 592).

4. When *a parabola rolls along a straight line,* the arc of the roulette of the vertex is equal to the arc of the cissoid which is the first positive pedal of the parabola with regard to the vertex.

Many other cases may be cited and many curves may be discovered as roulettes whose arcs can be found; this being so whenever the arc of the pedal of the rolling curve can be found.

In each of these cases we also find that the area swept out by the ordinate is double the corresponding sectorial area of the pedal.

678. **General Theorems with regard to Rolling on a Curve. Rectification of Roulette of a Carried Point P.**

We may prove the results for a carried point P as follows, directly and without deduction from the general formulae.

Let A be the point of contact,

B_2 an adjacent point on the fixed curve,

B_1 the point on the rolling curve which will come into contact with B_2,

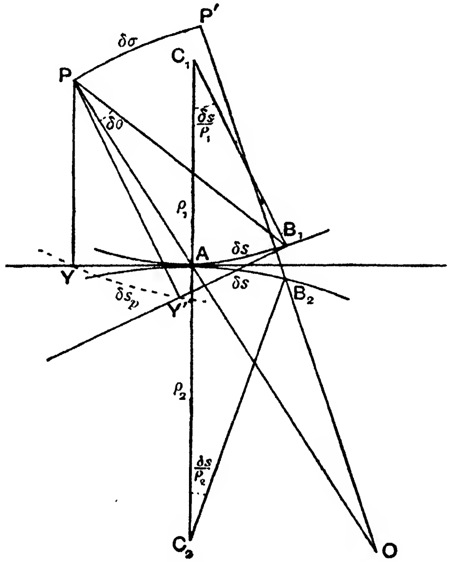


Fig. 196.

P, P' the two points on the roulette corresponding to the points of contact A and B_2, so that PA, $P'B_2$ are contiguous normals to the roulette. Let these meet in O.

Let C_1, C_2 be the centres of curvature of the rolling and fixed curves respectively at A, $P\hat{A}C_1 = \phi$,

ρ_1, ρ_2 the radii of curvature,

$r = AP$; PY, PY' perpendiculars on tangents at A and B_1,
δs, $\delta\sigma$, δs_p the elementary arcs of the fixed and rolling curves, the roulette, and the pedal of the rolling curve with regard to P; *i.e.*

$$AB_1 = AB_2 = \delta s, \quad PP' = \delta\sigma, \quad YY' = \delta s_p$$

Then when C_1B_1 comes into line with B_2C_2, PB_1 will come into line with B_2O. Let $APB_1 = \delta\theta$.

Then the angle turned through by the rolling curve is

$$A\hat{C}_1B_1 + A\hat{C}_2B_2 = \frac{\delta s}{\rho_1} + \frac{\delta s}{\rho_2}.$$

Also PB_1 turns through the same angle, and B_1B_2 is a second order small quantity. Hence, to the first order,

$$PP' = AP\left(\frac{\delta s}{\rho_1} + \frac{\delta s}{\rho_2}\right) = r\left(\frac{\delta s}{\rho_1} + \frac{\delta s}{\rho_2}\right).$$

Again, $$YY' = r\frac{\delta s}{\rho_1}, \text{ to the first order,}$$

since $YY'AP$ is ultimately a cyclic quadrilateral, as in Art. 674;

$$\therefore \ Lt\,\frac{PP'}{YY'} = \frac{\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}}{\dfrac{1}{\rho_1}} = 1 + \frac{\rho_1}{\rho_2},$$

i.e. $$\frac{d\sigma}{ds_p} = 1 + \frac{\rho_1}{\rho_2}$$

and $$\sigma = \int\left(1 + \frac{\rho_1}{\rho_2}\right) ds_p \quad \text{..........................(A)}$$

(the formula of Art. 667 for $\rho_1\, ds_p = r\, ds$).

679. Also, as in Art. 674,

$$\text{Area } PAB_2P' = \frac{1}{2} r(PP' + \delta s \sin\phi), \text{ to the first order,}$$

$$= \frac{1}{2} r\left\{\left(1 + \frac{\rho_1}{\rho_2}\right)\delta s_p + \delta s \sin\phi\right\}$$

$$= \frac{1}{2} r\left\{\left(1 + \frac{\rho_1}{\rho_2}\right) r\frac{\delta s}{\rho_1} + r\,\delta\theta\right\}$$

$$= \frac{1}{2} r^2\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\delta s + \frac{1}{2} r^2\,\delta\theta.$$

And integrating, **the area swept out by the normal to the roulette** between the roulette and the fixed curve

$$=\frac{1}{2}\int r^2 d\theta+\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds \text{ (the formula of Art. 667).} \quad \text{(B)}$$

680. When the *rolling curve is closed*, we have for the *whole* area swept by the normal in one turn of the curve, such that the original point of contact has again come into contact,

$$\textbf{Area swept} = \textbf{area of curve} + \frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds,$$

the limits of integration being from the initial to the final value of s.

681. It should be noted that in the investigations above, ρ_1 and ρ_2 are drawn in opposite directions. If the rolling curve be on the concave side of the fixed curve, the formulae will become

$$\textbf{Arc of roulette} = \sigma = \int\left(1-\frac{\rho_1}{\rho_2}\right)ds_p \quad \text{..............(A}')$$

$$\text{and } \left.\begin{array}{c}\textbf{Area swept}\\ \textbf{by normal}\end{array}\right\} = \frac{1}{2}\int r^2 d\theta+\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)ds. \quad \text{......(B}')$$

682. If $\rho_1=\rho_2$, as will always happen when a curve rolls upon an equal one, the rolling being started so that the points of contact are initially and always corresponding points, formula (A) shows that $\sigma=2s_p$,

i.e. the *length of any part of the roulette is double the corresponding part of the pedal.*

683. In the case of an ellipse rolling upon an equal ellipse and placed at starting with the ends of the major axes in contact, the paths of the foci are obviously circles of twice the radius of the auxiliary circle, which is the pedal of the ellipse, which is a verification of the general theorem.

In the case of the epi- and hypo-cycloids and the epi- and hypo-trochoids, ρ_1 and ρ_2 are the radii of the rolling and fixed circles and constant. Hence the arcs of such curves are proportional to the corresponding arcs of the first positive pedal of the rolling circle, *i.e.* to the arc of a cardioide or of a limaçon, and are therefore rectifiable in the same manner.

684. **Rolling along both sides of a Curve.**

If the rolling curve be allowed to roll first on the convex side of a fixed curve and then upon the concave side, starting with the same pair of points common and rolling in the same

manner as before, so that corresponding points again come into contact, formulae (A) and (B), (A′) and (B′) show that if σ, σ' be the arcs of the roulette, and A, A' the areas described by the normal in the two cases, and A_p the corresponding area of the pedal of the rolling curve, then

$$\sigma+\sigma'=\int\left(1+\frac{\rho_1}{\rho_2}\right)ds_p+\int\left(1-\frac{\rho_1}{\rho_2}\right)ds_p=2s_p$$

and
$$\begin{aligned}A+A'&=\frac{1}{2}\int r^2 d\theta+\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds\\&\quad+\frac{1}{2}\int r^2 d\theta+\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)ds\\&=\int r^2 d\theta+\int\frac{r^2}{\rho_1}ds=4A_p.\end{aligned}$$

And both results being independent of ρ_2, are independent of the nature of the fixed curve, and therefore in each case double of the results for rolling along a straight line.

685. *In the case of a curve carried by a second curve which itself slides in contact with two other curves, or moves in its own plane in any given manner, the same formulae as those established for a roulette can be used for the curvature and rectification of the envelope of the attached curve.*

For the motion being a case of rolling of the locus of the instantaneous centre I, traced on the moving lamina, upon the locus of the instantaneous centre I traced on a fixed plane, it is a matter in general of first determining these loci and their radii of curvature; or, what is equivalent, if δs be the arc of the fixed I-locus and ϕ the angle which the normal to the I-locus makes with the normal through I to the carried curve, and if $\delta\chi$ be the angle turned through by the moving curve whilst I travels over δs on its locus,

$$d\chi=\frac{ds}{\rho_1}+\frac{ds}{\rho_2},$$

and the formula
$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$$

may be written
$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{d\chi}{ds}$$

the various letters having the same meanings as before, ρ_1, ρ_2 referring to the two I-loci, the values being obtainable as explained in Art. 660.

When $\dfrac{d\chi}{ds}$, which is $\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}$, $\cos\phi$, r and ρ have been expressed in terms of ψ, the angle which the normal to the carried curve makes with a given line, the radius of curvature of the envelope is

$$\frac{d\sigma}{d\psi}=R=r+\frac{\cos\phi}{\dfrac{d\chi}{ds}-\dfrac{\cos\phi}{r+\rho}},$$

and $\sigma=\int R\,d\psi$ gives the intrinsic equation of the envelope of the carried curve.

Also, as before, the case of a carried point is included as that for which $\rho=0$, and the case of a carried straight line is included as that for which $\rho=\infty$, which respectively give

$$\sigma=\int\left(r+\frac{\cos\phi}{\dfrac{d\chi}{ds}-\dfrac{\cos\phi}{r}}\right)d\psi \quad\text{and}\quad \sigma=\int\left(r+\cos\phi\frac{ds}{d\chi}\right)d\psi$$

as the intrinsic equations required.

686. **When a Curve slides in such a manner as always to touch a Given Straight Line at a Given Point**, the glisette of any carried point is obtainable at once.

Let the carried point be taken as a pole, and let $p=f(\psi)$ be the tangential polar equation of the curve with regard to this pole.

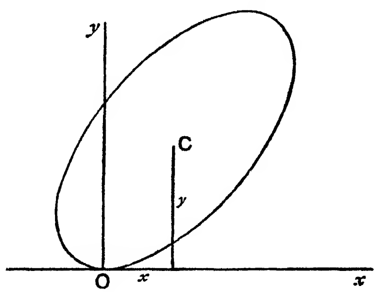


Fig. 197.

Then if the point of contact be taken as the origin and the given straight line as the x-axis, we have

$$\left.\begin{aligned} x&=\frac{dp}{d\psi}=f'(\psi),\\ y&=p=f(\psi),\end{aligned}\right\}$$

and the ψ-eliminant is the "glisette" required.

687. Illustrative Examples.

Ex. 1. If the curve be an equiangular spiral, we obviously have

$$p = r\sin\alpha \quad \text{and} \quad \frac{dp}{d\psi} = r\cos\alpha;$$

$\therefore\ y = x\tan\alpha$ is the path of the pole, as is geometrically obvious.

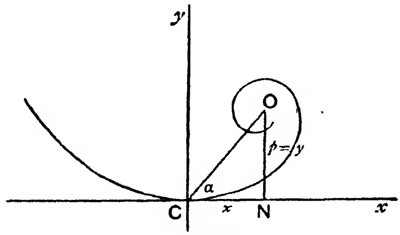


Fig. 198.

Ex. 2. If the curve be an ellipse,

$$\left.\begin{aligned} p^2 &= a^2\cos^2\psi + b^2\sin^2\psi, \\ p\frac{dp}{d\psi} &= -(a^2-b^2)\sin\psi\cos\psi, \end{aligned}\right\}$$

and the ψ-eliminant gives for the glisette of the centre the quartic

$$x^2y^2 = (a^2-y^2)(y^2-b^2).$$

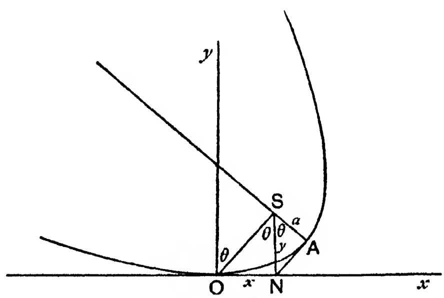


Fig. 199.

Ex. 3. In a parabola of latus rectum $4a$, we have for the glisette of the focus

$$\left.\begin{aligned} \frac{x}{y} &= \tan\theta, \\ \frac{y}{a} &= \sec\theta, \end{aligned}\right\}$$ 2θ being the angle subtended at the focus by the arc from the vertex to the point of contact (Fig. 199);

$$\therefore\ \frac{y^2}{a^2} - \frac{x^2}{y^2} = 1, \quad \text{or} \quad y^2(y^2-a^2) = a^2x^2,$$

i.e.
$$\frac{2a}{r} = (1+\cos 2\theta),$$

θ being the angle OS makes with the y-axis.

688. (ι, y) **Relations.**

In many curves the relation between the ordinate y and the angle ι between the ordinate and the tangent takes a very simple form, and is, moreover, very useful (1) in the determination of the envelope of a straight line carried by a curve which always touches a given straight line *at a given point* and also (2) in the problem of Brachistochronism for a law of force which is always in the same direction.

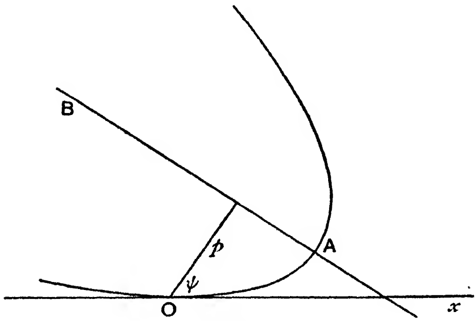


Fig. 200.

(1) Let O be the fixed point at which the curve always touches the fixed line Ox.

Let AB be the carried line.

Then if the equation of the curve has been expressed as $y=f(\iota)$, with AB as the x-axis, the **tangential polar equation of the envelope** of AB is clearly $p=f(\psi)$, for

$$y=p \quad \text{and} \quad \iota=\psi.$$

(2) The laws of force for the BRACHISTOCHRONOUS description of a curve,

(*a*) under a central force making $\int\frac{ds}{v}$ a minimum and $\frac{v}{p}=k$, a constant, v being the velocity;

(*b*) under a force parallel to a given straight line which we may take as the y-axis making $\int\frac{ds}{v}$ a minimum and $\frac{v}{\cos\psi}=u$, a constant,

are respectively

$$P=\frac{k^2}{2}\frac{dp^2}{dr} \quad \text{and} \quad P=\frac{u^2}{2}\frac{d}{dy}(\sin^2\iota).$$

These will be found in books treating of kinetics of a particle. They are placed here for the convenience of the student, and to illustrate further the use of the (ι, y) equation of a curve which is necessary for the glisette of a carried line with motion described above. The central force formula we are not now concerned with, but it will serve for practice in the use of (p, r) equations.

689. **To find the (ι, y) Equation.**

Let the tangent at P meet the x-axis at T.

The relation between ι and y is easy to get, for

$$\sin^2\iota = \cos^2 PTx = \frac{1}{1+\left(\frac{dy}{dx}\right)^2},$$

and if x be eliminated between this and the equation of the curve the relation between ι and y will result.

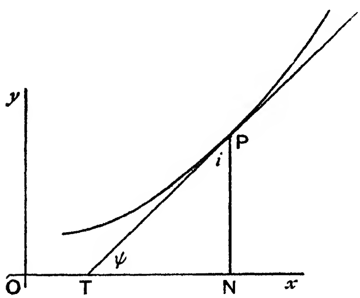


Fig. 201.

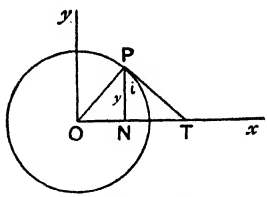


Fig. 202.

LIST OF COMMON (ι, y) EQUATIONS.

Circle, - - - - - -	$\sin^2\iota = \frac{y^2}{a^2}$.
Catenary, - - - - -	$\sin^2\iota = \frac{c^2}{y^2}$.
Tractrix, - - - - -	$\sin^2\iota = 1 - \frac{y^2}{c^2}$.
Cycloid, - - - - - -	$\sin^2\iota = 1 - \frac{y}{2a}$.
Evolute of a parabola,* - -	$\sin^2\iota = 1 - \frac{a}{y}$.

* Directrix for x-axis, Lat. Rect. $=4a/3$.

Evolute of a catenary, - - - $\sin^2\iota = 1 - \frac{4c^2}{y^2}$.

Four-cusped hypo-cycloid, - - $\sin^2\iota = 1 - \left(\frac{y}{a}\right)^{\frac{2}{3}}$.

Curves of the class $\frac{dy}{dx} = \frac{\sqrt{a^n - y^n}}{y^{\frac{n}{2}}}$, $\sin^2\iota = \frac{y^n}{a^n}$.

Curves of the class $\frac{dy}{dx} = \frac{y^{\frac{n}{2}}}{\sqrt{a^n - y^n}}$, $\sin^2\iota = 1 - \frac{y^n}{a^n}$.

Parabola, - - - - - - $\sin^2\iota = \frac{y^2}{4a^2 + y^2}$.

Rect. hyperbola, - - - - $\sin^2\iota = \frac{a^4}{a^4 + y^4}$.

Biaxal conic, - - - - - - $\sin^2\iota = \frac{a^2y^2}{b^4 + (a^2 - b^2)y^2}$.

The student should establish each of these results. It will be noted that in all cases they are expressed as $\sin^2\iota = f(y)$. This is obviously the form convenient in discussing Brachistochronism.

690. Ex. 1. If, for instance, a catenary slides in contact with a straight line Ox at a fixed point O, we have for the envelope of the directrix the tangential polar-equation $p = \frac{c}{\sin\psi}$, for $y = \frac{c}{\sin\iota}$ is the (ι, y) equation.

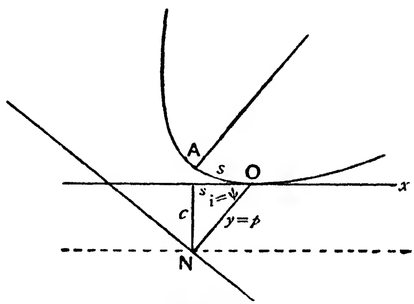


Fig. 203.

It is obvious from this equation that the directrix touches a parabola with O for focus and $4c$ for latus rectum. This is clear geometrically also, for the locus of the foot of the perpendicular upon the directrix is obviously a line at a distance c from the fixed line, and the envelope of the directrix is the first negative pedal of a fixed line, *i.e.* a parabola.

Since $\sin^2\iota=\frac{c^2}{y^2}$, the equation $P=\frac{u^2}{2}\frac{d}{dy}(\sin^2\iota)$ gives

$$P=-\frac{u^2}{2}\frac{2c^2}{y^3}=-\frac{c^2u^2}{y^3}.$$

Hence, the catenary is Brachistochronous for a law of force which acts perpendicularly towards the directrix and varying inversely as the cube of the distance from the directrix. The line of zero velocity in this case is at infinity.

Ex. 2. An ellipse slides, touching a straight line at a given point. What is the envelope of the axis major?

Here $$\cot^2\iota=\frac{b^4x^2}{a^4y^2}=\frac{b^2(b^2-y^2)}{a^2y^2};$$

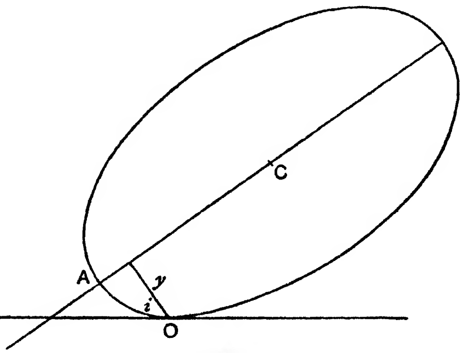


Fig. 204.

$\therefore$ the tangential polar equation of the envelope of the carried axis is

$$p^2(a^2\cos^2\psi+b^2\sin^2\psi)=b^4\sin^2\psi,$$

by writing p for y, ψ for ι, and reducing.

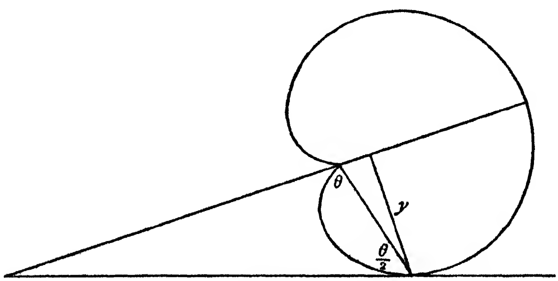


Fig. 205.

Ex. 3. A cardioide slides in contact with a fixed straight line at a fixed point. What is the envelope of the axis?

Here $$y = r\sin\theta = a(1-\cos\theta)\sin\theta$$
$$= 4a\sin^3\frac{\theta}{2}\cos\frac{\theta}{2},$$

and $$\iota = \frac{\theta}{2} + \left\{\frac{\pi}{2} - (\pi - \theta)\right\}$$
$$= \frac{3\theta}{2} - \frac{\pi}{2}.$$

Putting p for y and ψ for ι, the tangential polar equation of the envelope of the axis is

$$p = 4a\sin^3\frac{1}{3}\left(\frac{\pi}{2}+\psi\right)\cos\frac{1}{3}\left(\frac{\pi}{2}+\psi\right)$$
$$= a\sin\frac{\pi+2\psi}{3} - \frac{a}{2}\sin\frac{2\pi+4\psi}{3}.$$

691. **Two Curves in the Lamina touching Fixed Straight Lines.**

Let two curves be drawn upon a lamina, and let the lamina move so that the curves touch two given straight lines Ox, Oy

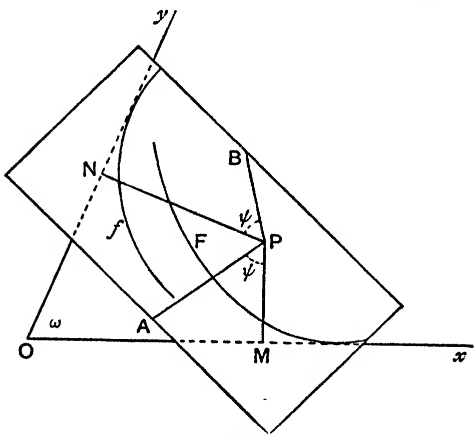


Fig. 206.

inclined at an angle ω, and let P be a point carried by the lamina. Let PM, PN be the perpendiculars upon Ox, Oy, and ψ the angle they respectively make with two initial lines PA, PB drawn upon the lamina, including an angle $\pi - \omega$, and initially at right angles to Ox and Oy respectively.

Then the path of P can be obtained at once.

Let $$p = f(\psi), \quad p = F(\psi)$$
be the tangential polar equations of the curves, with P for origin of measurement of p, and PA, PB respectively as initial lines.

Let x, y be the coordinates of P with regard to the lines Ox, Oy as coördinate axes.

Then $\quad x\sin\omega=f(\psi),\quad$ and $\quad y\sin\omega=F(\psi),$

and the ψ-eliminant furnishes the path of P.

It is clear that instead of the two curves on the lamina we might have one single curve drawn, *i.e.* $f(\psi)$ and $F(\psi)$ might be identical, except as regards the initial line from which ψ is measured in the two cases.

The rectification of the path of P follows from

$$dx=\operatorname{cosec}\omega\, f'(\psi)\,d\psi,\quad dy=\operatorname{cosec}\omega\, F'(\psi)\,d\psi,$$

and
$$ds^2=dx^2+2\,dx\,dy\cos\omega+dy^2,$$

whence
$$s=\operatorname{cosec}\omega\int\sqrt{f'^2+2f'F'\cos\omega+F'^2}\,d\psi,$$

where f' stands for $\dfrac{df(\psi)}{d\psi}$ and F' for $\dfrac{d}{d\psi}F(\psi)$.

692. Two Straight Lines in the Lamina touching Fixed Curves.

When three straight lines forming a triangle ABC are traced upon a lamina, and the lamina is made to move in such a manner that two of the sides AB, AC, say, touch given fixed curves, the third side BC will in its motion envelop a third curve, and there is a linear relation between the three arcs described by the points of contact. It has been shown (*Diff. Calc.*, Arts. 568-9) that the tangential-polar equation of the envelope of the carried side BC can be found at once.

If α, β, γ be the trilinear coordinates of any point O, fixed in space, with regard to the triangle ABC taken as a triangle of reference, we have the relation

$$a\alpha+b\beta+c\gamma=2\Delta,\qquad\text{(1)}$$

where a, b, c are the lengths of the sides and Δ the area of the triangle, with the usual trilinear convention that α, β, γ are positive when drawn from a point within the triangle.

Hence it follows that

$$a\frac{d\alpha}{d\psi}+b\frac{d\beta}{d\psi}+c\frac{d\gamma}{d\psi}=0,\quad a\frac{d^2\alpha}{d\psi^2}+b\frac{d^2\beta}{d\psi^2}+c\frac{d^2\gamma}{d\psi^2}=0,$$

where ψ is the angle any line fixed in the lamina makes with a line fixed in space;

$$\therefore\ a\left(\alpha+\frac{d^2\alpha}{d\psi^2}\right)+b\left(\beta+\frac{d^2\beta}{d\psi^2}\right)+c\left(\gamma+\frac{d^2\gamma}{d\psi^2}\right)=2\Delta.^*$$

And the increment of the angle of contingence being the same for all, we have $\pm a\rho_1 \pm b\rho_2 \pm c\rho_3 = 2\Delta$.

693. **Caution.**

Regarding O as the origin of measurement for perpendiculars for the tangential polar equations of the envelopes of BC, CA, AB, it is to be noted carefully that we are in the presence of two separate conventions with regard to the signs of the perpendiculars, which may be antagonistic.

(1) The trilinear convention is that stated above, that the perpendiculars are reckoned as positive when the point from which they are drawn lies *within* the triangle of reference.

(2) In the general treatment of curves, *i.e.* in establishing the formula $\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2}$, and others involving p, the perpendicular *from the origin* has always been reckoned positive.

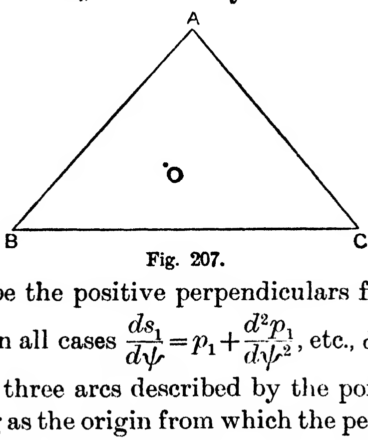


Fig. 207.

If p_1, p_2, p_3 be the positive perpendiculars from O upon the sides, we have in all cases $\frac{ds_1}{d\psi}=p_1+\frac{d^2p_1}{d\psi^2}$, etc., $\delta s_1, \delta s_2, \delta s_3$ being elements of the three arcs described by the points of contact.

Hence, so long as the origin from which the perpendiculars are measured lies within the triangle, we have $\alpha=p_1$, $\beta=p_2$, $\gamma=p_3$, and

$$\frac{ds_1}{d\psi}=\alpha+\frac{d^2\alpha}{d\psi^2},\quad \frac{ds_2}{d\psi}=\beta+\frac{d^2\beta}{d\psi^2},\quad \frac{ds_3}{d\psi}=\gamma+\frac{d^2\gamma}{d\psi^2}.$$

If, however, the origin lie between BC and AB produced and AC produced, $\alpha=-p_1$, $\beta=p_2$, $\gamma=p_3$, and

$$-\frac{ds_1}{d\psi}=\alpha+\frac{d^2\alpha}{d\psi^2},\quad \frac{ds_2}{d\psi}=\beta+\frac{d^2\beta}{d\psi^2},\quad \frac{ds_3}{d\psi}=\gamma+\frac{d^2\gamma}{d\psi^2},$$

* This method is stated by Mr. Besant to have been suggested by the late Master of Caius College, Dr. N. M. Ferrers.

with similar changes for other positions of the origin relative to the triangle of reference.

In addition to this, when we estimate the radius of curvature, it will be remembered that $\frac{ds}{d\psi}$, which is always

$$+\left(p+\frac{d^2p}{d\psi^2}\right),$$

is $\quad +\rho$ if s and ψ are increasing together,

but $\quad =-\rho$ if s and ψ are such that when one increases, the other decreases.

This point has been discussed in Art. 531.

Hence we have written

$$\pm a\rho_1 \pm a\rho_2 \pm a\rho_3 = 2\Delta,$$

the signs to be determined in each particular case. But in any case this equation is sufficient to prove that if two of the three quantities ρ_1, ρ_2, ρ_3 be constant, the third is also constant, *i.e.* if two sides of the triangle envelop circles or pass through fixed points, the third side also envelopes a circle, which is the theorem of Ex. 1, Art. 569, *Differential Calculus.*

694. The ambiguity as regards sign necessitates careful attention to the position of the origin relatively to the triangle.

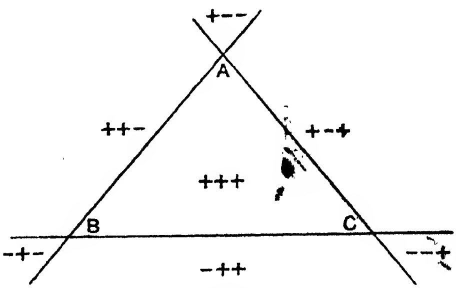


Fig. 208.

Three straight lines on a plane divide the plane into seven regions, and the signs of α, β, γ in these regions are indicated in the figure.

Accordingly we have, if we assume s_1, s_2, s_3 to be measured from points for which $\psi=0$, and to be each increasing when ψ increases,

$$as_1+bs_2+cs_3=2\Delta\psi$$

if, and so long as, the origin lies within the triangle;

$$-as_1+bs_2+cs_3=2\Delta\psi$$

if, and so long as, the origin lies in the region where the signs of α, β, γ are respectively $-++$, and so on for the other five regions.

Also, as the lamina moves the origin may pass from one region to another. Hence care must be taken in integrating between specified limits for ψ to observe the sweep of any of the three lines BC, CA, AB through the origin, and to take proper account thereof by using the appropriate case or cases of

$$\pm as_1\pm bs_2\pm cs_3=2\Delta\psi$$

for the intervening sweeps of the several sides.

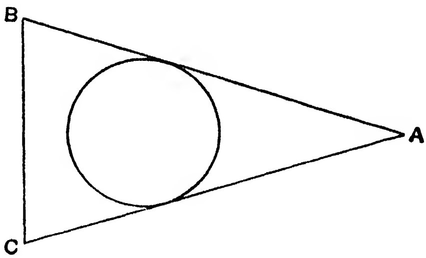


Fig. 209.

Thus, in integrating round an oval which the arms AB, AC touch, the oval lying within the triangle (Fig. 209), we have, taking the origin within the oval,

$$as_1+bs_2+cs_3=2\Delta\psi,$$

and for a complete revolution

$$aS_1+(b+c)S=4\pi\Delta,$$

where S is the perimeter of the oval

and S_1 that of the curve enveloped by BC.

695. In the same way, if the oval be always external to the triangle as in Fig. 210,

$$-aS_1'+(b+c)S=4\pi\Delta,$$

and similarly in other cases.

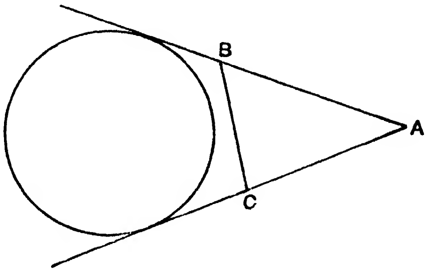


Fig. 210.

696. **A Limiting Case.**

If the triangle ABC becomes evanescent, we have the case of a line through A, viz. $B'C'$ (Fig. 211), carried by the pair of tangents AB, AQ, and making constant angles B, C with

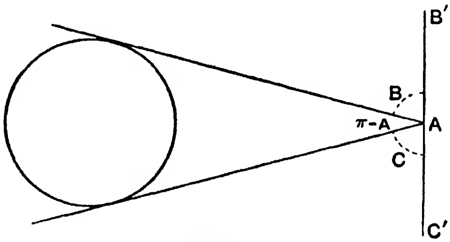


Fig. 211.

them respectively, the tangents making a constant angle A with each other. The sides a, b, c vanish in the ratio $\sin A:\sin B:\sin C$, and the theorem becomes

$$S_1'\sin A=(\sin B+\sin C)S,$$

i.e.
$$S'=\frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}}S.$$

697. If the carried line $B'C'$ lie within the angle PAQ, as shown in Fig. 212, it is the limit of the case in Fig. 213,

where the signs of the perpendiculars α, β, γ are respectively $-+-$, and $\alpha=-p_1$, $\beta=p_2$, $\gamma=-p_3$,

$$-as_1+bs_2-cs_3=2\Delta\psi \text{ (and ultimately } \Delta=0),$$

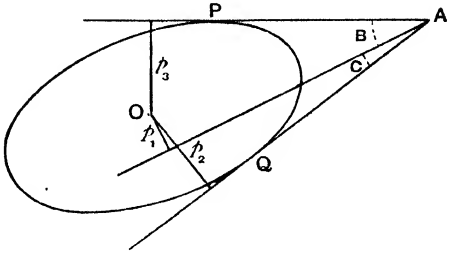


Fig. 212.

so long as BC does not sweep through the origin; and if it never does do so during the whole motion of the lamina during a complete revolution,

$$-aS''+(b-c)S=4\pi\Delta \quad \text{(and ultimately } \Delta=0),$$

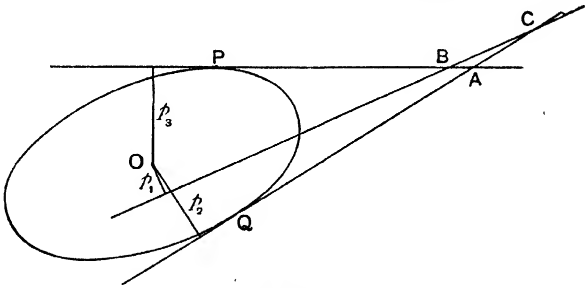


Fig. 213.

giving S'' the perimeter of the curve enveloped, or in the limit, when the triangle is evanescent,

$$S''=\frac{\sin B-\sin C}{\sin A}S=\frac{\sin\dfrac{B-C}{2}}{\cos\dfrac{A}{2}}S.$$

698. If, however, the line BC does sweep through the origin in the course of the revolution, the integration must be performed separately for the several complete portions for which the line BC moves without a sweep through the origin, and

the arcs of the envelope of BC being found thus, the positive results must be finally added together, using the formula

$$S'' = \frac{S_2 \sin B \sim S_3 \sin C}{\sin A}$$

for each portion.

699. Taking the case of any oval with two perpendicular axes of symmetry AOA', BOB', *e.g.* an ellipse, TP, TQ a pair of tangents at right angles, and the carried line being the bisector of the angle PTQ (Fig. 214), this line will obviously

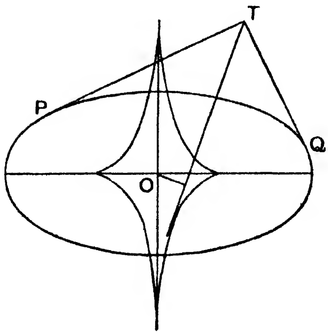


Fig. 214.

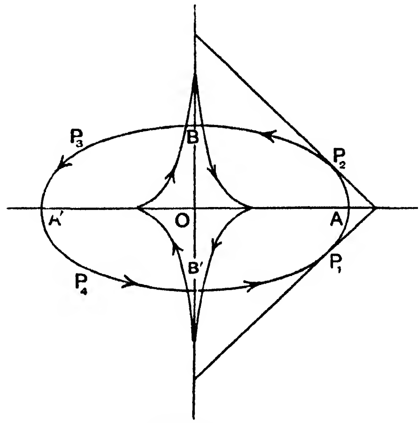


Fig. 215.

sweep through the centre every time the point T crosses one of the axes of symmetry, and whilst T travels along its locus over the first quadrant, the perimeter of the corresponding portion of the envelope of the carried line is

$$S'' = \frac{\sin\frac{\pi}{4}\,\text{arc}\,P_1P_2 \sim \sin\frac{\pi}{4}\,\text{arc}\,P_2P_3}{\sin\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}(\text{arc}\,P_1P_2 \sim \text{arc}\,P_2P_3)$$

$$= \sqrt{2}(\text{arc}\,BP_2 \sim \text{arc}\,AP_2),$$

where P_1, P_2, P_3, P_4 are the points of contact of tangents which make an angle of $\frac{\pi}{4}$ with the x-axis (Fig. 215).

It is to be noted that the arc in question is described in the opposite order to that of description of the ellipse by the several points of contact.

The whole perimeter is then $4\sqrt{2}(\text{arc } BP_2 - \text{arc } AP_2)$, and the curve is rectifiable in terms of arcs of an ellipse if the oval be elliptic, or in terms of arcs of whatever curve the doubly symmetric oval happens to be.

700. When the point A is at ∞, we have the case of parallel tangents to the oval, and the carried line AD is a line parallel to the tangents and dividing the chord of contact in the ratio

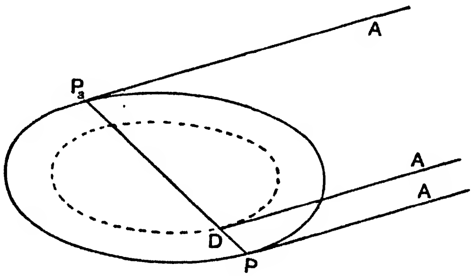


Fig. 216.

$\sin B : \sin C$ (see Fig. 213), where B and C are indefinitely small, *i.e.* in any definite ratio which we may assign, say $p:q$, and we then have

$$S'' = \frac{pS_2 - qS_3}{p+q}$$

for the perimeter of the envelope of AD replacing the result

$$S'' = \frac{S_2 \sin B - S_3 \sin C}{\sin(B+C)} \text{ of Art. 698.}$$

701. **A Case of Isoperimetric Companionship of Curves.**

Let us consider the form of a curve $O'PQ$ with pole N, which will be such that, when it rolls upon the fixed curve OP whose equation is known, $y = f(x)$, the pole N will travel along a straight line, say the x-axis.

Let O and O' be the points originally in contact, Ax, Ay the axes, P the point of contact, PN, OM ordinates, PT the tangent at P making an angle ψ with Ax, $O'N$ the radius vector of the rolling curve from which θ is measured and r the radius vector NP, ϕ the angle between the tangent to the rolling curve and its radius vector.

Then $$r = y,$$

$$\frac{r\,d\theta}{dr} = \tan\phi = \cot\psi = \frac{dx}{dy}.$$

Hence and $$\left.\begin{aligned} r\,d\theta &= dx \\ dr &= dy. \end{aligned}\right\} \qquad \text{(1)}$$

We therefore have $$\left.\begin{aligned} \theta &= \int\frac{dx}{r} = \int\frac{dx}{f(x)}, \\ r &= y \quad = f(x), \end{aligned}\right\} \qquad \text{(2)}$$

and if x be eliminated from equations (2), the polar equation of the rolling curve will result.

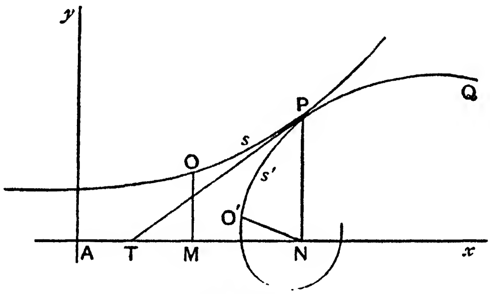


Fig. 217.

Again, if the form of the rolling curve had been given, say

$$r = F(\theta),$$

then $$\left.\begin{aligned} x &= \int F(\theta)\,d\theta, \\ y &= F(\theta), \end{aligned}\right\} \qquad \text{(3)}$$

and if θ be eliminated between these equations, the Cartesian equation of the fixed curve will result.

It follows that, since there is pure rolling without slipping, the corresponding arcs of the two curves must be equal.

This follows at once also from equation (1), for if s and s' be the respective arcs OP, $O'P$,

$$ds^2 = dx^2 + dy^2 = r^2\,d\theta^2 + dr^2 = ds'^2;$$

whence $ds = ds'$ and $s = s'$ if measured from such points as have originally been in contact.

It also follows that

$$\int y\,dx=\int r\,.\,r\,d\theta=\int r^2 d\theta,$$

i.e. the area swept over by the ordinate PN, that is $MNPO$, is double of the area swept out relatively to the rolling curve by its radius vector, that is the sectorial area $O'NP$.

The polar subtangent of the rolling curve is the Cartesian subtangent of the fixed curve, and the subnormals are the same.

Hence, given $$y=f(x),$$
we can, by the transformation $y=r$, $dx=r\,d\theta$, obtain another curve $r=F(\theta)$ for which

(1) corresponding arcs are equal;

(2) the area travelled over by the ordinate of the one is double the sectorial area swept out by the radius vector of the other;

and (3) if the second be allowed to roll upon the first, having been properly adjusted at the start, the locus of the pole of the rolling curve is the x-axis of the other.

702. **Generalisation.**

More generally, if we take any polar curve

$$r=F(\theta),$$

and construct from it a Cartesian locus, such that for each

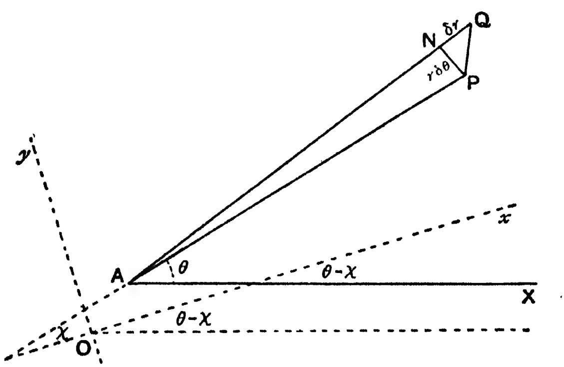


Fig. 218.

point (r, θ) on the one there is a point (x, y) on the other for which

$$dx = dr\cos\chi - r\,d\theta\sin\chi,$$
$$dy = dr\sin\chi + r\,d\theta\cos\chi,$$

where χ is any angle whatever at our choice, we have, upon elimination of r and θ, a new curve in which

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 = ds'^2,$$

where ds, ds' are corresponding elements of arcs in the two curves. It follows that

$$ds = ds' \quad \text{and} \quad s = s',$$

if the origins of measurement of arc are so chosen that s and s' vanish together.

703. The geometrical meaning of this is plain. We are projecting dr, $r\,d\theta$ upon a pair of perpendicular axes Ox, Oy with an arbitrary origin, and such that the x-axis makes an angle χ behind the radius vector of the polar curve, and therefore makes an angle $\theta - \chi$ with the initial line of the polar curve, or what is the same thing, with a fixed line through O parallel to the initial line of the polar curve; and by reserving choice of χ, we can make the new axes either fixed axes or moving in any given manner.

If we make $\chi = 0$, *i.e.* if we make the x-axis turn at the same angular rate as the radius vector of the polar curve, we have

$$dx = dr, \quad dy = r\,d\theta,$$

the transformation discussed in the last article, except that the axes of x and y are interchanged.

If we make $\chi = \theta$ or $\theta + \text{const.}$, we have fixed axes.

If we make $\theta - \chi = \dfrac{\theta}{n}$, we make our axes turn at $\dfrac{1}{n}$th the rate of the radius vector, and so on.

Moreover, either or both of the axes AX, Ox may be regarded as a fixed axis, the matter being purely a relative one.

These transformations establish a remarkable connection between many curves of common occurrence, and further will furnish us with a method of deriving new rectifications.

704. Reverting to the more elementary case of

$$\left.\begin{aligned} dx &= r\,d\theta, \\ y &= r, \end{aligned}\right\}$$

we shall find that,

A straight line $y=x\cot\alpha$ has for its analogue an equiangular spiral $r=ae^{\theta\cot\alpha}$.

A straight line $r=c\operatorname{cosec}\theta$ has a companion in a catenary.

A parabola - - - has as companion a spiral of Archimedes.

An ellipse - - - has as companion one of the Rhodoneae. (*Diff. Calc.*, Art. 385.)

A cardioide - - - has as companion a cycloid.

And when any curve is rectifiable, a companion is also rectifiable in the same manner, and even when neither curve is rectifiable in terms of arcs of a circle or an ellipse, arcs of the one can be expressed in terms of arcs of the other.

And in addition the property as to the relative magnitude of the area swept out by the radius vector of the one and the ordinate of the other holds good.

Such pairs of curves may perhaps be termed **Isoperimetric Companions.**

As illustrative examples, we consider these examples in detail.

705. 1. Taking the straight line $y=x\cot\alpha$ as the fixed "curve,"

$$dy=dr, \quad dx=r\,d\theta;$$
$$\therefore\ dr=r\,d\theta\cot\alpha,$$
$$\frac{dr}{r}=d\theta\cot\alpha,$$
$$r=ae^{\theta\cot\alpha}.$$

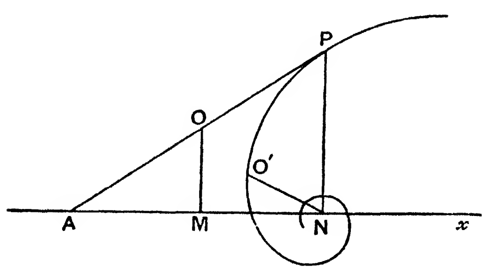


Fig. 219.

Hence an equiangular spiral $r=ae^{\theta\cot\alpha}$ and the straight line $y=x\cot\alpha$ correspond in the manner described, corresponding arcs being equal, and the Cartesian area bounded by the line, the x-axis and two ordinates

being equal to double the corresponding sectorial area of the spiral. (See *Diff. Calc.*, Art. 449.)

2. Take as the rolling polar curve the straight line $r = c \sec \theta$.

Then $$y = r = c \sec \theta, \quad dx = r\,d\theta = c \sec \theta\, d\theta;$$

$$\therefore \quad x = c \log \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = c \operatorname{gd}^{-1} \theta;$$

$$\therefore \quad \cos \theta \cosh \frac{x}{c} = 1 \quad \text{(Art. 69)},$$

or $$y = c \cosh \frac{x}{c}, \quad \text{the catenary,}$$

which is therefore the isoperimetric companion to the straight line, and rectifiable as has been seen (Art. 538). See also *Diff. Calc.*, Art. 444.

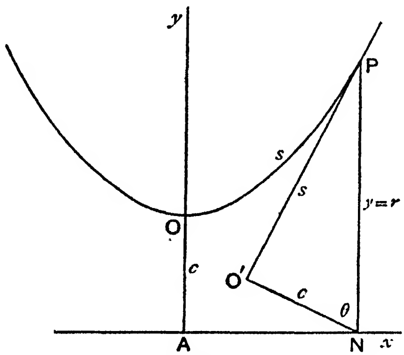


Fig. 220.

We note in addition to properties proved in *Diff. Calc.*, Art. 444, that

$$\text{Area } NO'P = \tfrac{1}{2} \text{ area } ANPO.$$

3. Take as the rolling polar curve the cardioide
$$r = a(1 - \cos \theta).$$

Then, for the Cartesian curve,
$$y = r = a(1 - \cos \theta),$$
$$x = \int r\,d\theta = a(\theta - \sin \theta),$$

i.e. a cycloid with cusp at the origin and vertex upward. These curves are therefore isoperimetric companions. When the cardioide is placed with its vertex in contact with the vertex of the cycloid on the concave side and allowed to roll inside the cycloid, the roulette of the pole is the line of cusps of the cycloid and the propositions of Art. 701 with regard to equality of corresponding arcs and the relative magnitudes of

the areas swept by the ordinates of the cycloid and the radius vector of the cardioide both hold good.

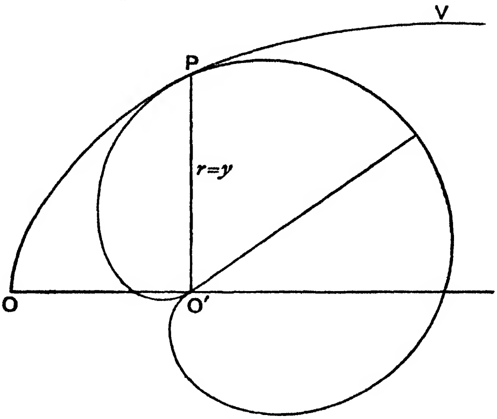


Fig. 221.

4. Take as fixed curve the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then $y=r$, $dx=r\,d\theta$ give

$$x=\frac{a}{b}\sqrt{b^2-r^2}, \quad \text{and} \quad r\,d\theta=-\frac{a}{b}\frac{r\,dr}{\sqrt{b^2-r^2}};$$

$$\therefore\ \frac{b\theta}{a}=\cos^{-1}\frac{r}{b},$$

i.e.

$$r=b\cos\frac{b}{a}\theta,$$

which is the isoperimetric companion of the ellipse. Hence the Rhodoneae $r=A\cos n\theta$ are rectifiable in terms of arcs of an ellipse.

PROBLEMS.

1. A circle of radius a rolls round the circumference of an equal circle. Prove that the area of the epitrochoid, described by a point carried with the rolling circle and distant c from its centre, is

$$(4a^2+2c^2)\pi. \qquad \text{[Oxf. I. P., 1918.]}$$

2. If a circle roll on the convex side of a parabola from one extremity of the latus rectum to the other, and can just pass between the vertex and the directrix, prove that four times the area traced out by that radius of this circle, which always passes through the

point of contact, exceeds the area of the circle by half the rectangle contained by the latus rectum and a line equal to the arc it cuts off. [R. P.]

3. An equiangular spiral rolls upon a straight line from a point P_1 to a point P_2 of the spiral. O, the pole of the spiral, traces out the path O_1O_2. From O_1O_2 are drawn perpendiculars O_1N_1, O_2N_2 on the straight line. Find the area of $O_1N_1N_2O_2$. [Colleges α, 1881.]

4. A closed oval curve rolls upon a fixed curve. Find an expression for the area of the roulette traced out by any carried point.

In a complete revolution of the closed oval curve, prove that the sum of the areas of the envelopes of two carried lines at right angles to one another which pass through a point fixed to the rolling curve is constant. Prove also that this sum exceeds the area of the roulette generated by the point, by the area of the rolling curve. [Colleges γ, 1887.]

5. If a closed oval curve roll with angular velocity ω on a straight line, while a point moves along its evolute with relative velocity $\omega\rho'$, prove that the area included in any portion of a revolution between the straight line, the curve generated by the moving point, and the perpendiculars to the former drawn through the extremities of the latter, is double the corresponding portion of the area between the curve and its evolute, bounded by the initial and final radii of curvature, provided the moving point is initially at the centre of curvature of the point of contact; ρ' being the radius of curvature of the evolute at the point corresponding to the point of the rolling curve in contact with the straight line. [Colleges δ, 1883.]

6. The cardioide $r=a(1-\cos\theta)$ rolls on a straight line; prove that the intrinsic equation of the roulette of the cusp is

$$2s=3a(2\psi-\sin 2\psi),$$

measuring from the point of contact of the cusp.

Prove also that its Cartesian equation is

$$\frac{4a-x}{2a}=\left\{2+\left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\}\sqrt{1-\left(\frac{y}{2a}\right)^{\frac{2}{3}}},$$

that its area is $\frac{15}{4}\pi a^2$, and that the radius of curvature of the roulette of the cusp is three times its distance from the point of contact. [Trinity, 1888.]

Find the evolute of the roulette of the pole and the intrinsic equation of the envelope of the axis.

7. A closed curve is moving in any manner in its own plane. Show that if ρ be the radius of curvature of the envelope of the tangent at any point of the curve, then

$$\int \rho\, ds$$

is equal to twice the area of the curve, the integral being taken all round the curve, ds being an element of arc of the moving curve.

[COLLEGES, 1879.]

8. A plane lamina moves in any given manner on a fixed plane: O is a fixed point on the fixed plane, P a point attached to the moving lamina and fixed upon it. If the area described by P about O be given, show that the locus of all points (P) in the moving plane for which the area is the same, is a circle, and that for different values of the area the corresponding circles are concentric.

[ST. JOHN'S, 1881.]

9. Examine the isoperimetric correspondence between the parabola $y^2 = 4ax$ and the Archimedean spiral $r = 2a\theta$, showing that the spiral can be made to roll upon the parabola in such manner that the pole of the spiral travels along the axis of the parabola.

10. Show that the reciprocal spiral $r\theta = a$ and the exponential curve $y = ae^{-\frac{x}{a}}$ are isoperimetric companions, both curves being rectifiable and corresponding arcs equal, and interpret the result by reference to the locus of the pole of the spiral when suitably started rolling.

11. Establish isoperimetric companionship between the curve

$$\left.\begin{aligned} \frac{x}{a} &= \log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) - \sin\phi, \\ \frac{y}{a} &= \frac{\sin^2\phi}{\cos\phi} \end{aligned}\right\}$$

and the cissoid $\qquad r = a\dfrac{\sin^2\theta}{\cos\theta}$.

12. Establish isoperimetric companionship between the semi-cubical parabola $ay^2 = x^3$ and the spiral $8au + \theta^3 = 0$.

13. Show that the curve

$$r = a\log\sec t,$$
$$\theta = \int \frac{dt}{\log\sec t}$$

is rectifiable and in isoperimetric companionship with the catenary of equal strength

$$y = a\log\sec\frac{x}{a}.$$

14. Show that the curves

$$\left.\begin{aligned} 4x &= a\left(\cos\phi - 9\cos\frac{\phi}{3}\right), \\ 4y &= a\left(3\sin\frac{\phi}{3} - \sin\phi\right) \end{aligned}\right\}$$

and $$r^{\frac{1}{3}} = a^{\frac{1}{3}}\sin\tfrac{1}{3}\theta$$

are rectifiable and isoperimetric companions.

15. Show that the curve

$$2\theta + 6\frac{\sqrt{a^{\frac{2}{3}} + r^{\frac{2}{3}}}}{r^{\frac{1}{3}}} = 3.\log\frac{\sqrt{a^{\frac{2}{3}} + r^{\frac{2}{3}}} + r^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + r^{\frac{2}{3}}} - r^{\frac{1}{3}}}$$

is rectifiable and in isoperimetric companionship with

$$x^{\frac{2}{3}} - y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

16. Show that the curve

$$\left.\begin{aligned} r &= 4a\sin\frac{t}{2}\cos^3\frac{t}{2}, \\ \theta &= \tan\frac{t}{2} - 2t \end{aligned}\right\}$$

is rectifiable and in isoperimetric companionship with the cardioide $r = a(1 + \cos\theta)$, its pole travelling along the axis of the cardioide as it rolls within the cardioide, the two poles being initially coincident.

17. Show, by taking $r = a\theta$ and $\chi = n\theta$ in Art. 702, that

$$x = \frac{a}{n^2}[n\theta\cos n\theta + (n-1)\sin n\theta],$$

$$y = \frac{a}{n^2}[n\theta\sin n\theta - (n-1)(\cos n\theta - 1)]$$

is an isoperimetric companion of the Archimedean spiral $r = a\theta$.

Hence show

(1) that $x^2 = 2ay$ is isoperimetric with $r = a\theta$;

(2) that $r = a\operatorname{cosec}\left\{\dfrac{\sqrt{r^2 - a^2}}{a} - \theta\right\}$ is rectifiable and in isoperimetric companionship with $r = 4a\theta$.

18. Show that an ellipse of semiminor axis b and eccentricity e can be made to roll upon the curve

$$\frac{b}{y} = \operatorname{dn}\frac{x}{b} \quad (\text{mod. } e),$$

so that the path of the centre of the ellipse is the x axis.

Show that if the origin be taken at the point for which the end of the major axis a is in contact with the curve, this may be reduced to the form

$$\frac{y}{a} = \mathrm{dn}\frac{x}{b}.$$

[Write $Kb - x$ for x and reduce, see Ch. XXXI., Art. 1352. See also Greenhill, *Elliptic Functions*, p. 72.]

19. Show that the perimeter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to twice the perimeter of one outer foil of the curve

$$r = b\cos\frac{b\theta}{a},$$

and that the area of the ellipse is equal to four times the area of one outer foil of the same curve.

Show furth that if the vertex of the foil be placed in contact with the inn of the ellipse at the end of the minor axis, and the foil roll liding upon the ellipse, the pole of the rolling foil will tr ajor axis of the ellipse.

Deduce proposition as to a circle rolling in the interior c double its radius.

20. An inv is made to slide, touching the rectangular axes Ox, w that the locus of the instantaneous centre on the plane x, y is a straight line. What is the locus of the instantaneous centre relatively to the curve.

Show that the glisettes of carried points are cycloids and trochoids, and the envelopes of carried straight lines are either cycloids or involutes of cycloids. [BESANT, *Roulettes and Glissettes.*]

21. A cycloid rolls along a straight line. Show that the intrinsic equations of the envelopes of (1) the axis, (2) the line of cusps, (3) the tangent at the vertex are respectively

$$(1)\quad s = a\psi^2 + 3a\sin^2\psi,$$

$$(2)\quad s = 3a(\psi + \tfrac{1}{2}\sin 2\psi),$$

$$(3)\quad s = a(\psi + \tfrac{3}{2}\sin 2\psi),$$

measuring s in each case from the point on its locus for which $\psi = 0$.

Trace each of these curves, supposing the cycloid to be continued both ways, and the rolling to continue with successive arches of the cycloid, and find the positions of their cusps.

Show that the whole perimeter of the last of these curves is

$$8a\sqrt{2} + 8a\sin^{-1}\sqrt{\tfrac{2}{3}} - 2\pi a,$$

and its area $= \frac{1}{2}\pi a^2$.

Show that the first evolutes of the second and third curves, and the second evolute of the first are four-cusped hypocycloids.

22. A parabola rolls on a straight line; show that

(1) the locus of the focus is a catenary (Art. 517),

(2) the envelope of the directrix is an equal catenary,

(3) the tangent at the vertex and the latus-rectum envelop parallels to a catenary,

(4) the intrinsic equation of the envelope of the axis is

$$s = a\,(2\log\sec\psi + \tan^2\psi).$$

23. If the cardioide $r = a(1 - \cos\theta)$ move so as to touch a straight line always at the same point, show that the locus [illegible] he pole is

$$r = 2a\sin^2\theta,$$

and that the intrinsic equation of the envelop[illegible] is is

$$\frac{3s}{a} = 12\sin^2\frac{\psi}{3} - 7\sin [illegible]$$

24. If an ellipse slide in contact wi[illegible] line at a given point, the glisette of the foci is

$$(x^2 + y^2)(b^2 + y^2)^2 = [illegible] a^2y^{[illegible]},$$

and that of the centre is $x^2y^2 = (a^2 - y^2)(y^2 - b^2)$.

25. A lamina moves in such manner that a certain point in it describes the path

$$\left.\begin{aligned} \xi &= c\sin\psi - c\cos\psi\log(\sec\psi + \tan\psi), \\ \eta &= c\cos\psi + c\sin\psi\log(\sec\psi + \tan\psi) - c, \end{aligned}\right\}$$

referred to fixed axes OX, OY in its plane, whilst a straight line through this point attached to the lamina makes an angle ψ with the Y-axis.

Reduce this motion to rolling. Also show that the difference of the curvatures of the loci of the instantaneous centre on the lamina and on the fixed plane is $\dfrac{\cos^2\psi}{c}$.

Show further that the intrinsic equation of the envelope of the line attached to the lamina is

$$\frac{ds}{d\psi} = c\sec\psi\tan\psi + c\log(\sec\psi + \tan\psi).$$

26. A lamina moves in its own plane, so that a point O' upon it traces out a cissoid,

$$\left.\begin{aligned} \xi &= -2a\cos^2\frac{\theta}{2}, \\ \eta &= 2a\frac{\cos^3\frac{\theta}{3}}{\sin\frac{\theta}{2}}, \end{aligned}\right\} \quad i.e.\ \eta^2(2a+\xi)+\xi^3=0,$$

upon a fixed plane with reference to a pair of fixed rectangular axes OX, OY in that plane, whilst a straight line $O'x$ attached to the moving lamina rotates, making an angle θ with OX. Show that the motion is that of rolling of one parabola upon another equal parabola, and deduce from the formula of Art. 660, for the difference of curvature of the I-loci, the radius of curvature of a parabola.

27. A catenary moves in its own plane so as always to touch a given straight line at a given point. Show that the tangential polar equation of the envelope of the axis is

$$\frac{p}{c}=\mathrm{gd}^{-1}\psi,$$

where c is the parameter of the catenary.

28. The centre of a circular disc of radius a travels along a parabolic path $y^2=2ax$, spinning at an angular velocity ω in a clockwise direction, the centre receding from the axis with a velocity $a\omega$. Show that the motion thus produced is that of the rolling of an involute of the circle upon the axis of the parabola, and that the velocity of the point of contact is the same as the velocity with which the centre of the circle recedes from the tangent at the vertex.

29. A Bernoulli's lemniscate moves so as to touch a fixed axis at a given point. Show that the tangential polar equation of the envelope of the axis is

$$p^2=a^2\sin^2\frac{\psi}{3}\cos\frac{2\psi}{3},$$

and that the glisette of the pole is

$$r^2=a^2\sin\theta.$$

30. A circle rolls on an equal circle and carries with it a fixed tangent. Find the intrinsic equation of the envelope of the carried tangent. [OXFORD II. P., 1887.]

31. A triangle of area Δ moves so that two of its sides (a, b) touch an oval of perimeter l at points where the radii of curvature are ρ, ρ'; prove that the radius of curvature and the perimeter of the envelope of the third side c are

$$\frac{1}{c}(2\Delta - a\rho - b\rho') \quad \text{and} \quad \frac{1}{c}\{4\pi\Delta - (a+b)l\}.$$

[ST. JOHN'S, 1883.]

32. An ellipse rolls on a fixed horizontal straight line (the axis of x). Show that the locus of the highest point of the ellipse will be

$$x = \int \frac{y^4 - 8a^2b^2}{y^2\sqrt{(4a^2 - y^2)(y^2 - 4b^2)}}\, dy,$$

and reduce the integral to the standard form.

[ST. JOHN'S COLL., 1881.]

33. Prove that the intrinsic equation of the envelope of the directrix of a catenary of parameter c, rolling on a circle of radius c, will be found by eliminating α between the equations

$$\left.\begin{aligned} \frac{s}{c} &= \tfrac{1}{2}\tan\alpha\sec\alpha + \tfrac{1}{4}\log\frac{1+\sin\alpha}{1-\sin\alpha} \\ \text{and} \qquad \psi &= \alpha + \tan\alpha. \end{aligned}\right\}$$

[ST. JOHN'S, 1886.]

34. A given right-angled triangle is made to slide round the outside of a fixed oval curve with the point P on the curve, the side PR touching it and the side PQ normal to it. If s be the perimeter of the oval, prove that the length of the curve enveloped by QR is equal to $\quad (s + 2\pi PQ)\sin PQR.$ [ST. JOHN'S, 1889.]

35. When a curve rolls on a straight line, show how to find the locus of the centre of curvature at the point of contact, and prove that, in the case of a cardioide, the locus is an ellipse.

[ST. JOHN'S, 1889.]

36. When a curve rolls on a fixed curve, prove that the locus of the centre of curvature is inclined to the common tangent at the angle

$$\tan^{-1}\{\rho\, d\rho'/(\rho+\rho')\, ds\},$$

where ρ, ρ' are the radii of curvature of the fixed and rolling curves at the point of contact. [ST. JOHN'S, 1889.]

37. A cardioide $r = a(1-\cos\theta)$ rolls upon an equal cardioide, the vertices coinciding during the roll. Show that the roulette of the pole of the rolling curve is

$$r = 4a\sin^2\left(\frac{\pi}{6} + \frac{\theta}{2}\right),$$

that the tangential polar equation of the envelope of the axis is

$$p = 4a \sin \psi \sin^3 \frac{\pi + \psi}{6},$$

and that the area of the roulette of the pole is

$$\frac{a^2}{2}(3\pi + 4\sqrt{3}).$$

38. A cardioide of perimeter $8a$ rolls on the outer side of a cycloid of equal perimeter from cusp to cusp, the vertices coinciding during the roll. Show that the area of the roulette of the cusp of the cardioide between the roulette and the cycloid $= \frac{9}{2}\pi a^2$.

Show also that the arc of any portion of the roulette of the cusp measured from the vertex of the curve is double the distance of the point of contact of the two curves from the axis of the cycloid.

Show further that the tangential polar equation of the envelope of the axis of the cardioide is

$$p = 2a \sin \psi (\psi + 2 \cos^3 \psi),$$

where p is drawn from the vertex of the cycloid and ψ is measured from its axis.

39. A cycloid of length $8a$ rolls on the outside of a cardioide of equal length, a cusp of the cycloid starting from the cusp of the cardioide. Show that the intrinsic equation of the envelope of the line joining the cusps of the cycloid is

$$2s = 3a\psi + 6a \sin \frac{\psi}{2},$$

ψ being measured from the tangent at the vertex of the cardioide.

[OXF. II. P., 1913.]

CHAPTER XX.

RECTIFICATION OF TWISTED CURVES.

706. Let PQ be any elementary arc δs of the curve. Let the coordinates of P and Q be respectively

$$(x, y, z) \quad \text{and} \quad (x+\delta x, y+\delta y, z+\delta z)$$

with regard to any three fixed rectangular axes Ox, Oy, Oz. Then

$$(\text{chord } PQ)^2 = \delta x^2 + \delta y^2 + \delta z^2.$$

Now, if Q be made to travel along the curve so as to approach indefinitely near to P, the chord PQ and the arc PQ ultimately differ by an infinitesimal of higher order than the arc PQ itself, *i.e.* the chord PQ and the arc PQ ultimately vanish in a ratio of equality.* Hence we have to the second order of small quantities,

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now suppose the curve to be specified in one of the two usual ways,

(a) as the line of intersection of two specified surfaces

$$f(x, y, z) = 0, \quad F(x, y, z) = 0,$$

or (b) the coordinates of any point x, y, z upon it expressed in terms of some fourth variable t, and defined by the equations $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$.

The First Case.

In Case (a) choice must be made of one of the three variables x, y, z to be considered as the independent variable, say x, and the equations $f=0$, $F=0$ are then to be solved to find

* For a discussion of this point see De Morgan, *Differential and Integral Calculus*, p. 445. See also *Diff. Calc.*, Art. 34, for a plane curve.

the other two, y and z, in terms of x. Then differentiating, we express $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$ in terms of x; say

$$\frac{dy}{dx}=\phi(x), \quad \frac{dz}{dx}=\psi(x).$$

We then have $s=\int\sqrt{dx^2+dy^2+dz^2}$

$$=\int\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}}\,dx$$

$$=\int\sqrt{[1+\{\phi(x)\}^2+\{\psi(x)\}^2]}\,dx.$$

And when the integration has been effected, the length of the arc between the points specified by any particular limits which may be assigned to x, will have been obtained.

707. A more Symmetrical Mode of Procedure.

We might also proceed as follows:

Along the line of intersection of $f=0$ and $F=0$ we have

$$f_x\,dx+f_y\,dy+f_z\,dz=0$$

and $$F_x\,dx+F_y\,dy+F_z\,dz=0,$$

giving $$\frac{dx}{J_1}=\frac{dy}{J_2}=\frac{dz}{J_3}=\frac{ds}{\sqrt{J_1^2+J_2^2+J_3^2}}=\frac{ds}{\sqrt{\phi}}, \text{ say,}$$

J_1, J_2, J_3 being the Jacobians

$$\begin{vmatrix} f_y, & f_z \\ F_y, & F_z \end{vmatrix}, \quad \begin{vmatrix} f_z, & f_x \\ F_z, & F_x \end{vmatrix}, \quad \begin{vmatrix} f_x, & f_y \\ F_x, & F_y \end{vmatrix},$$

i.e. $$\frac{\partial(f, F)}{\partial(y, z)}, \quad \frac{\partial(f, F)}{\partial(z, x)}, \quad \frac{\partial(f, F)}{\partial(x, y)}.$$

Then $$s=\int\sqrt{\phi}\frac{dx}{J_1} \text{ or } \int\sqrt{\phi}\frac{dy}{J_2} \text{ or } \int\sqrt{\phi}\frac{dz}{J_3},$$

making use of the one which is most convenient; and whichever is used, both the dependent variables occurring must be expressed in terms of the independent one before integration.

708. The Second Case.

In Case (b) we have

$$x=f_1(t), \quad y=f_2(t), \quad z=f_3(t)$$

and $$\frac{dx}{dt}=f_1'(t), \quad \frac{dy}{dt}=f_2'(t), \quad \frac{dz}{dt}=f_3'(t),$$

whence $$s=\int\sqrt{[\{f_1'(t)\}^2+\{f_2'(t)\}^2+\{f_3'(t)\}^2]}\,dt;$$

and we obtain the arc by integration, as before, between any two points corresponding to the limits assigned for the variable t.

709. If the equations of the curve be presented in the form

$$\frac{x}{f_1(t)}=\frac{y}{f_2(t)}=\frac{z}{f_3(t)}=\frac{1}{f(t)},$$

we have $$\frac{dx}{dt}=\frac{f_1'(t)f(t)-f_1(t)f'(t)}{\{f(t)\}^2}=\frac{J_1}{f^2}, \text{ say.}$$

Similarly $$\frac{dy}{dt}=\frac{J_2}{f^2},\quad \frac{dz}{dt}=\frac{J_3}{f^2},$$

where J_2 and J_3 have meanings corresponding to J_1.

Hence $$\frac{dx}{J_1}=\frac{dy}{J_2}=\frac{dz}{J_3}=\frac{dt}{f^2}=\frac{ds}{\sqrt{\phi}},$$

where $$\phi=J_1^2+J_2^2+J_3^2.$$

Hence $$s=\int\frac{\sqrt{J_1^2+J_2^2+J_3^2}}{f^2}\,dt=\int\frac{\sqrt{\phi}}{f^2}\,dt.$$

710. The rectification of a curve therefore depends upon the possibility of performing the integration $\int\frac{\sqrt{\phi}}{f^2}\,dt.$

When f_1, f_2, f_3, f are rational integral and algebraic functions of t, we have the case of a unicursal twisted curve.

The advanced student is referred to the very important memoir by Mr. R. A. Roberts, "On the Rectification of Certain Curves," in vol. xviii. of the *Proceedings of the London Mathematical Society*, which has already been referred to in other places.

711. Ex. 1. Find the length of an arc of the curve which is the line of intersection of the parabolic cylinder $y^2=4ax$ and the cylinder

$$z=\sqrt{x(x-a)}-\frac{a}{2}\cosh^{-1}\frac{2x-a}{a}.$$

Here we take x as the independent variable and obtain

$$\frac{dy}{dx}=\sqrt{\frac{a}{x}},$$

$$\frac{dz}{dx}=\frac{2x-a}{2\sqrt{x(x-a)}}-\frac{1}{\sqrt{\left(\frac{2x-a}{a}\right)^2-1}}=\frac{2x-a}{2\sqrt{x(x-a)}}-\frac{a}{2\sqrt{x(x-a)}}=\sqrt{\frac{x-a}{x}};$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \frac{a}{x} + 1 - \frac{a}{x} = 2\,;$$

$$\therefore s = \sqrt{2}\int_{x_1}^{x_2} dx = \sqrt{2}(x_2 - x_1),$$

where x_1 and x_2 are the lower and upper limits of integration.

Hence, in this curve any portion of the arc is $\sqrt{2}$ times its projection upon the x-axis. In other words, at every point of this curve the tangent makes an angle of $\frac{\pi}{4}$ with the x-axis.

Taking the same curve, let us put

$$x = \frac{a}{2}(1 + \cosh u), \quad \textit{i.e. } a\cosh^2\frac{u}{2}.$$

Then $$y = 2a\cosh\frac{u}{2}, \quad z = \frac{a}{2}(\sinh u - u);$$

we then have a case such as that discussed in (*b*) of the preceding article, having expressed x, y and z in terms of an auxiliary fourth variable u.

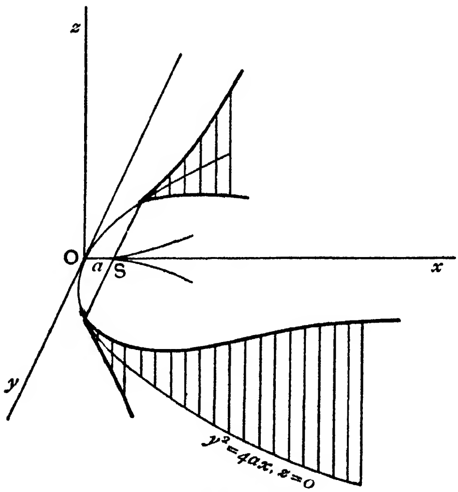


Fig. 222.

Then $$\frac{dx}{du} = \frac{a}{2}\sinh u, \quad \frac{dy}{du} = a\sinh\frac{u}{2}, \quad \frac{dz}{du} = \frac{a}{2}(\cosh u - 1).$$

Therefore $$\left(\frac{ds}{du}\right)^2 = \frac{a^2}{4}\left[\sinh^2 u + 4\sinh^2\frac{u}{2} + (\cosh u - 1)^2\right]$$

$$= \frac{a^2}{4}\left[\sinh^2 u + 2(\cosh u - 1) + (\cosh u - 1)^2\right]$$

$$= \frac{a^2}{2}\sinh^2 u\,;$$

whence $$s=\left[\frac{a}{\sqrt{2}}\cosh u+C\right]$$
$$=\frac{a}{\sqrt{2}}\left[\frac{2x}{a}-1\right]_{x_1}^{x_2}=\sqrt{2}(x_2-x_1) \text{ as before.}$$

The curve of intersection of the two cylinders is represented in Fig. 222.

Ex. 2. To find an expression in the form of an integral for the rectification of the line of intersection of two right circular cylinders whose axes intersect at right angles.

If we take the axes of the cylinders as the axes of z and x respectively, we may write the equations of the cylinders as

$$x^2+y^2=a^2, \quad y^2+z^2=b^2.$$

Let us take $a>b$.

From the equations

$$x=\sqrt{a^2-y^2}, \qquad z=\sqrt{b^2-y^2},$$

we have $$dx=\frac{-y\,dy}{\sqrt{a^2-y^2}}, \quad dz=\frac{-y\,dy}{\sqrt{b^2-y^2}},$$

and $$ds^2=dx^2+dy^2+dz^2$$
$$=\frac{a^2b^2-y^4}{(a^2-y^2)(b^2-y^2)}\,dy^2.$$

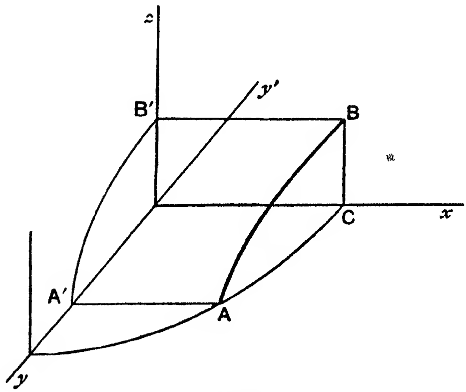


Fig. 223.

Put $y=b\sin\theta$, and let $b=ka$, $k<1$.

Then $$ds^2=\frac{k^2a^4-k^4a^4\sin^4\theta}{(a^2-k^2a^2\sin^2\theta)}\,d\theta^2,$$

$$s=b\int_0^{\theta}\frac{\sqrt{1-k^2\sin^4\theta}}{\sqrt{1-k^2\sin^2\theta}}\,d\theta.$$

When the cylinders are of equal radius, $k=1$, and this becomes

$$s=b\int\sqrt{1+\sin^2\theta}\,d\theta$$

$$=\int\sqrt{(b\sqrt{2})^2\sin^2\theta+b^2\cos^2\theta}\,d\theta,$$

i.e. the result of Art. 573, for an ellipse whose axes are in the ratio $\sqrt{2}:1$, to which the curve of intersection then reduces.

It is interesting in this connexion to note more generally that when the axes of two equal cylinders cut at right angles, and a sphere rolls completely round in contact with both cylinders, the locus of its centre is two ellipses. In our case the rolling sphere has a zero radius.

712. In the "**right circular Helix**" or "**Helicoidal curve,**" which is an ordinary thread on a screw, we have a curve traced on a right circular cylinder and cutting all the generators of the cylinder at the same angle.

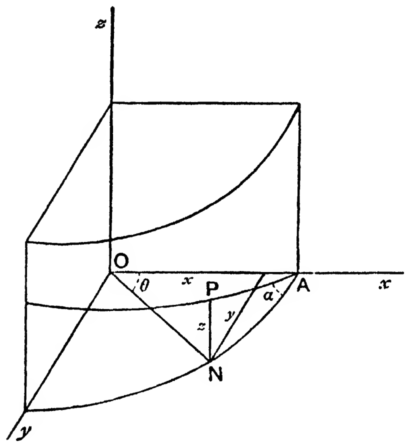


Fig. 224.

Let α be the angle the screw-thread makes with a circular section of the cylinder, P any point on the curve, coordinates x, y, z referred to rectangular axes, the z-axis being the axis of the cylinder and the x-axis taken to cut the curve at a point A. Let θ be the angle the plane OPN through P and the axis makes with the plane of xz, and let a be the radius of the cylinder.

We have $\qquad x=a\cos\theta,\quad y=a\sin\theta,\quad z=a\theta\tan\alpha.$

Hence $\qquad ds^2=dx^2+dy^2+dz^2=a^2\sec^2\alpha\,d\theta^2$

and $\qquad s=a\theta\sec\alpha.$

This is obvious from the fact that in this case the surface may be developed into a plane, and the triangle ANP becomes a right-angled triangle with sides $a\theta$, $a\theta\tan\alpha$ and s, with one of its acute angles α.

713. Since the curve develops into a straight line when the surface is developed into a plane, the surface itself being supposed entirely inextensible, the distance between any two points which it connects upon the cylinder is a minimum distance on the cylinder between those two points. Such lines of minimum length on any surface are termed **Geodesics** (see Smith's *Solid Geom.*, Art. 259).

Hence geodesic lines on a right circular cylinder are helices.

714. A Property of Geodesic Lines.

It is an obvious property of such curves that if P, Q be any points upon a geodesic line upon any surface, the path from P to Q *via* this line being less than from P to Q *via* any contiguous supposititious paths from P to Q, viz. PBQ, or PCQ, on opposite sides of it and of the same length, and the three

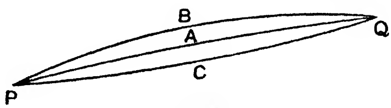


Fig. 225.

lengths PAQ the geodesic, and PBQ, PCQ the supposititious paths being unaltered in length by any deformation of the surface on which they are drawn, supposed inextensible, the deformed path to which PAQ is changed will still be in length intermediate between the lengths of the contiguous paths to which PBQ and PCQ are changed and which are equal. Hence, in the limit when PBQ and PCQ and their deformed lengths are made to close up to ultimate coincidence with PAQ and its deformed length, it will be clear that the deformed PAQ is still a line of minimum length on the deformed surface, being entrapped between two supposititious paths which are both of greater length on opposite sides of it. Thus geodesics on inextensible surfaces remain geodesics after any deformation of the surface on which they are drawn.

715. It follows that a right circular helix remains a right circular helix if the paper on which it is drawn be transferred from the cylinder upon which it was wrapped to a cylinder of different radius. Let a and b be the radii of the first and second cylinders and β the angle the new helix makes with the circular section. Then $s=\frac{a}{\cos\alpha}\theta=\frac{b}{\cos\beta}\theta'$, where θ' is the angle in the new helix corresponding to θ in the original one;

$$\therefore\ \theta'=\frac{a\cos\beta}{b\cos\alpha}\theta,$$

and the new coordinates of P can be written down, the axes being placed as described for the first helix.

716. **Cylindrical Coordinates.**

For many cases, particularly for curves drawn upon cylinders, it is desirable to use cylindrical coordinates, viz. r, θ, z, *i.e.* the ordinary Cartesians are transformed to the polar system as regards the x, y plane, and the z-coordinate is left unaltered.

Taking r, θ, z and $r+\delta r, \theta+\delta\theta, z+\delta z$ as the coordinates of contiguous points P, Q on a curve, we have, since $\delta r, r\,\delta\theta, \delta z$ are mutually perpendicular elements,

$$PQ^2=\delta r^2+(r\,\delta\theta)^2+\delta z^2.$$

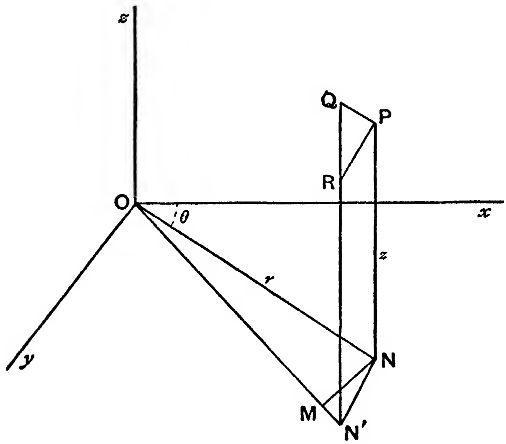


Fig. 226.

For if N, N' be the feet of the perpendiculars from P, Q upon the plane of x-y, we have, to the second order,

$$NN'^2=(r\,\delta\theta)^2+\delta r^2,$$

and plainly $$PQ^2=NN'^2+\delta z^2.$$

Hence, if the distance measured along the arc PQ be δs, we have, to the second order,

$$\delta s^2=\delta r^2+(r\,\delta\theta)^2+\delta z^2,$$

whence $$s=\int\sqrt{dr^2+(r\,d\theta)^2+dz^2},$$

which we may write in any of the forms

$$s=\int\sqrt{\left[\left(\frac{dr}{d\theta}\right)^2+r^2+\left(\frac{dz}{d\theta}\right)^2\right]}d\theta,$$

$$s=\int\sqrt{\left[1+r^2\left(\frac{d\theta}{dr}\right)^2+\left(\frac{dz}{dr}\right)^2\right]}dr$$

or $$s=\int\sqrt{\left[\left(\frac{dr}{dz}\right)^2+r^2\left(\frac{d\theta}{dz}\right)^2+1\right]}dz,$$

according as it is convenient to take θ, r or z as the independent variable; or we may also write it, as in Cartesians, as

$$s=\int\sqrt{\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]}dt,$$

in case r, θ, z are expressed in terms of a fourth auxiliary variable t.

The most common case is when θ is taken as the independent variable.

717. Curves on a Right Circular Cylinder.

When we are discussing a curve drawn upon the surface of a right circular cylinder of radius a, we have

$$r=a \quad \text{and} \quad dr=0,$$

and the rectification formula at once reduces to

$$s=\int\sqrt{dz^2+a^2d\theta^2}=\int\sqrt{a^2+\left(\frac{dz}{d\theta}\right)^2}\,d\theta.$$

718. If we apply this to the case of the helix already considered, viz.

$$x=a\cos\theta, \quad y=a\sin\theta, \quad z=a\theta\tan\alpha,$$

we have $$r=a, \quad z=a\theta\tan\alpha,$$

$$s=\int a\sqrt{1+\tan^2\alpha}\,d\theta=a\theta\sec\alpha, \quad \text{as before (Art. 712).}$$

It will be at once remarked, however, that in all cases of curves drawn upon a right circular cylinder, the length of the arc may as readily be considered by first developing the cylindrical surface into a plane, and in fact the formula above is merely the Cartesian formula

$$\int\sqrt{dz^2+dx^2}$$

for the developed surface, dx replacing $a\,d\theta$.

719. Ex. Find the length of an arc of the curve of intersection of the cylinders

$$x^2+y^2=a^2, \quad xe^{\frac{z}{a}}=a.$$

Putting $x=a\cos\theta$, we have

$$y=a\sin\theta \quad \text{and} \quad z=a\log\sec\theta.$$

Hence $$\frac{dz}{d\theta}=a\tan\theta \quad \text{and} \quad \frac{ds}{d\theta}=a\sec\theta\,;$$

whence $$s=a\,\mathrm{gd}^{-1}\theta \quad \text{or} \quad s=a\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

In this case the developed curve is the Catenary of Equal Strength, viz. $\zeta=a\log\sec\frac{\xi}{a}$, in which $\xi=a\psi$ and $s=a\,\mathrm{gd}^{-1}\psi$ (see Ex. 5, Art. 519).

720. General Polar Formulae.

The general polar formula for rectification in terms of the radius vector r, the co-latitude θ, and the azimuthal angle, or longitude, ϕ, is easily obtained.

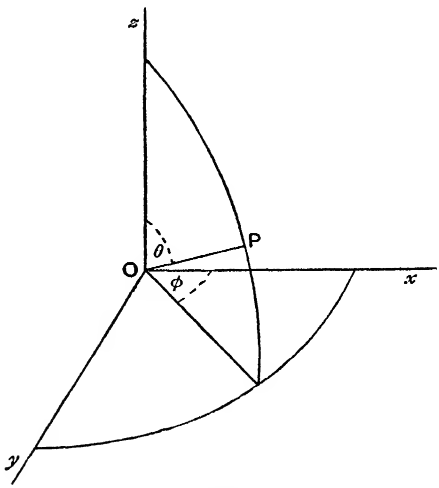


Fig. 227.

In passing from the point $P(r, \theta, \phi)$ to a contiguous point $Q(r+\delta r, \theta+\delta\theta, \phi+\delta\phi)$ along an elementary arc δs of a curve, the projections of the chord PQ in the three directions,

(a) along the radius vector, increasing r;

(b) in the meridian plane, increasing θ;

(c) perpendicular to the meridian plane, increasing ϕ,

are respectively $\quad \delta r, \quad r\,\delta\theta, \quad r\sin\theta\,\delta\phi$;

and these being mutually perpendicular elements we have, to the second order,

$$\delta s^2 = \delta r^2 + r^2 \delta\theta^2 + r^2 \sin^2\theta\, \delta\phi^2,$$

and as either r, θ, ϕ or a fourth variable t can be regarded as the independent variable to suit circumstances, we have

$$s = \int \sqrt{\left[1 + r^2\left(\frac{d\theta}{dr}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dr}\right)^2\right]}\, dr,$$

or $$s = \int \sqrt{\left[\left(\frac{dr}{d\theta}\right)^2 + r^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right]}\, d\theta,$$

or $$s = \int \sqrt{\left[\left(\frac{dr}{d\phi}\right)^2 + r^2\left(\frac{d\theta}{d\phi}\right)^2 + r^2\sin^2\theta\right]}\, d\phi,$$

or $$s = \int \sqrt{\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]}\, dt.$$

721. **Modification for Curves on the Sphere and the Cylinder.** There are two important cases to consider.

(1) If the curve under discussion **lie on a sphere of radius** a,

$$r = a, \quad dr = 0,$$

and $$s = a\int \sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

or $$s = a\int \sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta}\, d\phi;$$

or if it be deemed desirable to use the latitude l instead of the co-latitude θ $\left(l = \frac{\pi}{2} - \theta\right)$,

$$s = a\int \sqrt{1 + \cos^2 l\left(\frac{d\phi}{dl}\right)^2}\, dl$$

or $$s = a\int \sqrt{\left(\frac{dl}{d\phi}\right)^2 + \cos^2 l}\, d\phi.$$

(2) If the curve under discussion lie on the surface of **a right circular** cone whose semivertical angle is a, and whose axis is the z-axis and vertex the origin, we have

$$\theta = a, \quad d\theta = 0,$$

$$s = \int \sqrt{1 + r^2\sin^2 a\left(\frac{d\phi}{dr}\right)^2}\, dr$$

or $$s = \int \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2\sin^2 a}\, d\phi.$$

722. Ex. 1. **"Rhumb" Line** or **"Loxodrome"** on a sphere.

This is a curve on the surface of a sphere which cuts all the meridians at the same angle.

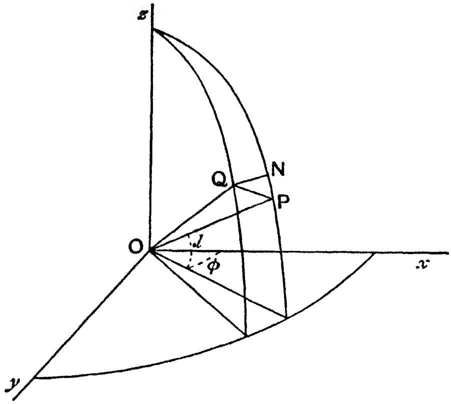


Fig. 228.

Let PQ be an element ds of such a line, zOP, zOQ meridian planes. Let a small circle of the sphere parallel to the equatorial plane x-y pass through Q and cut the meridian plane of P in N. Let l and ϕ be the latitude and longitude of P, a the radius of the sphere and α the constant angle NPQ.

Then $$\tan\alpha = Lt\,\frac{NQ}{PN} = Lt\,\frac{a\cos l\,\delta\phi}{a\,\delta l}, \quad i.e.\ \cos l\,\frac{d\phi}{dl} = \tan\alpha,$$

or $$\cot\alpha\,d\phi = \sec l\,dl\,;$$

whence $$\phi\cot\alpha = \mathrm{gd}^{-1}l, \quad i.e.\ \log\tan\left(\frac{\pi}{4}+\frac{l}{2}\right),$$

which, with $r=a$, form the equations of the curve.

Also $$s = a\int\sqrt{1+\cos^2 l\left(\frac{d\phi}{dl}\right)^2}\,dl = a\sec\alpha\,.\,l.$$

Hence in this curve we have

$$r=a, \quad l=\mathrm{gd}(\phi\cot\alpha) \quad \text{and} \quad s = al\sec\alpha.$$

Ex. 2. In the case of a spiral traced on a sphere and defined by the equation $l=\phi\tan\alpha$, where α is constant, we have

$$\begin{aligned} s &= a\int\sqrt{1+\cos^2 l\left(\frac{d\phi}{dl}\right)^2}\,dl \\ &= a\int\sqrt{1+\cot^2\alpha\cos^2 l}\,dl \\ &= a\int\sqrt{\operatorname{cosec}^2\alpha-\cot^2\alpha\sin^2 l}\,dl \\ &= a\operatorname{cosec}\alpha\int\sqrt{1-\cos^2\alpha\sin^2 l}\,dl \\ &= a\operatorname{cosec}\alpha\,E(l,\cos\alpha), \end{aligned}$$

and the arc of this spiral is therefore expressible as an arc of an ellipse of semi-major axis $a \operatorname{cosec} a$ and eccentricity $\cos a$ (see Art. 567).

Ex. 3. In the case of a curve drawn upon a conical surface to cut all the generators at the same constant angle a, we have, taking the origin

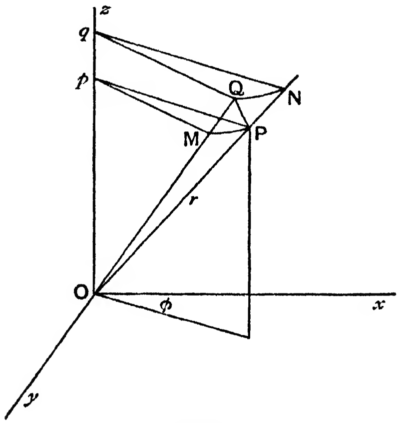


Fig. 229.

at the vertex and the axis of the cone as the z-axis and β for the semi-vertical angle of the cone,

$$\frac{r \sin \beta \, d\phi}{dr} = \tan a,$$

as in Example (1), for the sphere, and therefore

$$\frac{dr}{r} = \sin \beta \cot a \, d\phi \, ;$$

whence $$r = Ae^{\phi \sin \beta \cot a},$$

where A is an arbitrary constant, determinable when some one point on the curve is specified.

The projection of the curve upon the x-y plane is therefore an equi-angular spiral of angle $\cot^{-1}(\sin \beta \cot a)$.

We also have $$s = \int \sqrt{1 + r^2 \sin^2\beta \left(\frac{d\phi}{dr}\right)^2} \, dr$$

$$= \int \sqrt{1 + \tan^2 a} \, dr = \Big[r \sec a \Big]_{r_1}^{r_2}$$

between limits r_1, r_2.

If the spiral passes through the origin, and s be measured from that point, $$s = r \sec a,$$

which is also obvious from the consideration that if the curve be developed upon a plane it will become an equiangular spiral of angle a.

723. The p, r Formula.

The p, r formula of Art. 547, viz. $s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}$, still holds for curves of double curvature.

For, with the same notation as before,

$$\frac{p}{r}=\sin\phi \quad \text{and} \quad \frac{dr}{ds}=\cos\phi,$$

ϕ being the angle which the tangent makes with the radius vector from the origin; whence

$$\frac{ds}{dr}=\sec\phi=\frac{1}{\sqrt{1-\frac{p^2}{r^2}}}=\frac{r}{\sqrt{r^2-p^2}}$$

and
$$s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}.$$

For cases of curves drawn upon a sphere, the centre being at the origin, the formula is useless. For in that case, the tangent being necessarily at all points at right angles to the radius vector, $\frac{dr}{ds}=0$ and $p=r$ throughout.

In the case of a curve drawn upon a right circular cone whose vertex is at the origin, we may use the formula with advantage; but it is to be remembered that we are doing no more than if we regarded the conical surface as developed upon a plane.

Ex. For the case already considered of a curve cutting all the generators of a cone at a constant angle a, we have at once $p=r\sin a$ and $s=\int\frac{dr}{\cos a}=r\sec a$, as in the last article.

There are but few curves of double curvature, however, for which the p, r relation is known, with the exception of course of such as, having been originally plane curves, have been laid upon a developable surface. For such cases the formula is useful, as also of course whenever the relation can be readily found.

724. Ex. Let $BAA'B'$ be a strip of thin inextensible ribbon lying upon a plane. Let OAA' be a perpendicular from any point O of the plane upon AB and $A'B'$ and OPP' any other radius vector from O.

Let $OA=l_0$, $OP=l$, $PA=s$.

Then obviously $$l^2=s^2+l_0^2.$$

Now imagine this ribbon wrapped tightly without folding or creasing upon a right circular cone of vertex O with OAA' as a generator, the semivertical angle being a, the wrapping commencing with OA in con-

tact with the cone. When the wrapping has been completed, OP coming into contact and becoming a generator, let us unwrap the triangle from the cone, keeping OP in contact and starting the unwrapping with

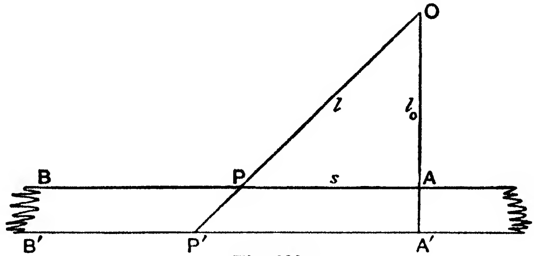


Fig. 230.

releasing the generator OA, keeping O fixed at the vertex; when the unwrapping is just complete, the triangle has taken the position OYP, and is the same triangle as we started with, $O\hat{Y}P$ being a right angle.

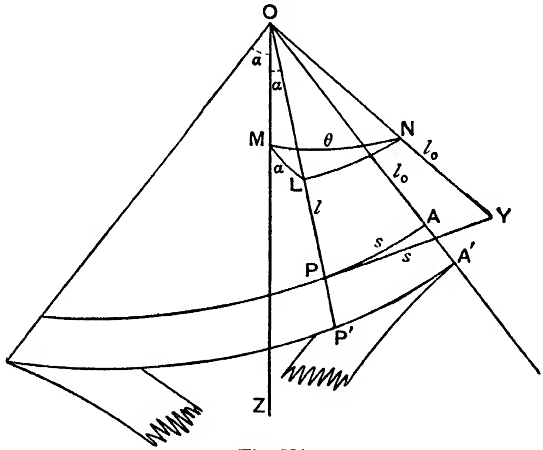


Fig. 231.

It appears

(1) that the arc AP upon the cone has a length $\sqrt{l^2 - l_0^2}$;

(2) that the arc AP upon the cone is a geodesic;

(3) that the locus of Y in the unwrapping lies on a sphere of radius l_0 and vertex at O;

(4) that the p, r equation of this geodesic on the cone is $p = l_0$, for this is so on the plane from which it was constructed;

(5) the formula $s = \int \frac{r\,dr}{\sqrt{r^2 - p^2}}$ is merely

$$s = \int \frac{r\,dr}{\sqrt{r^2 - p_0^2}} = \left[\sqrt{r^2 - p_0^2}\right]_{l_0}^{l} = \sqrt{l^2 - l_0^2};$$

(6) the Y locus is an involute of the geodesic;

(7) taking a sphere of any radius with centre at O, cutting the axis OZ at M, the generator OP at L and OY the perpendicular on the tangent at N, LMN is a right-angled spherical triangle, where

$$ML=a, \quad LN=\tan^{-1}\frac{s}{l_0} \quad \text{and} \quad M\hat{L}N=\frac{\pi}{2};$$

whence $\qquad \cos MN=\cos a \cos LN$

and $\qquad \sin a=\cot L\hat{M}N \tan LN=\frac{s}{l_0}\cot L\hat{M}N.$

If ϕ be the angle between the plane ZOY and the plane ZOA, and θ the angle ZOY, we have thus shown that

$$\cos\theta=\cos a\cos LN, \quad \text{and therefore } \frac{s}{l_0}=\frac{\sqrt{\cos^2 a-\cos^2\theta}}{\cos\theta}.$$

Now, if we take a circle on the plane OPY with centre O and radius OP, and consider the arc bounded by OP and OY produced, this arc will wrap upon the cone and will coincide with the corresponding arc of the circular section of the cone through P; whence if χ be the angle between the plane ZOP and the plane ZOA,

$$l\sin a\,.\,\chi=l\times\text{angle } POY,$$

and $\qquad \chi\sin a=\tan^{-1}\frac{s}{l_0}.$

Hence $\qquad \phi=\chi-L\hat{M}N=\frac{1}{\sin a}\tan^{-1}\frac{s}{l_0}-\tan^{-1}\frac{s}{l_0\sin a},$

i.e. $\qquad \phi=\frac{1}{\sin a}\tan^{-1}\frac{\sqrt{\cos^2 a-\cos^2\theta}}{\cos\theta}-\tan^{-1}\frac{1}{\sin a}\frac{\sqrt{\cos^2 a-\cos^2\theta}}{\cos\theta},$

is the equation of a cone which by its intersection with the sphere of radius l_0 and centre O gives the Y locus, which is also an involute of the geodesic on the cone.

725. Inversion.

The process of inversion may sometimes be employed with advantage. This is particularly the case when a twisted curve lies on the surface of a sphere. By inverting with regard to a point on the surface of the sphere, the spherical surface is inverted into a plane and the twisted curve into a plane curve, and *vice versa.*

Let O be the pole of inversion and k the constant, and let the diameter OA of the sphere meet the plane into which the sphere inverts at C. Then $OA\,.\,OC=k^2$,

$$OC=\frac{k^2}{OA}=c, \text{ say.}$$

Let the element PQ, viz. δs, of a twisted curve on the spherical surface invert into $P'Q'$, viz. $\delta s'$, an element of the plane inverse curve.

Then $$PQ=k^2\frac{P'Q'}{OP'.OQ'},$$

or ultimately $$ds=k^2\frac{ds'}{OP'^2}.$$

Let $$CP'=r.$$

Then $$s=k^2\int\frac{ds'}{c^2+r^2},$$

and if this integral for the plane curve can be found, the rectification of the twisted curve on the sphere will have been effected.

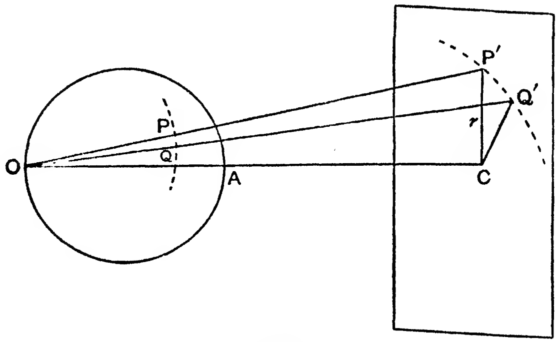


Fig. 232.

The method may also be used to *discover* rectifiable twisted curves which lie on a spherical surface.

726. **Extension of Art. 230, *Diff. Calc.*, for Present Purposes.**

The angle between intersecting curves is unaffected by inversion. (Extension of Art. 230 of *Diff. Calc.*)

If two planes $QPP'Q'$, $RPP'R'$ intersect in the line PP' and if PQ, $P'Q'$ make the same angle with PP' in opposite directions as also PR and $P'R'$, then the angle $Q\hat{P}R=Q'\hat{P'}R'$. For, take distances PN and $P'N'$ equal to each other in opposite directions from P and P' respectively on PP' produced, and let two planes perpendicular to the line PP' be drawn through N and N' to cut PQ and PR at Q and R, and to cut $P'Q'$ and $P'R'$ in Q' and R' respectively.

Then, from the congruent pairs of triangles PNQ and $P'N'Q'$, and PNR and $P'N'R'$ respectively, we have $NQ=N'Q'$ and $NR=N'R'$, whilst $Q\hat{N}R=Q'\hat{N}'R'$, and therefore the triangles QNR, $Q'N'R'$ are congruent and $QR=Q'R'$; whence the angles $Q\hat{P}R$, $Q'\hat{P}'R'$ are also equal.

It follows therefore that if PQ, $P'Q'$ be the directions of the tangents at P and P' to inverse elements of curves in the

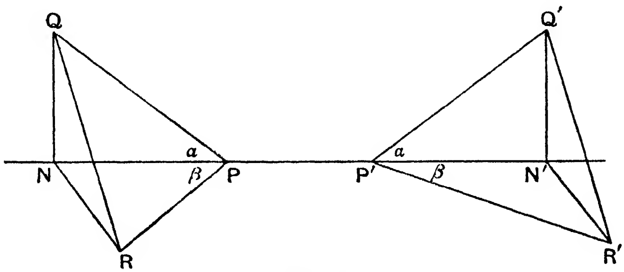


Fig. 233.

plane $PP'Q'Q$ and PR, $P'R'$ be the directions of the tangents at P and P' to inverse elements of curves in the plane $PP'R'R$, then, as in this case PQ and $P'Q'$ make equal angles with PP' in opposite directions, as also do PR and $P'R'$ (as proved in *Differential Calculus*, Art. 229, for curves in a plane), it will follow that the angle between two curves meeting at P is equal to the angle between the inverses meeting at P'. Hence the result of Art. 230 of *Diff. Calc.* is now extended to any case of inversion, the curves not being necessarily plane, and the pole of inversion now lying anywhere.

727. **Stereographic Projection, etc.**

If we take as constant of inversion the diameter of the sphere, and the pole of inversion a point O on the sphere, the sphere inverts into the tangent plane at the opposite end of the diameter through the pole.

If the constant of inversion be taken as

$$\frac{\text{diameter}}{\sqrt{2}}, \quad i.e.\ \sqrt{2}\,.\,\text{radius},$$

the sphere inverts into the equatorial plane of which the origin of inversion is a pole.

In all such cases the inversion amounts to a conical projection with the origin O as pole of projection.

When the projection is upon an equatorial plane with O for pole, it is called a Stereographic Projection.

In any of these cases, the angles of intersection of any spherical curves project or invert into equal angles of intersection of the projected or inverted curve. Orthogonal intersection remains orthogonal intersection in the projected curves; curves which touch on the sphere project or invert into curves which touch; circular arcs which pass through the pole O invert into straight lines; all other circles, great or small, into circles.

Ex. Consider the rectification of the line of intersection of the sphere

$$x^2+y^2+z^2=cz$$

with the elliptic cone $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}.$$

Inverting with regard to the origin, and with c for constant of inversion, the sphere becomes the plane $z=c$, and the cone remains unaltered, but cutting the plane $z=c$ in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

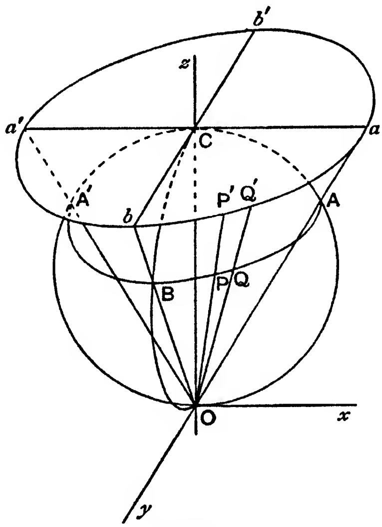


Fig. 234.

If PQ, $P'Q'$ be corresponding elements ds, ds' of the original and the inverse curves,

$$ds=c^2\frac{ds'}{OP'^2}=\frac{c^2}{c^2+r^2}ds',$$

where r is the central radius vector of the ellipse to the point P'.

Hence, taking θ as the complement of the eccentric angle of P', we have for the ellipse,

$$x=a\sin\theta,\quad y=b\cos\theta,\quad r^2=a^2\sin^2\theta+b^2\cos^2\theta,$$
$$ds'^2=(a^2\cos^2\theta+b^2\sin^2\theta)\,d\theta^2.$$

and

$$ds=\frac{c^2\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}{(c^2+b^2)\cos^2\theta+(c^2+a^2)\sin^2\theta}d\theta$$
$$=c^2\frac{a^2\cos^2\theta+b^2\sin^2\theta}{\{(c^2+b^2)\cos^2\theta+(c^2+a^2)\sin^2\theta\}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}d\theta$$
$$=c^2\frac{-\{(c^2+b^2)\cos^2\theta+(c^2+a^2)\sin^2\theta\}+(a^2+b^2+c^2)}{\{(c^2+b^2)\cos^2\theta+(c^2+a^2)\sin^2\theta\}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}d\theta$$
$$=c^2\left[-\frac{1}{a\sqrt{1-e^2\sin^2\theta}}+\frac{a^2+b^2+c^2}{c^2+b^2+a^2e^2\sin^2\theta}\,\frac{1}{a\sqrt{1-e^2\sin^2\theta}}\right]d\theta\,;$$
$$\therefore\ s=-\frac{c^2}{a}F(\theta,e)+\frac{c^2}{a}\cdot\frac{a^2+b^2+c^2}{b^2+c^2}\,\Pi\left(\theta,e,\frac{a^2-b^2}{b^2+c^2}\right),$$

where e is the eccentricity. And thus the arc of this curve is expressible by the elliptic integrals of the first and third kinds.

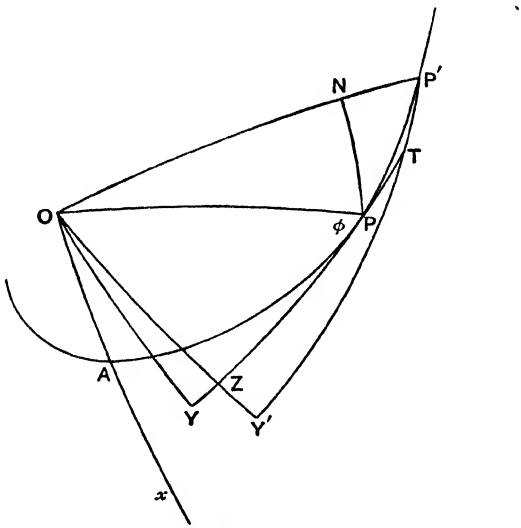


Fig. 235.

728. Curves on Spherical Surfaces in particular. Formulae for the Rectification of Curves on a Spherical Surface, analogous to the p, r and p, ψ Formulae for a Plane Curve.

Let APP' be any curve drawn upon the surface of a sphere of radius unity. Let P, P' be contiguous points, and let

arc $PP' = \delta s$. Let O be any fixed pole on the sphere, and let PY, $P'Y'$ be the great circle tangents at P and P'; OY, OY' the great circle perpendiculars to them from O, and OAx a fixed great circle cutting the curve at A, the point from which s is measured.

Let $x\hat{O}Y = \psi$, $\quad Y\hat{O}Y' = \delta\psi$, $\quad OY = p$, $\quad OY' = p + \delta p$,

$$PY = t, \quad P'Y' = t + \delta t.$$

Let OP, OP' be the great circle radii vectores of P and P', and let $O\hat{P}Y = \phi$.

Then, from the spherical triangle OYP, we have

$$\cos r = \cos p \cos t \quad \text{and} \quad \sin p = \sin r \sin \phi.$$

Let PN be the great circle perpendicular upon OP'. Thus, as in plane geometry, we have

$$\frac{dr}{ds} = \cos\phi \left(\text{viz. } Lt\, \frac{NP'}{PP'}\right),$$

and
$$s = \int \frac{dr}{\cos\phi} = \int \frac{dr}{\sqrt{1 - \dfrac{\sin^2 p}{\sin^2 r}}},$$

i.e.
$$s = \int \frac{\sin r\, dr}{\sqrt{\sin^2 r - \sin^2 p}}. \quad \text{.........................(1)}$$

Let OY' intersect PY at Z, then, from the right-angled triangle YOZ,
$$\sin OY = \cot YOZ \tan YZ,$$

i.e. to the first order, $\quad YZ = \delta\psi \sin p.$

Also to the first order,

$$P'Y' = P'P + PZ = \delta s + t - YZ$$

i.e.
$$t + \delta t = \delta s + t - \sin p\, \delta\psi.$$

And in the limit, $\quad \dfrac{ds}{d\psi} = \dfrac{dt}{d\psi} + \sin p,$

i.e.
$$s = t + \int \sin p\, d\psi. \quad \text{.......................(2)}$$

Formulae (1) and (2) are analogous to

$$s = \int \frac{r\, dr}{\sqrt{r^2 - p^2}} \quad \text{and} \quad s = t + \int p\, d\psi$$

for plane curves.

729. Convention of Sign of t. Closed Oval.

In regard to t it is necessary to make a convention with regard to sign. It will be in agreement with the convention for plane curves, Art. 531, if we fix that t is to be reckoned positive when, as in the case of Fig. 185, PY is measured from the point of contact in the direction opposite to that of increase of the arc s.

As in plane curves, it appears that if the curve considered be a closed oval on the sphere, t returns to its original value when integration is taken round the oval. Hence for a closed curve surrounding the pole, encircling it once,

$$s=\int_0^{2\pi}\sin p\,d\psi.$$

If the radius of the sphere be a instead of unity, which has been taken for convenience, the absolute length of the arc will be changed in the ratio $a:1$, so that if s' and t' be *lengths*, whilst p and r are measured by the *angles* subtended at the centre of the sphere, formulae (1) and (2) become respectively

$$s'=a\int\frac{\sin r\,dr}{\sqrt{\sin^2 r-\sin^2 p}}\quad\text{and}\quad s'=t'+a\int\sin p\,d\psi.$$

730. Ex. In the case of a **Loxodrome** cutting meridians at a constant angle α, let r, θ be the co-latitude and azimuthal angle of any current point P upon the curve.

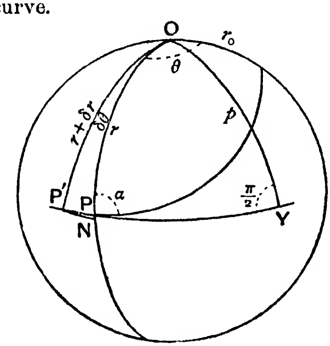


Fig. 236.

Then $\phi=\alpha$ and $\sin p=\sin r\sin\alpha$.

Hence $$s'=a\int\frac{\sin r\,dr}{\sqrt{\sin^2 r-\sin^2 p}}=\frac{a}{\cos\alpha}\cdot r,$$

a being the radius of the sphere, *i.e.*

Arc of curve measured from the pole $=\dfrac{\text{arcual radius vector } OP}{\cos\alpha}$, ...(a)
as in the case of the equiangular spiral upon a plane. (See also Art. 548.)

We also have in this curve

$$Lt\,\frac{a\sin r\,\delta\theta}{a\,\delta r}=\tan\alpha,$$

i.e.
$$\frac{dr}{\sin r}=\cot\alpha\,d\theta,$$

i.e.
$$\log\left(\frac{\tan\frac{r}{2}}{\tan\frac{r_0}{2}}\right)=\theta\cot\alpha,$$

if $r=r_0$, when $\theta=0$, *i.e.* $\tan\frac{r}{2}=\tan\frac{r_0}{2}e^{\theta\cot\alpha}$, ...(b)
which is another form of the property $l=\text{gd}(\theta\cot\alpha)$, ...(c)
already established in Art. 722, a relation between the latitude and longitude analogous to that between y and x in a Cartesian equation.

731. To find $\sin p$.

The expression for $\sin p$ in terms of ψ which is required in the integration of Art. 729 may be found as follows. Take the z-axis through O, the pole of the curve. Let C be the

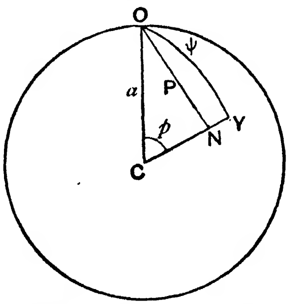


Fig. 237. (See also Fig. 235.)

centre of the sphere and $F(x, y, z)=0$ be the equation of the cone which cuts the sphere $x^2+y^2+z^2=a^2$ in the given curve.

Then F is a homogeneous function of x, y and z.

The tangent plane to the cone at the point x', y', z' of the curve is
$$xF_{x'}+yF_{y'}+zF_{z'}=0.$$

The equation of a perpendicular plane COY through the z-axis is
$$xF_{y'}-yF_{x'}=0.$$

Hence $\quad \tan\psi = \dfrac{F_{y'}}{F_{x'}}, \quad$ *i.e.* $\dfrac{F_{x'}}{\cos\psi} = \dfrac{F_{y'}}{\sin\psi}$. ....(A)

And the perpendicular P ($=ON$, Fig. 237), upon the tangent plane from the pole O, whose coordinates are $(0, 0, a)$, is

$$P = \frac{aF_{z'}}{\sqrt{F_{x}'^2 + F_{y}'^2 + F_{z}'^2}}. \quad \text{......................(B)}$$

From $F=0$ and equations (A) and (B), the ratios $x':y':z'$ are to be eliminated, and there will result a relation between P and ψ, say,
$$P = af(\psi).$$

Again,
$$\frac{P}{a} = \sin p.$$

Hence the relation required is $\sin p = f(\psi)$.

732. **Relation with the Polar Curve.**

Let any curve be drawn upon a sphere of centre O and radius r, and let the cone with vertex O, and passing through

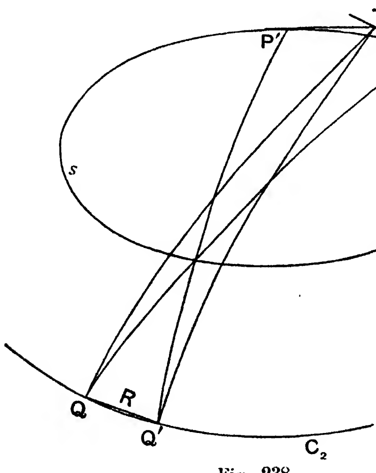


Fig. 238.

the curve, be drawn. Let a plane through the centre of the sphere, and therefore cutting the sphere in a great circle, roll upon the surface of the cone. The poles of this plane then trace out two equal loci on the surface of the sphere. Either of these equal and similar loci is called the polar curve

of the given curve. The great circle arcs which are the lines of intersection of the sphere and the plane touch the curve as the plane rolls, and are great circle tangents.

Let Q, Q' be two positions of one of the poles corresponding to the great circles PT, $P'T$, intersecting at T and touching a curve C_1 drawn upon the sphere. Let the curve locus of Q be referred to as the curve C_2. Drawing the great circles PQ, TQ, TQ', $P'Q'$, we have

$$PQ=TQ, \text{ both quadrants,}$$

$$TQ'=P'Q', \text{ both quadrants,}$$

and $$TQ=TQ', \text{ both being quadrants.}$$

Hence, in the limit when P' and P are indefinitely close, T ultimately lies upon C_1, and is the pole of a tangent plane to the cone with vertex at O, which cuts the sphere in C_2. Hence the relation between the two curves is reciprocal. Each one is the locus of the poles of tangent planes of the cone which defines the other. If QRQ' be the great circle arc joining Q and Q', T is its pole, and the poles of all great circles which pass through T lie on QRQ' or QRQ' produced, that is the great circle chord QRQ' of the arc QQ' of C_2 is the path of the poles of great circles through T.

The figure bounded by the arc QQ' of the C_2 locus and the great circle arc $Q'RQ$ is thus the reciprocal of the figure bounded by the arc PP' of the C_1 locus and the two great circle tangents TP, TP'. Also the angle between two great circles being the same as that subtended at the centre by their poles, we have

$$\text{Angle } PTP'=\pi-Q\hat{O}Q', \text{ i.e. } \pi-Q\hat{R}Q'.$$

733. **A Theorem given by Schulz.**

Let a circumscribed polygon consisting of an infinitely large number of infinitesimal great circle tangents be drawn to the one curve C_1, and let the reciprocal inscribed polygon of great circle chords be drawn in C_2. Then, if the angles of the one be A, B, C, D, ..., and the angular measures of the corresponding sides of the other be a', b', c', d', ..., we have

$$A=\pi-a', \quad B=\pi-b', \text{ etc.}$$

We have Area of the polygon $ABCD\ldots = [\Sigma A - (n-2)\pi]r^2$
(Todhunter and Leathem, *Spherical Trigonometry*, Art. 129)

$$= [\Sigma(\pi - a') - (n-2)\pi]r^2$$
$$= (2\pi - \Sigma a')r^2$$
$$= 2(\pi - s')r^2,$$

if s' be the angular semiperimeter of the polygon $A'B'C'D'\ldots$.

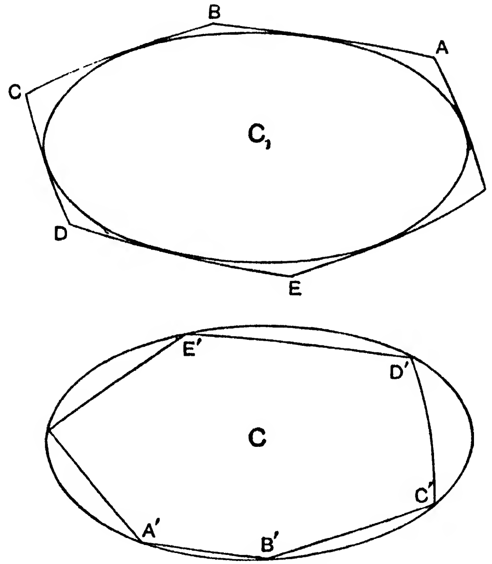


Fig. 239.

This remarkable relation is stated by Todhunter and Leathem as "referred to" by Schulz, *Sphärik*, ii., p. 241.* The authorship does not appear to be clear. Proceeding to the limit when the sides are indefinitely small, if (C_1), (P_1) be the area and linear perimeter of C_1, and (C_2), (P_2) the area and linear perimeter of C_2, we have

$$(C_1) + r(P_2) = 2\pi r^2 = \text{half the surface of the sphere,}$$

and similarly $$(C_2) + r(P_1) = 2\pi r^2,$$

that is $2\pi r^2 - (C_1) = r(P_2)$ and $2\pi r^2 - (C_2) = r(P_1)$.

Thus when the area of the one curve can be found, the perimeter of the other can be found and *vice versa*.

* See also Williamson's *Integral Calculus*, Art. 188.

It appears also that the area included between either curve and any great circle which it does not cut is equal to a

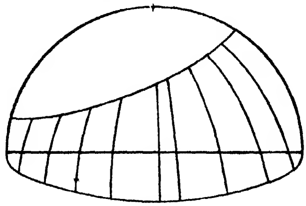


Fig. 240.

rectangle of length the perimeter of the other curve and breadth the radius of the sphere.

734. **Formula analogous to that for the Area of a Plane Curve in Polars.**

It is a well-known result in the mensuration of a spherical surface that the area of any belt on a sphere is equal to the corresponding belt on the enveloping cylinder whose axis is perpendicular to the bounding planes of the belt. Let APA'

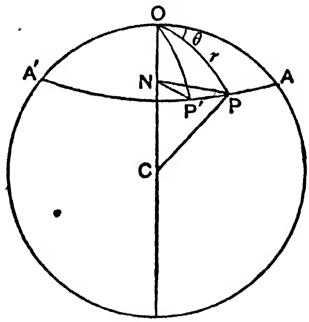


Fig. 241.

be any small circle of a sphere of radius a. Let O be the pole of the circle and OP any great circle radius vector from O of length r, subtending an angle ρ at the centre. Then the area of the spherical cap cut off by the small circle

$$=2\pi a(a-a\cos\rho)=2\pi a^2(1-\cos\rho).$$

Let the azimuthal angle of OP be θ. Then we have for the area between OP and OP' for which θ is increased to $\theta+\delta\theta$,

$$\text{Area } OPP'=\frac{\delta\theta}{2\pi}\times 2\pi a^2(1-\cos\rho)$$
$$=a^2(1-\cos\rho)\delta\theta,$$

analogous to the result $\frac{1}{2}r^2\delta\theta$ for a plane (and indeed becoming $\frac{1}{2}r^2\delta\theta$ when we put $\frac{r}{a}$ for ρ and the radius a becomes ∞).

Hence, taking ρ, θ as coordinates, we have for the area of any portion of the spherical surface bounded by a curve on the sphere, and the meridians $\theta=\theta_1$, $\theta=\theta_2$,

$$A=a^2\int_{\theta_1}^{\theta_2}(1-\cos\rho)\,d\theta,$$

in the same way as $A=\frac{1}{2}\int r^2 d\theta$ for a plane area (Art. 407).

If the curve be an oval encircling the pole O once,

$$A=a^2\int_0^{2\pi}(1-\cos\rho)\,d\theta=2\pi a^2-a^2\int_0^{2\pi}\cos\rho\,d\theta.$$

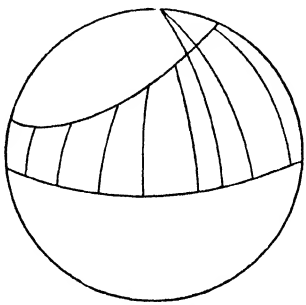


Fig. 242.

The area therefore between the curve and the equatorial plane of O is

$$a^2\int_0^{2\pi}\cos\rho\,d\theta,$$

or if we use l for the latitude, *i.e.* the complement of ρ, and θ for the longitude or azimuthal angle,

$$\text{Area}=a^2\int_0^{2\pi}\sin l\,d\theta.$$

If, then, this integral be evaluated for the polar or reciprocal curve C_2, the result will be aP_1, *i.e.*

$$\text{Perimeter}=P_1=a\int_0^{2\pi}\sin l\,d\theta,$$

$(l,\ \theta)$ being the latitude and longitude of a point on the reciprocal curve.

Illustrative Examples.

Ex. 1. To test this result in a known case, take C_1 as a small circle with pole at O and of angular radius ρ. Its perimeter is obviously

$$2\pi a \sin \rho.$$

The polar curve is another small circle of angular radius $\frac{\pi}{2} - \rho$, and therefore the latitude of any point on it is ρ, in this case a constant. The formula gives

$$P_1 = a\int_0^{2\pi} \sin \rho \, d\theta = 2\pi a \sin \rho,$$

which is in agreement with the stated result.

Ex. 2. Find the length of the spiral, traced on a sphere, whose reciprocal is defined by the equation $4\rho = \theta$ corresponding to limits for θ from 0 to 2π, ρ and θ having the meanings assigned to them in Art. 734.

The area between the reciprocal spiral and the equatorial plane is

$$a^2 \int_0^{2\pi} \cos \frac{\theta}{4} \, d\theta = 4a^2.$$

Hence the perimeter required $= 4a$, *i.e.* twice the diameter of the sphere. (Fig. 243.)

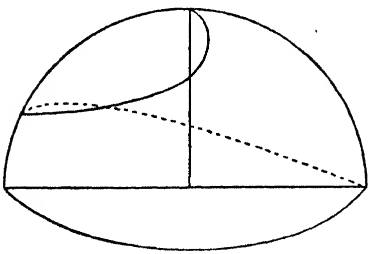


Fig. 243.

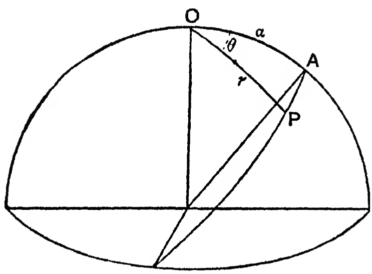


Fig. 244.

Ex. 3. To find the area bounded by any arc of a great circle and two spherical radii vectores.

Let the plane of the great circle be at right angles to the plane of the paper and cut the meridian in the plane of the paper at a point A whose co-latitude is α. (Fig. 244.)

Then the equation of the great circle is

$$\cos \theta = \cot \rho \tan \alpha,$$

from the spherical triangle OPA, right angled at A.

Then we have

$$\text{Area} = a^2 \int_{\theta_1}^{\theta_2} (1 - \cos \rho) \, d\theta$$

$$= a^2(\theta_2 - \theta_1) - a^2 \int_{\theta_1}^{\theta_2} \frac{\cos \theta \cot \alpha}{\sqrt{\cos^2 \theta \cot^2 \alpha + 1}} \, d\theta,$$

and the integral

$$\int \frac{\cos\theta}{\sqrt{\cot^2\alpha\cos^2\theta+1}}\,d\theta=\int\frac{d\sin\theta}{\sqrt{\operatorname{cosec}^2\alpha-\cot^2\alpha\sin^2\theta}}$$

$$=\tan\alpha\int\frac{d\sin\theta}{\sqrt{\sec^2\alpha-\sin^2\theta}}$$

$$=\tan\alpha\sin^{-1}(\sin\theta\cos\alpha).$$

Hence the area between two radii making angles θ_1 and θ_2 with the meridian in the plane of the paper is

$a^2(\theta_2-\theta_1)-a^2[\sin^{-1}(\sin\theta_2\cos\alpha)-\sin^{-1}(\sin\theta_1\cos\alpha)]$. (See Art. 781.)

735. The Case of a Sphero-conic.

DEF. A sphero-conic is the line of intersection of a cone of the second degree with a sphere whose centre is at the vertex of the cone.

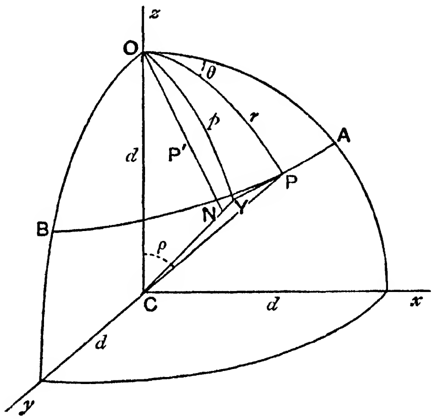


Fig. 245.

Let the equation of the sphere be $x^2+y^2+z^2=d^2$, and that of the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2},\quad (a>b).$$

The reciprocal cone has for equation

$$a^2x^2+b^2y^2=c^2z^2.$$

Putting ρ for the co-latitude and θ for the azimuthal angle of any point, we have $x=d\sin\rho\cos\theta$, $y=d\sin\rho\sin\theta$, $z=d\cos\rho$, and the equations of the sphero-conic and its reciprocal become respectively

$$\frac{\cot^2\rho}{c^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\quad\text{and}\quad c^2\cot^2\rho=a^2\cos^2\theta+b^2\sin^2\theta$$

in ρ, θ coordinates.

The area A_1 bounded by the arc of the sphero-conic

$$\frac{\cot^2\rho}{c^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$$

and the meridians $\theta=0$, $\theta=\theta$ is given by

$$A_1=d^2\int_0^\theta(1-\cos\rho)\,d\theta$$

$$=d^2\int_0^\theta\left\{1-c\sqrt{\frac{a^2\sin^2\theta+b^2\cos^2\theta}{a^2(b^2+c^2)\sin^2\theta+b^2(a^2+c^2)\cos^2\theta}}\right\}d\theta$$

$$=d^2(\theta-cI_1),\text{ say};$$

and putting

$$\frac{a\sin\theta}{\sin\chi}=\frac{b\cos\theta}{\cos\chi},\quad i.e.\ \tan\theta=\frac{b}{a}\tan\chi,$$

$$d\theta=\frac{ab\,d\chi}{a^2\cos^2\chi+b^2\sin^2\chi};$$

whence

$$I_1=\int_0^\chi\frac{ab\,d\chi}{a^2-(a^2-b^2)\sin^2\chi}\,\frac{1}{\sqrt{(a^2+c^2)-(a^2-b^2)\sin^2\chi}}$$

$$=\frac{b}{a\sqrt{a^2+c^2}}\Pi\left(\chi,\ \sqrt{\frac{a^2-b^2}{a^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right)$$

and

$$A_1=d^2.\,\theta-\frac{d^2bc}{a\sqrt{a^2+c^2}}\Pi\left\{\tan^{-1}\left(\frac{a}{b}\tan\theta\right),\ \sqrt{\frac{a^2-b^2}{a^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right\},$$

and is therefore expressed in terms of a Legendrian integral of the third species.

For the reciprocal sphero-conic $c^2\cot^2\rho=a^2\cos^2\theta+b^2\sin^2\theta$ the area A_2 bounded by the arc and the meridians $\theta=\theta$ and $\theta=\frac{\pi}{2}$ is given by

$$A_2=d^2\int_\theta^{\frac{\pi}{2}}(1-\cos\rho)\,d\theta$$

$$=d^2\int_\theta^{\frac{\pi}{2}}\left\{1-\sqrt{\frac{a^2\cos^2\theta+b^2\sin^2\theta}{(a^2+c^2)\cos^2\theta+(b^2+c^2)\sin^2\theta}}\right\}d\theta$$

$$=d^2\left(\frac{\pi}{2}-\theta-\Big[I_2\Big]_\theta^{\frac{\pi}{2}}\right),\text{ say};$$

and putting

$$\frac{b\sin\theta}{\cos\chi}=\frac{a\cos\theta}{\sin\chi},\quad i.e.\ \tan\theta=\frac{a}{b}\cot\chi,$$

we have

$$\left[I_2\right]_\theta^{\frac{\pi}{2}}=\frac{b^2}{a\sqrt{b^2+c^2}}\,\Pi\left(\chi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right)\quad\text{(Art. 388, Ex. 7)};$$

whence

$$A_2=d^2\left[\frac{\pi}{2}-\theta-\frac{b^2}{a\sqrt{b^2+c^2}}\,\Pi\left\{\tan^{-1}\left(\frac{a}{b}\cot\theta\right),\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{a^2+b^2}},\ -\frac{a^2-b^2}{a^2}\right\}\right];$$

and the area of the same curve from $\theta=0$ to $\theta=\theta$ is

$$d^2\left(\theta-\left[I_2\right]_0^\theta\right)=d^2\{\theta-(\Pi_1-\Pi)\},$$

where Π is the same elliptic integral as occurs in the value of A_2 and Π_1 is its complete value.

736. Again, for the Rectification of

$$\frac{\cot^2\rho}{c^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2},$$

the tangent plane to the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

at any point $P(x', y', z')$ of the sphero-conic APB (Fig. 245) is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{zz'}{c^2}, \quad\text{..........................(1)}$$

and the perpendicular plane OCY through the z-axis is

$$x\frac{y'}{b^2}-y\frac{x'}{a^2}=0, \quad\text{..........................(2)}$$

giving
$$\tan\psi=\frac{a^2y'}{b^2x'},$$

where ψ is the azimuthal angle of the plane OCY, *i.e.*

$$\frac{x'}{a^2\cos\psi}=\frac{y'}{b^2\sin\psi}.$$

Also the perpendicular ON $(=P')$ from the pole O upon the tangent plane at P, viz. CPY, is given by

$$\frac{c}{d}P'=\frac{\dfrac{z'}{c}}{\sqrt{\dfrac{x'^2}{a^4}+\dfrac{y'^2}{b^4}+\dfrac{z'^2}{c^4}}}=\sqrt{\frac{\dfrac{x'^2}{a^2}+\dfrac{y'^2}{b^2}}{\dfrac{x'^2}{a^4}+\dfrac{y'^2}{b^4}+\left(\dfrac{x'^2}{a^2}+\dfrac{y'^2}{b^2}\right)\dfrac{1}{c^2}}}$$

Therefore, if p be the angle OCY subtended at C by the great circle arc OY, $P'=d\sin p$, and we have

$$\sin p=\sqrt{\frac{a^2\cos^2\psi+b^2\sin^2\psi}{(a^2+c^2)\cos^2\psi+(b^2+c^2)\sin^2\psi}};$$

$$\therefore \int_\psi^{\frac{\pi}{2}}\sin p\,d\psi=\frac{b^2}{a\sqrt{b^2+c^2}}\Pi\left(\psi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right).$$

(Art. 388, Ex. 7.)

Hence, if s and t be the *lengths* of the arcs of the spheroconic from P to B, and of the 'tail' PY respectively (Fig. 245),

$$s=t+d\int_\psi^{\frac{\pi}{2}}\sin p\,d\psi$$

and $$s=t+\frac{b^2 d}{a\sqrt{b^2+c^2}}\Pi\left(\psi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right),$$

and t remains to be found

Now t is the arcual measure of the great circle arc PY.

The equations of CY, CP (C being the centre of the sphere) are, from (1) and (2),

$$\frac{x}{\frac{x'}{a^2}}=\frac{y}{\frac{y'}{b^2}}=\frac{z}{\frac{c^2}{z'}\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)}\quad\text{and}\quad\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}.$$

Hence

$$\cos\left(\frac{t}{d}\right)=\cos Y\hat{C}P=\frac{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+c^2\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)}{d\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}\sqrt{1+\frac{c^4}{z'^2}\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)}}$$

$$=\frac{c^2\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}\right)}{d\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}\cdot\frac{c^2}{z'}\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}}$$

$$=\frac{z'}{d}\sqrt{\frac{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}}.$$

Also $\dfrac{\frac{x'}{a}}{a\cos\psi}=\dfrac{\frac{y'}{b}}{b\sin\psi}=\dfrac{\frac{z'}{c}}{\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}}$

$$=\frac{d}{\sqrt{a^2(a^2+c^2)\cos^2\psi+b^2(b^2+c^2)\sin^2\psi}};$$

$$\therefore\ \cos\left(\frac{t}{d}\right)=\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}\,\sqrt{\frac{(a^2+c^2)\cos^2\psi+(b^2+c^2)\sin^2\psi}{a^2(a^2+c^2)\cos^2\psi+b^2(b^2+c^2)\sin^2\psi}}.$$

Hence t is found, viz.

$$t=-d\cos^{-1}\left\{\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}\,\sqrt{\frac{(a^2+c^2)\cos^2\psi+(b^2+c^2)\sin^2\psi}{a^2(a^2+c^2)\cos^2\psi+b^2(b^2+c^2)\sin^2\psi}}\right\},$$

the negative sign being prefixed because PY is measured from P in the direction of the measurement of the arc increasing from P to B. (See Art. 729.) Finally then we have

$$\frac{\text{arc } PB}{d}=-\cos^{-1}\left\{\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}\right.$$
$$\left.\times\sqrt{\frac{(a^2+c^2)\cos^2\psi+(b^2+c^2)\sin^2\psi}{a^2(a^2+c^2)\cos^2\psi+b^2(b^2+c^2)\sin^2\psi}}\right\}$$
$$+\frac{b^2}{a\sqrt{b^2+c^2}}\,\Pi\left\{\psi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right\}.$$

737. Mr. Burstall's Theorem.

A remarkable property of the curve is established by Mr. Burstall, in vol. xviii. of the *Proceedings of the London Mathematical Society*, giving a result analogous to that of Fagnano for the ellipse.

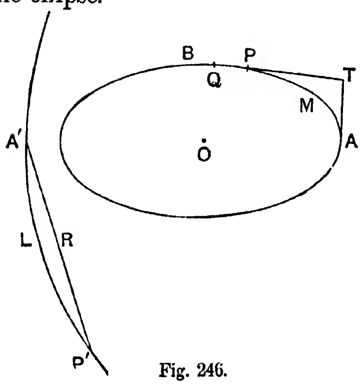


Fig. 246.

Let AB be the sphero-conic represented for convenience upon a plane, and let $A'B'$ be an arc of the reciprocal

sphero-conic, A being an end of the major axis of the one and A' being the corresponding point on the reciprocal curve. Let P and P' be corresponding points of the sphero-conic and its reciprocal; and let $A'RP'$ be the great circle chord of the reciprocal sphero-conic; and AT, PT the great circle arcs tangential at A and P.

Then, since the areas $ATPMA$ and $A'LP'RA'$ are reciprocal areas, we have

$$d(\text{Arc } AP + \text{tangent } PT + \text{tangent } TA) = 2\pi d^2 - \text{area of } A'LP'RA'.$$

Now, putting Δ and Δ' for the spherical areas $OA'LP'$ and $OA'RP'$ respectively, $A'\hat{O}P' = \theta'$, and

$$I = \int \frac{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{\sqrt{(a^2+c^2)\cos^2\theta + (b^2+c^2)\sin^2\theta}}\, d\theta,$$

the same indefinite Legendrian integral that has occurred both in the rectification above and in the quadrature of the reciprocal curve with specified limits, we have

$$\text{Arc } AP + \text{tang. } PT + \text{tang. } TA = 2\pi d - (\Delta - \Delta')/d,$$

and

$$\Delta = d^2 \int_0^{\theta'} (1 - \cos\rho)\, d\theta, \quad \text{where } c^2\cot^2\rho = a^2\cos^2\theta + b^2\sin^2\theta,$$
$$= d^2(\theta' - I_0^{\theta'}),$$

whilst Δ' can be found free from elliptic integrals (Art. 734, Ex. 3).

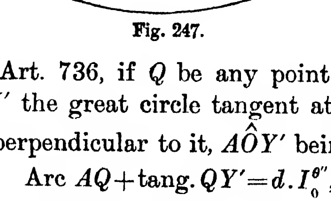


Fig. 247.

Again, as in Art. 736, if Q be any point of the original sphero-conic, QY' the great circle tangent at Q, and OY' the great circle arc perpendicular to it, $A\hat{O}Y'$ being θ'',

$$\text{Arc } AQ + \text{tang. } QY' = d \,.\, I_0^{\theta''},$$

and $\quad \text{Arc } AP + \text{tang. } TP + \text{tang. } TA = 2\pi d - (\Delta - \Delta')/d$

$$= 2\pi d + \frac{\Delta'}{d} - d\,.\,\theta' + d\,.\,I_0^{\theta'}.$$

If then we take the angles $AOQ(\theta'')$ and $A'OP'(\theta')$ equal and eliminate the integral, we have

$$\text{Arc } AP + \text{tang. } TA + \text{tang. } TP + d\,.\,\theta' - \frac{\Delta'}{d}$$
$$= 2\pi d + \text{arc } AQ + \text{tang. } QY',$$

or

$$\text{Arc } AQ - \text{arc } AP = \text{tang. } TA + \text{tang. } TP - \text{tang. } QY' + d\,.\,\theta' - \Delta'/d$$
$$- \text{circumf. of a great circle,}$$

giving the difference of two arcs in terms of certain arcs of circles and Δ'.

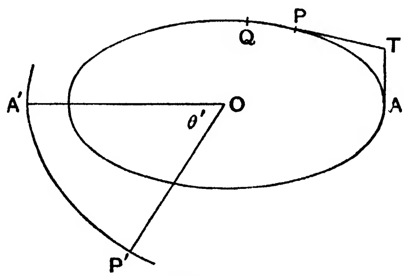


Fig. 248.

Hence we have the difference of the arcs AP, AQ expressed in terms of elementary functions, free from elliptic integrals, which is Mr. Burstall's result, and in its peculiarity resembles Fagnano's result for a plane ellipse.

738. **Artifices for the Construction of Rectifiable Twisted Curves.**

Some artifices for the construction of rectifiable twisted curves may be noted.

1. If we take

$$x = \int u^2 dt, \quad y = \sqrt{2}\int uv\, dt, \quad z = \int v^2 dt,$$

where u, v are any functions of t at our choice, we have

$$\left(\frac{ds}{dt}\right)^2 = u^4 + 2u^2v^2 + v^4 \quad \text{and} \quad \frac{ds}{dt} = u^2 + v^2.$$

Hence $$s = \int (u^2 + v^2)\, dt = x + z + \text{const.}^*$$

* For a very similar method, viz. taking

$$y = \int \sqrt{2f(x)}\, dx, \quad z = \int f(x)\, dx,$$

see Williamson's *Int. Calc.*, p. 244.

E.g. consider the line of intersection of the cylinders

$$3x=z^3, \quad y\sqrt{2}=z^2.$$

Putting $\qquad z=t, \quad y=\sqrt{2}\,\dfrac{t^2}{2}, \quad x=\dfrac{t^3}{3},$

$$\frac{dx}{dt}=t^2, \quad \frac{dy}{dt}=\sqrt{2}\,t, \quad \frac{dz}{dt}=1,$$

we have the case

$$u=t, \quad v=1 \quad \text{and} \quad s=x+z+\text{const.}$$

2. If we take $\qquad x=\int(u-v)(u-w)\,dt,$

$$y=\int(v-w)(v-u)\,dt,$$

$$z=\int(w-u)(w-v)\,dt,$$

where u, v, w are any functions of t at our choice, then, since

$$\Sigma(u-v)^2(u-w)^2=(\Sigma u^2-\Sigma vw)^2,$$

we have $\qquad \dfrac{ds}{dt}=\Sigma u^2-\Sigma vw=\dfrac{dx}{dt}+\dfrac{dy}{dt}+\dfrac{dz}{dt},$

and $\qquad s=x+y+z+C.$

E.g. taking $\qquad u=0, \quad v=1, \quad w=t,$

$$\frac{dx}{dt}=t, \quad \frac{dy}{dt}=1-t, \quad \frac{dz}{dt}=t^2-t,$$

$$x=\frac{t^2}{2}, \quad y=t-\frac{t^2}{2}, \quad z=\frac{t^3}{3}-\frac{t^2}{2};$$

whence we have $\qquad x+y=t\,; \quad x+z=t^3/3\,;$

$$\therefore\ (x+y)^2=2x, \quad 3(z+x)=(x+y)^3,$$

the equations of the curve.

And for the rectification,

$$s=\frac{t^3}{3}-\frac{t^2}{2}+t+\text{const.}=x+y+z+C,$$

and any specified limits may be taken.

3. Again, if we take

$$x=\int(v-w)^2dt, \quad y=\int(w-u)^2dt, \quad z=\int(u-v)^2dt,$$

we have $\qquad \left(\dfrac{ds}{dt}\right)^2=\Sigma(v-w)^4=2(\Sigma u^2-\Sigma vw)^2,$

and $\qquad s=\sqrt{2}\left[\int\Sigma u^2dt-\int\Sigma vw\,dt\right]+\text{const.}$

$$=\frac{1}{\sqrt{2}}\int\left(\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}\right)dt+C$$

$$=\frac{1}{\sqrt{2}}(x+y+z)+C,$$

and the values of u, v, w are at our choice, as before.

In all these cases if u, v, w be chosen as rational integral algebraic functions of t, the equations of the curve can be found and its length between any specified limits.

4. Similarly, other algebraic identities which express the sum of three squares as a constant multiple of the square of a fourth expression may be used in the same manner to construct rectifiable twisted curves.

E.g. $$\left[u^2-\left(\frac{v^2+w^2}{2u}\right)^2\right]^2+(v^2-w^2)^2+4v^2w^2=\left[u^2+\left(\frac{v^2+w^2}{2u}\right)^2\right]^2.$$

Hence, putting

$$\frac{dx}{dt}=u^2-\left(\frac{v^2+w^2}{2u}\right)^2,\quad \frac{dy}{dt}=v^2-w^2,\quad \frac{dz}{dt}=2vw,$$

with any arbitrary choice of u, v, w as functions of t, we have

$$\frac{ds}{dt}=u^2+\left(\frac{v^2+w^2}{2u}\right)^2 \quad\text{and}\quad s+x=2\int u^2\,dt.$$

It will be noted that all these methods proceed with a view to making $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2$, a perfect square and avoiding the necessity of integrating an irrational expression.

5. One type more may be given illustrating the construction of a rectifiable twisted curve upon the same plan, but of non-algebraic character. Taking u, v, w any arbitrary functions of t, put

$$x=\int\frac{du}{dt}\sin v\sin w\,dt,\quad y=\int\frac{du}{dt}\sin v\cos w\,dt,\quad z=\int\frac{du}{dt}\cos v\,dt.$$

Then $$\frac{ds}{dt}=\frac{du}{dt}\quad\text{and}\quad s=u+\text{const}.$$

E.g. taking $$v=w=t\quad\text{and}\quad u=\frac{t^2}{2},$$

$$\frac{dx}{dt}=t\sin^2 t,\quad \frac{dy}{dt}=t\sin t\cos t,\quad \frac{dz}{dt}=t\cos t.$$

Then $$\frac{ds}{dt}=t,\quad s=\frac{t^2}{2}+C,$$

the curve being
$$\left.\begin{aligned} 8x&=2t^2-2t\sin 2t-\cos 2t,\\ 8y&=-2t\cos 2t+\sin 2t,\\ z&=\phantom{2t^2-2t\cos 2t+{}}t\sin t+\cos t.\end{aligned}\right\}$$

Methods 1, 2, 3, 4 either give rise to unicursal twisted curves, viz. those in which the coordinates x, y, z can be expressed as rational algebraic functions of a single parameter t or may be made to give rise to curves in which x, y, z and s are irrational functions of t, this depending upon the choice made for u, v, w.

739. Generalised Formulae.

If the Cartesian coordinates of a point x, y, z be expressed as functions of any other three independent parameters u, v, w, as

$$x=f_1(u, v, w), \quad y=f_2(u, v, w), \quad z=f_3(u, v, w),$$

then $$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw, \quad dy=\text{etc.}, \quad dz=\text{etc.}$$

And if we write

$$a\equiv\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2, \quad b\equiv\left(\frac{\partial x}{\partial v}\right)^2++, \quad c\equiv\left(\frac{\partial x}{\partial w}\right)^2++,$$

$$f\equiv\frac{\partial x}{\partial v}\frac{\partial x}{\partial w}+\frac{\partial y}{\partial v}\frac{\partial y}{\partial w}+\frac{\partial z}{\partial v}\frac{\partial z}{\partial w}, \quad g=\frac{\partial x}{\partial w}\frac{\partial x}{\partial u}++, \quad h=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}++,$$

we have, for the element of distance ds between x, y, z and $x+dx, y+dy, z+dz$,

$$ds^2=a\,du^2+b\,dv^2+c\,dw^2+2f\,dv\,dw+2g\,dw\,du+2h\,du\,dv,$$

and for *two* assigned relations between u, v and w, defining a linear path for x, y, z, we have the rectification formula

$$s=\int[a\,du^2+b\,dv^2+c\,dw^2+2f\,dv\,dw+2g\,dw\,du+2h\,du\,dv]^{\frac{1}{2}}.$$

740. If *one relation only* between u, v and w be assigned, x, y, z travels on an assigned surface. Let the relation be

$$\chi(u, v, w)=0.$$

Then $$\frac{\partial\chi}{\partial u}du+\frac{\partial\chi}{\partial v}dv+\frac{\partial\chi}{\partial w}dw=0,$$

and this being a linear relation between du, dv, dw, one of the letters u, v, w, and one of the differentials du, dv, dw may be eliminated, and the square of the linear element ds may then be expressed as

$$ds^2=E\,du^2+2F\,du\,dv+G\,dv^2$$

where the forms of x, y, z are now

$$x=\phi_1(u, v), \quad y=\phi_2(u, v), \quad z=\phi_3(u, v).$$

The values of E, F, G derived from these equations are

$$E=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2, \quad F=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$

$$G=\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2.$$

741. The quantity $EG-F^2$ is essentially positive.

For
$$EG-F^2=\begin{vmatrix} \dfrac{\partial y}{\partial u}, & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial y}{\partial v}, & \dfrac{\partial z}{\partial v} \end{vmatrix}^2 + \text{two similar expressions}$$
$$=\left\{\frac{\partial(y, z)}{\partial(u, v)}\right\}^2+\left\{\frac{\partial(z, x)}{\partial(u, v)}\right\}^2+\left\{\frac{\partial(x, y)}{\partial(u, v)}\right\}^2$$
$$=J_1^2+J_2^2+J_3^2, \text{ say, and is positive.}$$

742. Eliminating du, dv from the equations
$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv,\quad dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv,\quad dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{dv}dv,$$
we have $J_1dx+J_2dy+J_3dz=0$, identically, viz. the differential equation of the surface on which the curve lies.

743. Dr. Salmon (*Solid Geom.*, p. 252) shows that the differential equation of the lines of curvature is
$$\begin{vmatrix} dx & dy & dz \\ J_1 & J_2 & J_3 \\ dJ_1 & dJ_2 & dJ_3 \end{vmatrix}=0,$$
and obtains in terms of u and v a formula for the evaluation of the principal radii of curvature.

744. Now ds^2 is the square of the linear element connecting the point u, v with the point $u+\delta u$, $v+\delta v$, and lies on the surface $\quad x=\phi_1(u, v), \quad y=\phi_2(u, v), \quad z=\phi_3(u, v)$.

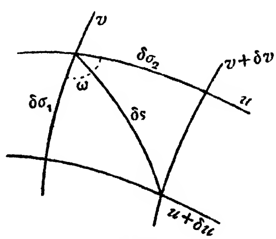


Fig. 249.

If we travel along a line for which v is constant, we have $d\sigma_1=\sqrt{E}\,du$, and if we travel along a line for which u is constant, we have $d\sigma_2=\sqrt{G}\,dv$, and ds is the corresponding

diagonal of the infinitesimal parallelogram whose adjacent edges are $d\sigma_1$, $d\sigma_2$. Let ω be the angle between them.

Then $$ds^2 = E\,du^2 + 2\sqrt{EG}\,du\,dv\cos\omega + G\,dv^2;$$
whence it appears that
$$\cos\omega = \frac{F}{\sqrt{EG}} \quad\text{and}\quad \therefore\ \sin\omega = \frac{\sqrt{EG - F^2}}{\sqrt{EG}},$$
and that the area of the elementary parallelogram
$$= d\sigma_1\,d\sigma_2\sin\omega = \sqrt{EG - F^2}\,du\,dv.$$

We therefore have also a formula for the quadrature of the surface, viz.
$$S = \iint\sqrt{EG - F^2}\,du\,dv$$
$$= \iint\sqrt{J_1^2 + J_2^2 + J_3^2}\,du\,dv.$$

When the two families of curves on the surface, viz. $u = \text{const.}$, $v = \text{const.}$, cut orthogonally, we have
$$\cos\omega = 0 \quad\text{and}\quad F = 0,$$
and $$s = \int\sqrt{E\,du^2 + G\,dv^2}, \quad S = \iint\sqrt{EG}\,du\,dv.$$

This will necessarily be so, for instance, when $u = \text{const.}$, $v = \text{const.}$ are the equations of the lines of curvature on the surface.

PROBLEMS.

1. Show that the equations of a Rhumb line on a sphere of radius r may be written as
$$x^2 + y^2 + z^2 = r^2,$$
$$(x^2 + y^2)^{\frac{1}{2}}\cosh\left(n\tan^{-1}\frac{y}{x}\right) = r.$$

2. Show that the curve of intersection of the cylinders
$$y^2 = 8ax, \quad x = ae^{\frac{z}{a}},$$
is given by $$s = x + z + \text{const.}$$

3. A sphere of diameter K touches the plane of an ellipse of principal axes a, b at its centre C. A is the other end of the diameter of the sphere through C. The ellipse is projected on to the sphere by lines through A. Show that the length of the curve so described will be
$$4\int_0^{\frac{\pi}{2}} \frac{K^2\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}{K^2 + a^2\cos^2\phi + b^2\sin^2\phi}\,d\phi.$$ [St. John's, 1884.]

4. A curve is drawn upon the surface of a sphere such that

$$\phi \sin \theta = \text{const.},$$

ϕ being the longitude and θ the co-latitude of any point.

Show that $s = a \log\left(\dfrac{\tan \theta_1/2}{\tan \theta_2/2}\right)$ is the length of the arc between points where $\theta = \theta_1$ and $\theta = \theta_2$, and a is the radius of the sphere.

Give a sketch showing the nature of the curve $\phi \sin \theta = 1$ upon the sphere $r = a$.

5. Show that the line of intersection of the sphere

$$r = c \cos \theta$$

and the cone $$\tan \theta = \frac{a}{c} e^{\phi \cot a}$$

is rectifiable, and that $$s = c\theta \sec a.$$

Also show that the conical projection of this curve on the sphere upon the tangent plane at the end of the diameter remote from the origin, the origin being the pole of projection, is an equiangular spiral. Hence deduce the same result by inversion.

6. Show that the curve of intersection of the sphere

$$x^2 + y^2 + z^2 - 2az = 0$$

and the cone $$(2x^2 + 2y^2 + zx)^2 = z^2(x^2 + y^2)$$

projects conically from the origin into a cardioide upon the plane $z = 2a$. Hence obtain the rectification of the twisted curve.

7. Show that the length of the arc of intersection of the cylinders

$$\left.\begin{aligned} x^2 &= 2y, \\ x^3 &= 6z, \end{aligned}\right\}$$

measured from the origin to any point x, y, z, is $x + z$.

8. Show that for the curve

$$\frac{60x}{-45 - 40t^2 - 12t^4} = \frac{3y}{3 - t^2} = \frac{z}{t} = t,$$

the arc measured from the origin, is given by

$$s + x = \tfrac{1}{2}\sqrt{z} + \text{const.}$$

9. In the curve for which

$$\frac{dx}{dt} = (1 - t)(1 - t^2), \quad \frac{dy}{dt} = -t(1 - t)^2, \quad \frac{dz}{dt} = t(1 - t)^2(1 + t),$$

show that $$s = x + y + z + C.$$

10. Show that in the curve of intersection of

$$r = a\cos\theta \quad \text{and} \quad \cos 2\theta = \tan^2\phi,$$

$$s = \frac{a}{2\sqrt{2}}\left[2E\left(\chi, \frac{1}{\sqrt{2}}\right) - \frac{\sin\chi\cos\chi}{\sqrt{1-\frac{1}{2}\sin^2\chi}}\right],$$

where $\sin\chi = \sqrt{2}\sin\phi.$

Show that the inverse of this curve with regard to the origin is a lemniscate, the constant of inversion being a.

11. Show that the rectification of the line of intersection of

$$x^2 + y^2 + z^2 = cz \quad \text{and} \quad y^2 = 4cx$$

is given by

$$s = \frac{c}{2^{\frac{1}{4}}3^{\frac{1}{2}}}\left\{\frac{1}{\sqrt{\cos\frac{\pi}{12}}}\tan^{-1}\left(\frac{\sin\theta}{2^{\frac{1}{4}}\sqrt{\cos\frac{\pi}{12}}}\right) + \frac{1}{\sqrt{\sin\frac{\pi}{12}}}\left(\tanh^{-1}\frac{\sin\theta}{2^{\frac{1}{4}}\sqrt{\sin\frac{\pi}{12}}}\right)\right\}$$

where $\tan\theta = \sqrt{\frac{x}{c}},$

and show that this curve can be inverted into a parabola lying upon a tangent plane to the sphere.

12. A Loxodrome is drawn on a sphere to cut all the meridians at the same constant angle α; show that the area of the surface of the sphere, included between any arc of this curve and the two meridians through its ends, is

$$a^2\tan\alpha\log\frac{1+\sin\psi_1}{1+\sin\psi_2},$$

where ψ_1 and ψ_2 are the latitudes of the ends of the arc and a is the radius of the sphere. [Ox. II. P., 1900.]

CHAPTER XXI.

VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION, AND THEIR CENTROIDS.

745. Volumes.

Supposing the z-axis to be the axis of revolution, the typical equation of such a surface is

$$x^2+y^2=f(z).$$

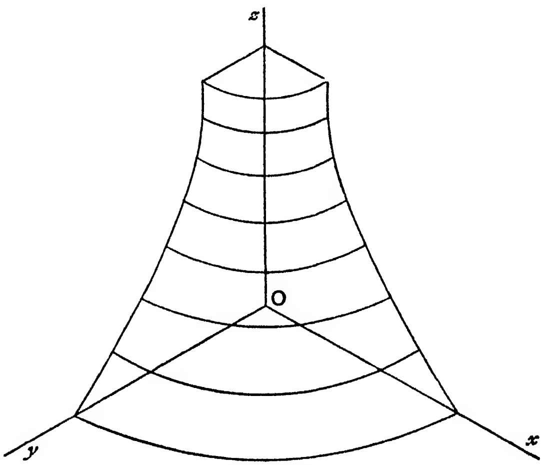


Fig. 250.

It is formed by the revolution about the z-axis of the curve $y^2=f(z)$ which lies in the y-z plane.

It was shown in Art. 24 that the solid contained by this surface and the planes $z=z_1$, $z=z_2$, is to be obtained by the formula

$$V=\int_{z_1}^{z_2}\pi y^2\,dz,$$

y being the perpendicular from any point of the revolving curve upon the axis of revolution.

It is obvious that if we regard the surface as defined by its three-dimension equation $x^2+y^2=f(z)$, we must replace the y^2 and the dx of Art. 12 by x^2+y^2 and dz respectively. The formula therefore will stand as

$$V=\pi\int_{z_1}^{z_2}(x^2+y^2)\,dz,$$

i.e.

$$\pi\int_{z_1}^{z_2}f(z)\,dz.$$

746. More generally, if the revolution be about any line AB in the plane of the curve, and if PN be any perpendicular drawn from a point P of the curve upon the line AB, and $P'N'$ be a contiguous perpendicular, the volume is expressed as

$$Lt_{NN'=0}\Sigma\pi PN^2 . NN',$$

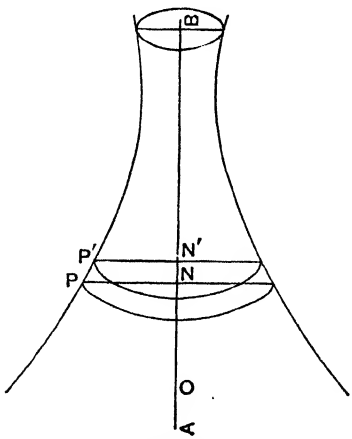


Fig. 251.

or if O be a given point on the line AB,

$$V=\int\pi PN^2\,d(ON),$$

the limits being the values of ON which mark the terminal planes of the solid formed.

747. Illustrative Examples.

1. Find the volume formed by the revolution of the loop of the curve $y^2=x^2\dfrac{a-x}{a+x}$ (Art. 403, Ex. 3) about the x-axis, *i.e.* the volume bounded by the closed portion of the surface

$$(y^2+z^2)(a+x)=x^2(a-x).$$

Here volume$=\pi\displaystyle\int_0^a x^2\frac{a-x}{a+x}\,dx.$

Putting $a+x=u$, this becomes

$$=\pi\int_a^{2a}\frac{(u-a)^2(2a-u)}{u}\,du$$

$$=\pi\int_a^{2a}\left(\frac{2a^3}{u}-5a^2+4au-u^2\right)du$$

$$=\pi\left[2a^3\log u-5a^2u+2au^2-\frac{u^3}{3}\right]_a^{2a}$$

$$=2\pi a^3[\log 2-\tfrac{2}{3}].$$

2. Find the volume of the spindle formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of the latus rectum.

Let the parabola be $y^2=4ax$.

Then the axis of revolution is $y=2x$, and $PN=\dfrac{y-2x}{\sqrt{5}}$.

Also $\quad AN=\sqrt{x^2+y^2-\dfrac{(y-2x)^2}{5}}=\sqrt{x^2+4y^2+4xy}/\sqrt{5}$

$$=\frac{2y+x}{\sqrt{5}};$$

$$\therefore\ d(AN)=\frac{dx+2dy}{\sqrt{5}}=\frac{dx+2\sqrt{\dfrac{a}{x}}\,dx}{\sqrt{5}},$$

and $\qquad PN=\dfrac{2(\sqrt{ax}-x)}{\sqrt{5}}.$

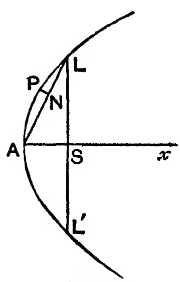


Fig. 252.

Hence

$$\text{Volume}=\int\pi PN^2\,d(AN)=\pi\int_0^a\tfrac{4}{5}x(\sqrt{a}-\sqrt{x})^2\left(1+2\frac{\sqrt{a}}{\sqrt{x}}\right)\frac{1}{\sqrt{5}}\,dx$$

$$=\frac{4\pi}{5\sqrt{5}}\times\frac{a^3}{6}=\frac{2\pi a^3\sqrt{5}}{75}.$$

748. Surfaces of Revolution.

Again, if S be the area of the curved surface of the solid traced out by the revolution of any arc AB about a given line XY in its plane, let PN, QM be two adjacent perpendiculars from points P, Q of the arc upon the axis of revolution, δs

the elementary arc PQ, δS the area of the elementary zone or belt traced out by the revolution of PQ about XY.

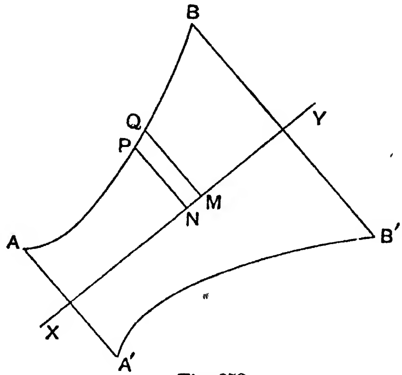


Fig. 253.

Let p_1 and p_2 be the greatest and the least of the perpendicular distances of points on the arc PQ from the axis of revolution. Then we may take it as axiomatic that the area traced out by PQ in its revolution is greater than it would be if each point of PQ were at the distance p_2 from the axis, and less than if each point were at a distance p_1 from the axis, *i.e.* δS lies between

$$2\pi p_1\,\delta s \quad \text{and} \quad 2\pi p_2\,\delta s.$$

Also p_1 and p_2 differ by a small quantity of at least the first order from PN. Hence $2\pi p_1\,\delta s$ and $2\pi p_2\,\delta s$ differ by a small quantity of at least the second order from $2\pi PN\,\delta s$ Therefore in the limit we have

$$\frac{dS}{ds}=2\pi PN$$

or

$$S=\int 2\pi PN\,ds.$$

749. Various Forms of the Formula.

If the axis of revolution be the x-axis, this may be written as

$$S=\int 2\pi y\,ds, \quad \int 2\pi y\frac{ds}{dx}dx, \quad \int 2\pi y\frac{ds}{dy}dy,$$

$$\int 2\pi y\frac{ds}{d\theta}d\theta, \quad \int 2\pi y\frac{ds}{dr}dr, \text{ etc.},$$

as may happen to be convenient in any particular example, the values of $\frac{ds}{dx}, \frac{ds}{dy}, \frac{ds}{d\theta}$, etc., being obtained according to the rules of the Differential Calculus, viz.

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}, \quad \frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}, \text{ etc.}$$

750. **Centroids.**

The centroids, both of the surface and volume of a solid of revolution bounded by planes perpendicular to the axis of revolution, are plainly upon the axis of revolution, supposing the surface density and the volume density in the respective cases to be either constant or some function of the distance from a point on the axis of revolution, *i.e.* so that the distribution of density is symmetrical about the axis.

Take the x-axis as the axis of revolution, σ the surface density and ρ the volume density, both symmetrical as to the axis, and functions of x alone, so that the elementary zones in the one case and the elementary discs in the other case, into which the surface or volume is divided, have their own centroids upon the axis of x, and we have, on application of the formula $\bar{x}=\frac{\Sigma mx}{\Sigma m}$,

(1) For the Surface,

$$\bar{x}=\frac{\int(\sigma 2\pi y\,ds)x}{\int(\sigma 2\pi y\,ds)}=\frac{\int \sigma xy\,ds}{\int \sigma y\,ds};$$

(2) For the Volume,

$$\bar{x}=\frac{\int(\rho\pi y^2 dx)x}{\int(\rho\pi y^2 dx)}=\frac{\int \rho xy^2 dx}{\int \rho y^2 dx}.$$

It is to be noted that in the first case s is left as the independent variable; in the second case, x.

If it be desirable to take x or θ as the independent variable in the first case, we must replace ds by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \text{ or } \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta,$$

as the case may be.

In cases where ρ or σ are constants, they of course disappear from the formulae.

751. Ex. 1. Find the surface of a zone of a sphere bounded by parallel planes $z=z_1$, $z=z_1+h$.

If a be the radius of the sphere, and θ be the latitude of any point P on the sphere, we have (Fig. 254)

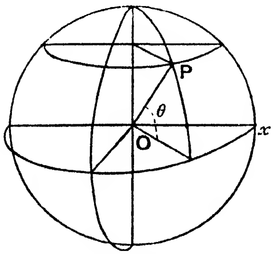


Fig. 254.

$$S=\int 2\pi a\cos\theta . ds \quad \text{and} \quad ds=a\,d\theta$$

$$=2\pi a^2\Big[\sin\theta\Big]_{\theta_1}^{\theta_2}$$

$$=2\pi a^2[\sin\theta_2-\sin\theta_1]$$

$$=2\pi a^2\left[\frac{z_1+h}{a}-\frac{z_1}{a}\right]=2\pi ah,$$

and therefore equal to the corresponding belt intercepted upon the enveloping cylinder by the same planes, the z-axis being the axis of the cylinder. This is the result usually arrived at in a Newtonian manner in books on Mensuration. It has already been used in Art. 734.

Ex. 2. Find the surface of a belt of the paraboloid formed by the revolution of the curve $y^2=4ax$ about the x-axis.

Here $$\frac{dy}{dx}=\sqrt{\frac{a}{x}},\quad \frac{ds}{dx}=\sqrt{1+\frac{a}{x}},$$

and $$S=2\pi\int_{x_1}^{x_2} y\frac{ds}{dx}\,dx=4\pi\sqrt{a}\int_{x_1}^{x_2}\sqrt{x+a}\,dx$$

$$=\tfrac{8}{3}\pi a^{\frac{1}{2}}\{(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}\};$$

and since for the parabola the radius of curvature is given by

$$\rho=\frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}},$$

we have $$S=\frac{4\pi a}{3}(\rho_2-\rho_1),$$

where ρ_1, ρ_2 are the radii of curvature of the generating curve at the points where it is cut by the planes bounding the belt.

Ex. 3. The curve $r=a(1+\cos\theta)$ revolves about the initial line. Find the volume and surface of the figure formed.

Here $$V=\int \pi y^2\,dx=\pi\int r^2\sin^2\theta\, d(r\cos\theta)$$

$$=\pi\int a^2(1+\cos\theta)^2\sin^2\theta . a d(\cos\theta+\cos^2\theta),$$

the limits being such that the radius vector sweeps over the upper half of the cardioide.

Hence $\quad V = \pi a^3 \int_0^\pi (1+\cos\theta)^2(1+2\cos\theta)\sin^3\theta\, d\theta$

$$= \pi a^3 \int_0^\pi (1+4\cos\theta+5\cos^2\theta+2\cos^3\theta)\sin^3\theta\, d\theta$$

$$= 2\pi a^3 \int_0^{\frac{\pi}{2}} (1+5\cos^2\theta)\sin^3\theta\, d\theta$$

$$= 2\pi a^3 \left\{\frac{2}{3} + 5\frac{\Gamma(\frac{3}{2})\,\Gamma(2)}{2\Gamma(\frac{7}{2})}\right\} = 2\pi a^3 \cdot \frac{4}{3}$$

$$= \frac{8\pi a^3}{3}.$$

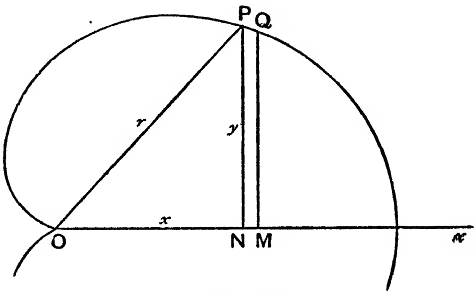


Fig. 255.

Again,

$$S = 2\pi \int y\, ds = 2\pi \int_0^\pi r\sin\theta \frac{ds}{d\theta}\, d\theta$$

$$= 2\pi \int_0^\pi a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= 2\pi a^2 \int_0^\pi (1+\cos\theta)\sin\theta \,.\, 2\cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta$$

$$= 32\pi a^2 \left[-\frac{1}{5}\cos^5\frac{\theta}{2}\right]_0^\pi = \tfrac{32}{5}\pi a^2.$$

Ex. 4. Find the centroid of the solid formed in the last example, the volume density being uniform.

The centroid obviously lies upon the axis. To find its abscissa $\bar{x}$ we have

$$\bar{x} = \frac{\int x \,.\, \pi y^2 dx}{\int \pi y^2 dx}.$$

The denominator has just been calculated, viz. $= \frac{8}{3}\pi a^3$.

The numerator

$$= \pi \int r\cos\theta \,.\, r^2\sin^2\theta\, d(r\cos\theta)$$

$$= \pi a^4 \int (1+\cos\theta)^3 \cos\theta\sin^2\theta\, d(\cos\theta + \cos^2\theta),$$ the limits being π and 0

$$=\pi a^4\int_0^\pi (1+\cos\theta)^3\cos\theta(1+2\cos\theta)\sin^3\theta\,d\theta$$

$$=\pi a^4\int_0^\pi (\cos\theta+5\cos^2\theta+9\cos^3\theta+7\cos^4\theta+2\cos^5\theta)\sin^3\theta\,d\theta$$

$$=2\pi a^4\int_0^{\frac{\pi}{2}}(5\cos^2\theta+7\cos^4\theta)\sin^3\theta\,d\theta$$

$$=2\pi a^4\left[5\,\frac{\Gamma(2)\,\Gamma(\frac{3}{2})}{2\Gamma(\frac{7}{2})}+7\cdot\frac{\Gamma(2)\,\Gamma(\frac{5}{2})}{2\Gamma(\frac{9}{2})}\right]$$

$$=2\pi a^4\left[\frac{5}{2}\cdot\frac{2}{5}\cdot\frac{2}{3}+\frac{7}{2}\cdot\frac{2}{7}\cdot\frac{2}{5}\right]=\frac{32\pi a^4}{15}.$$

Hence $$\bar{x}=\frac{32\pi a^4}{15}\bigg/\frac{8\pi a^3}{3}=\tfrac{4}{5}a.$$

Ex. 5. Find the centroid of the surface formed by the revolution of the cardioide, as in Ex. 3, the surface density being uniform.

Here $$\bar{x}=\frac{\int x\,2\pi y\,ds}{\int 2\pi y\,ds}.$$

The denominator was calculated in Ex. 3.

The numerator $$=2\pi\int_0^\pi r\cos\theta\,.\,r\sin\theta\,\frac{ds}{d\theta}\,d\theta$$

$$=2\pi\int_0^\pi a^2(1+\cos\theta)^2.\cos\theta.\sin\theta.\,2a\cos\frac{\theta}{2}\,d\theta$$

$$=32\pi a^3\int_0^\pi \cos^6\frac{\theta}{2}\sin\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$=64\pi a^3\left[\frac{\cos^7\frac{\theta}{2}}{7}-2\frac{\cos^9\frac{\theta}{2}}{9}\right]_0^\pi$$

$$=\tfrac{320}{63}\pi a^3.$$

Hence $$\bar{x}=\tfrac{320}{63}\pi a^3/\tfrac{32}{5}\pi a^2=\tfrac{50}{63}a.$$

Ex. 6. As an example of the case when x and y are given in terms of a third variable, consider the case of the surface of the solid formed by the revolution of a cycloid about the line of cusps.

Here $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, $ds=2a\cos\frac{\theta}{2}d\theta$, and the perpendicular from x, y upon the line of cusps$=a(1+\cos\theta)$.

Hence $$S=2\int_0^\pi 2\pi a(1+\cos\theta)\,2a\cos\frac{\theta}{2}\,d\theta$$

$$=16\pi a^2\int_0^\pi \cos^3\frac{\theta}{2}\,d\theta$$

$$=32\pi a^2\int_0^{\frac{\pi}{2}}\cos^3\phi\,d\phi,\quad\text{where } \phi=\frac{\theta}{2},$$

$$=\frac{64\pi a^2}{3}.$$

752. THE THEOREMS OF PAPPUS OR GULDIN.

When any plane closed curve revolves about a straight line in its own plane which does not cut the curve we have the following theorems:

I. *The* VOLUME *of the ring formed is equal to that of a cylinder whose base is the revolving curve and whose height is the length of the path of the centroid of the* AREA *af the curve.*

II. *The* SURFACE *of the ring formed is equal to that of a cylinder whose base is the revolving curve and whose height is the length of the path of the centroid of the* PERIMETER *of the curve.*

These theorems were given by PAPPUS, in his *Mathematical Collections*, in the latter half of the fourth century, and rediscovered by GULDIN, and published in his *Centrobaryca* early in the seventeenth century.[1]

753. THEOREM I.

Let the x-axis be the axis of rotation. Divide the area (A) into infinitesimal rectangular elements with sides parallel to the coordinate axes, such as $P_1P_2P_3P_4$ in the accompanying figure, each of area δA.

Let the ordinate $P_1N_1=y$.

Let rotation take place through an infinitesimal angle $\delta\theta$.

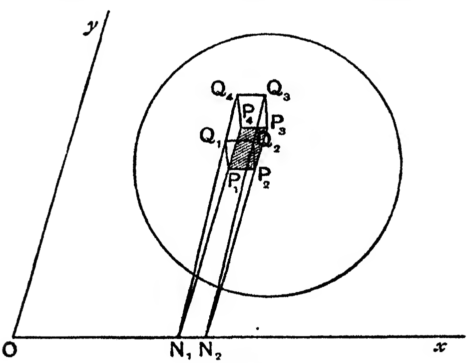


Fig. 256.

Then the elementary solid formed is on base δA, and its height to first order infinitesimals, is $y\,\delta\theta$, and therefore to infinitesimals of the third order its volume is $\delta A \,.\, y\,\delta\theta$.

[1] Cajori, *History of Mathematics*, pages 59, 167.

If the rotation be through any finite angle a, we obtain by summation or integration,

$$\delta A \,.\, y \,.\, a.$$

If this be integrated over the whole area of the curve, we have for the volume of the solid formed,

$$a\int y\,dA.$$

Now the formula for the ordinate of the centroid of a number of masses $m_1, m_2, \ldots$, with ordinates $y_1, y_2, \ldots$, is $\bar{y}=\frac{\Sigma my}{\Sigma m}$.

Hence the ordinate of the centroid of the *area* of the revolving curve is

$$\bar{y}=\frac{\int y\,dA}{\int dA}=\frac{\int y\,dA}{A},$$

and therefore $$\int y\,dA=A\bar{y}.$$

Hence the volume formed $=A(a\bar{y})$.

But A is the area of the revolving figure, and $a\bar{y}$ is the length of the path of the centroid of the *area*. Hence the theorem is established.

If the curve perform a complete revolution and form a solid ring, we have

$$a=2\pi \quad\text{and}\quad V=A(2\pi\bar{y}).$$

754. THEOREM II.

Again, take the axis of revolution as the x-axis. Divide the perimeter s into infinitesimal elements, such as P_1P_2, of length δs.

Let the ordinate P_1N_1 be called y.

Let rotation take place through an infinitesimal angle $\delta\theta$.

Then the elementary area formed, $P_1P_2Q_2Q_1$, is ultimately a rectangle with sides δs and $y\,\delta\theta$, and to infinitesimals of the second order its area is $\delta s \,.\, y\,\delta\theta$.

If the rotation be through any finite angle a we obtain, by summation or integration, $\delta s \,.\, ya$.

If this be integrated over the whole perimeter of the curve, we have for the curved surface of the solid formed,

$$a\int y\,ds.$$

If $\bar{y}$ be the ordinate of the centroid of the *perimeter* of the curve in the plane of x-y, we have

$$\bar{y}=\frac{\int y\,ds}{\int ds}=\frac{\int y\,ds}{s}.$$

Then $$\int y\,ds=s\bar{y}$$

and $$\text{the surface formed}=s(a\bar{y}).$$

But s is the perimeter of the revolving figure, and $a\bar{y}$ is the length of the path of the centroid of the *perimeter* of the revolving curve. Hence the theorem is established.

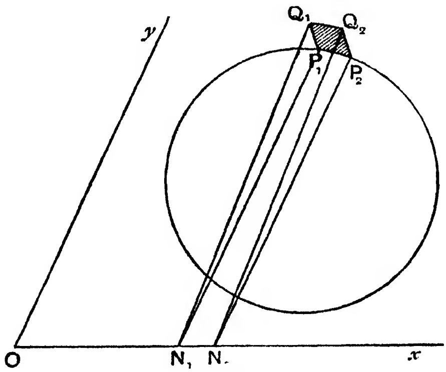


Fig. 257.

If the curve perform a complete revolution and form a solid ring, we have $a=2\pi$, and its surface is

$$S=s(2\pi\bar{y}).$$

Illustrative Example.

The volume and surface of an "**Anchor-ring**" or "**Tore**" formed by the revolution of a circle of radius a about a line in the plane of the circle at distance d from the centre ($d>a$) are respectively,

$$\text{Volume} = \pi a^2\times 2\pi d=2\pi^2a^2d,$$

$$\text{Surface} =2\pi a\times 2\pi d=4\pi^2ad.$$

In this case the centroid of the perimeter and the centroid of the area are at the same point, viz. the centre of the revolving figure. This of course *would not generally be the case.*

755. **Precautions.**

In these theorems it has been stated that the axis of revolution does not cut the curve. If the curve consists of more than one closed oval, it is to be further noted that the whole portion to which the rules apply must lie on one side of the axis of revolution.

When the axis of revolution cuts the curve, or when regions bounded by the curve lie on opposite sides of the axis of revolution, the theorems, both as to volume and surface, give the difference of the volumes or surfaces traced by the portions on opposite sides of the axis of revolution.

756. **Note by Mr. Routh.**

Again, it has been pointed out by Mr. E. J. Routh (*Anal. Statics*, vol. i., p. 293) that during any elementary rotation through an angle $\delta\theta$, the axis of revolution *need only be an instantaneous axis of revolution.* Let G be the centre of gravity of the revolving area A, G' a contiguous position of the centre

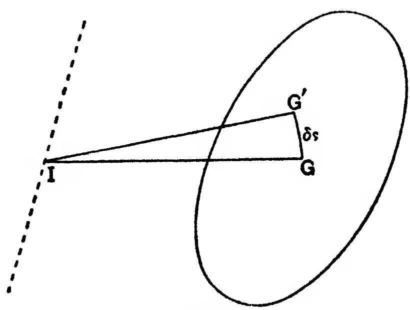


Fig. 258.

of gravity, $\delta s = GG'$, and let the plane of A be always at right angles to the tangent to the path of G. Let I be the centre of curvature of G's path. The rotation through $\delta\theta$ may be regarded as about a straight line through I perpendicular to the plane GIG', and the volume generated is

$$A \times IG\,\delta\theta \quad \text{or} \quad A\,\delta s.$$

And integrating, the volume generated is

Area × length of the path of the *centroid of the area.*

And further, for the theorem with regard to the surface; if the plane of the revolving curve be always at right angles to the tangent to the path of the centroid of the *perimeter*, the surface generated is the perimeter of the revolving curve × length of the path of the *centroid of the perimeter.*

Ex. A circle of radius c ($<b$) moves with its centre on the ellipse $x^2/a^2+y^2/b^2=1$, the plane of the circle being perpendicular to the direction of the tangent to the ellipse at the centre of the circle. The volume and surface of the ring generated are $\pi c^2 P$ and $2\pi c P$ respectively, where P is the perimeter of the ellipse, *i.e.* $4aE\left(\frac{\pi}{2}, e\right)$ when $b^2=a^2(1-e^2)$.

In this case the centroids of *area* and *perimeter* of the moving curve are the same point, viz. the centre of the circle.

757. Axis not in the Plane of the Curve. Extension for the Theorem as to the Volume.

Consider next the case when the rotation is about a line *not* in the plane of the area, but *parallel to it.*

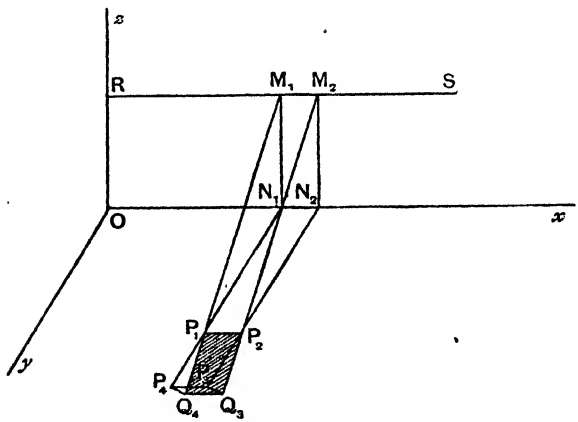


Fig. 259.

Let $\delta x\,\delta y$ be an element of the revolving plane, the x-axis being taken parallel to the axis of revolution and the z-axis cutting it at R, and the area lying entirely on one side of the x-axis.

Let $P_1P_2P_3P_4$ be the element $\delta x\,\delta y$, and let P_1N_1, P_2N_2 be perpendiculars on the x-axis and P_1M_1, P_2M_2 perpendiculars on the axis of revolution, and let θ be the angle $N_1P_1M_1$.

Let $P_1P_2Q_3Q_4$ be the projection of $\delta x\,\delta y$ on the plane $M_1M_2P_2P_1$, that is, the normal section of the elementary ring formed, and let a represent the angular extent of the revolution.

Then, to the second order the volume traced out by the revolution of $\delta x\,\delta y$ about RS is

$$P_1P_2Q_3Q_4 \times (aP_1M_1),$$

i.e. $$\delta x\,\delta y \cos\theta \times \left(a\,\frac{P_1N_1}{\cos\theta}\right) \quad \text{or} \quad \delta x\,\delta y \times aP_1N_1,$$

and is the same as that of $\delta x\,\delta y$ about the x-axis.

Hence, taking the limit when δx, δy are infinitesimally small, and integrating over any area which lies on one side of the x-axis in the x-y plane, we have the theorem that the volume generated by the area revolving about a line parallel to the plane of the area, but not in its own plane, is the same as would be traced out if the revolution were about the projection of the axis of revolution upon the plane of the area through the same angle.

758. **Axis not parallel to the Plane of the Area.**

Finally, suppose the axis of revolution *not parallel to the plane of the area*. Let the area lie in the x-y plane and

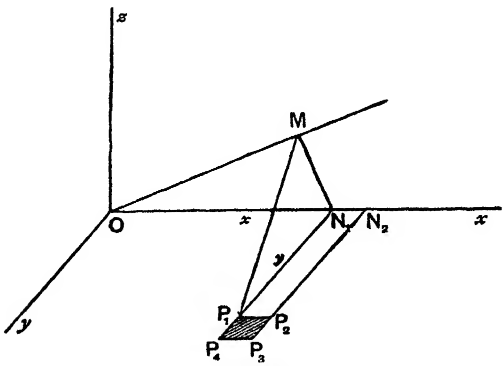


Fig. 260.

entirely on one side of the x-axis, and let the x-axis be the projection of the axis of revolution upon the x-y plane, and

the origin the point where that axis cuts the x-y plane and θ its inclination to the plane. Then the equations of the axis of revolution are

$$\frac{x}{\cos\theta}=\frac{y}{0}=\frac{z}{\sin\theta}.$$

The perpendicular upon this from P_1, $(x, y, 0)$, viz. P_1M, is

$$P_1M=\sqrt{x^2+y^2-x^2\cos^2\theta}=\sqrt{x^2\sin^2\theta+y^2}.$$

The equation of the plane OMP_1 is

$$-X\sin\theta+\frac{x}{y}\sin\theta\, Y+Z\cos\theta=0.$$

The direction ratios of the normal are

$$-\sin\theta,\quad +\frac{x}{y}\sin\theta,\quad +\cos\theta.$$

The direction cosines of the normal to the element $P_1P_2P_3P_4$, *i.e.* $\delta y\,\delta z$, are $(0, 0, 1)$.

The angle between these normals is

$$\cos^{-1}\frac{\cos\theta}{\sqrt{1+\dfrac{x^2}{y^2}\sin^2\theta}}=\cos^{-1}\left(\frac{y\cos\theta}{P_1M}\right).$$

The projection of $\delta x\,\delta y$ upon the plane OMP_1 is therefore

$$\delta x\,\delta y\,.\,\frac{y\cos\theta}{P_1M}.$$

The volume generated by the revolution of $\delta x\,\delta y$ through any angle a about OM is therefore

$$\left(\delta x\,\delta y\,\frac{y\cos\theta}{P_1M}\right)(P_1Ma),\quad \textit{i.e.}\ (\delta x\,\delta y)(ya\cos\theta)$$

to the second order, and is therefore the same as if the rotation took place about the projection of the axis of rotation upon the plane of the area, the angle of rotation being $a\cos\theta$ instead of a.

And integrating over the whole area, we have the theorem that the volume generated by the revolving area, the revolution being through an angle a, is the same as the volume generated by revolution about the projection of the axis on the plane of the area through an angle a which is multiplied by the cosine of the angle between the axis of revolution and the plane of the area (see Ex. 1, p. 294, *Anal. Statics*, E. J. Routh); or,

which is the same thing, the volume may be found by revolution through an angle α about the projection and then multiplied by $\cos\theta$. This supposes the revolving area to be entirely on one side of the projection of the axis on its plane.

759. Ex. 1. A quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves (1) about its semiminor axis. The area is $\frac{\pi ab}{4}$. The abscissa of the centroid is $\frac{4a}{3\pi}$. The volume traced out in a complete revolution is

$$\frac{\pi ab}{4}\cdot 2\pi\frac{4a}{3\pi}=\frac{2}{3}\pi a^2 b.$$

(2) If the revolution were about a straight line outside the plane of x-y but parallel to the minor axis, and which projects upon the minor axis, the volume would still be $\frac{2}{3}\pi a^2 b.$

(3) If the revolution were about a straight line through the centre at right angles to the major axis, and making an angle θ with the minor axis, the volume would be $\frac{2}{3}\pi a^2 b\cos\theta.$

Ex. 2. An ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about its tangent

$$x\cos\alpha+y\sin\alpha=p,$$

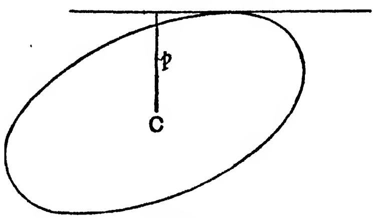


Fig. 261.

The volume generated is

$$\pi ab\times 2\pi p,\quad\text{where}\quad p^2=a^2\cos^2\alpha+b^2\sin^2\alpha.$$

If the revolution were about a line making an angle θ with this tangent, and which projects upon the tangent, the volume generated would be

$$\pi ab\times 2\pi p\times\cos\theta.$$

PROBLEMS.

1. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis is a mean proportional between those generated by the revolution of the ellipse and of the auxiliary circle round the major axis. [I. C. S., 1881.]

2. Find the volume of the solid formed by the revolution of a cycloid round a tangent at the vertex.

3. The loop of the curve $2ay^2 = x(x-a)^2$ revolves about the straight line $y=a$; find the volume of the solid generated. [Oxford I. P., 1890.]

4. Show that the volume of the solid formed by the revolution of the cissoid $y^2(2a-x)=x^3$ about its asymptote is equal to $2\pi^2a^3$. [Trinity, 1886.]

5. Find the volume of the solid produced by the revolution of the loop of the curve $y^2 = x^2\dfrac{a+x}{a-x}$ about the axis of x. [I. C. S., 1882.]

Prove that the areas of the oblate and prolate spheroids formed by rotating an ellipse of major axis $2a$ and eccentricity e about its principal axes are

$$2\pi a^2\left(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right)$$

and $$2\pi a^2\left(1-e^2+\frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right).$$

[Oxford II. P., 1914.]

Prove also that of all prolate spheroids formed by the revolution of an ellipse of given area, the sphere has the greatest surface. [I. C. S., 1891.]

Find the surface of any zone of an ellipsoid of revolution cut off by planes perpendicular to the axis of revolution. [Colleges α, 1888.]

7. If the evolute of a catenary revolve about the directrix of the catenary, show that the area of any portion of the surface generated, cut off by two planes perpendicular to the directrix, varies as the difference of the cubes of the radii of its bounding circles. [Colleges α, 1892.]

8. Find the volume of the solid formed by the revolution about the prime radius of the loop of the curve

$$r^3 = a^3\theta\cos\theta$$

between $\theta=0$ and $\theta=\dfrac{\pi}{2}$. [Oxford II. P., 1890.]

9. If the cardioide $r=a(1-\cos\theta)$ revolve round the line $p=r\cos(\theta-\gamma)$, prove that the volume generated is

$$3p\pi^2a^2+\tfrac{5}{2}\pi^2a^3\cos\gamma,$$

assuming that the line does not cut the cardioide. [St. John's, 1882.]

10. Prove that the area of the surface generated by the revolution of a portion of the arc of a cycloid about the normal at one extremity is equal to the area of the cycloid multiplied by

$$\tfrac{16}{3}\left[(\beta-\gamma)\sin\beta\cos\gamma+\tfrac{3}{4}\cos(\beta+\gamma)-\tfrac{1}{12}\cos(3\beta-\gamma)-\tfrac{2}{3}\cos 2\gamma\right],$$

where γ and β are the angles of inclination of the axis of revolution, and of the normal at the other extremity of the arc, to the axis of the cycloid.

Deduce the areas of the surfaces generated by the revolution of the whole cycloid about its axis and about its base.

[COLLEGES ϵ, 1884.]

11. Find the volume of the solid formed by revolving a loop of a lemniscate of Bernoulli about the straight line in its plane which passes through the pole and is perpendicular to the axis.

[OXFORD I. P., 1901.]

12. The lemniscate $r^2=a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the volume and surface of the solid generated are respectively $\pi^2a^3/4$ and $4\pi a^2$.

13. A surface is the locus of points which have their distances from a fixed plane inversely proportional to the fifth power of their distances from a fixed point O in that plane. Prove that its volume equals twice that of the sphere which, with its centre at O, touches the surface. [OXFORD II. P., 1880.]

14. Find the volume of the solid formed by the revolution of the curve $(a-x)y^2=a^2x$ about its asymptote. [I. C. S., 1883.]

15. Show that the rate of increase of the volume of an anchor ring when the radius of the generating circle is increased while its centre remains at a constant distance a from the axis of revolution is

$$2\pi^2ad,$$

the diameter of the generating circle being d, increasing at unit rate. [TRINITY COLL., 1881.]

16. A loop of the curve $r=a\sin n\theta$ revolves about the initial line. Find the volume of the solid thus generated, and verify the result by deducing the volume of the ring formed by the revolution of a circle about a tangent. [COLLEGES α, 1889.]

17. If the curve $r=a+b\cos\theta$ revolve about the initial line, show that the volume generated is

$$\tfrac{4}{3}\pi a(a^2+b^2),$$

provided a be $<b$, [COLLEGES α, 1884.

18. The curve $r=a(1-\epsilon\cos\theta)$, when ϵ is very small, revolves about a tangent parallel to the initial line; prove that the volume of the solid thus generated is approximately

$$2\pi^2a^3(1+\epsilon^2).$$ [I. C. S., 1892.]

19. The curve $r^3=a^3\cos 3\theta$ revolves about $\theta=0$. Prove that the loop in the third quadrant generates a volume

$$\frac{3\pi a^3}{8}.$$ [OXFORD I. P., 1902.]

20. A loxodrome is drawn from a point A on the earth's surface to a point B. If θ_1, ϕ_1 be the longitude and co-latitude of B, and θ_2, ϕ_2 the corresponding quantities for A, show that the area contained between the meridians of A and B and the loxodrome is

$$\frac{2(\theta_1-\theta_2)\log(\cos\frac{1}{2}\phi_1\div\cos\frac{1}{2}\phi_2)}{\log(\tan\frac{1}{2}\phi_1\div\tan\frac{1}{2}\phi_2)},$$

the radius of the earth being taken as unity. [ST. JOHN'S, 1884.]

21. Prove that the whole area bounded by the curve

$$x^4+y^4=2axy^2$$

is $\dfrac{\pi a^2}{2\sqrt{2}}$. Also show that if the area revolves about the x-axis, either loop generates a solid whose volume is $\frac{2}{3}\pi a^3$. When the area revolves about the y-axis, the whole volume generated is $\dfrac{\pi^2a^3}{4}\sqrt{2}$.

22. Determine the curve which generates, by revolving about the axis of x, a volume proportional to the length cut off from the axis by the terminal bounding planes. [TRIN. HALL AND MAGD., 1886.]

23. The axes of two cylinders of radius a intersect at an angle α; show that the whole volume common to the two is

$$\tfrac{16}{3}a^3\operatorname{cosec}\alpha.$$ [TRIN. H. AND MAGD., 1886.]

24. Evaluate $\displaystyle\iint\frac{dS}{p^n}$, taken over the surface of a sphere of radius a, p being the perpendicular on the tangent plane from a fixed point within the sphere at a distance b from the centre; showing that

$$\iint\frac{dS}{p^n}=\frac{2\pi a^2}{(n-1)b}\left[(a-b)^{-n+1}-(a+b)^{-n+1}\right].$$

[OXFORD II. P., 1892.]

25. Show that the volume traced out by the part of the area of the curve $r=f(\theta)$ which lies between $\theta=\beta$ and $\theta=\alpha$, when the curve revolves about the line $\theta=\gamma$, taking $\alpha>\beta>\gamma$, is

$$\frac{2\pi}{3}\int_\beta^\alpha [f(\theta)]^3 \sin(\theta-\gamma)\,d\theta.$$ [OXFORD I. P., 1902.]

26. In the case of any portion of a surface revolving about an axis, prove that the volume generated is the sum with the proper signs of the corresponding volumes generated by the projections of the surface on any two planes at right angles to one another through the axis of rotation. [γ, 1900.]

27. A point O is taken on a diameter of a sphere (centre C, radius a) so that $OC=c$ $(c<a)$; the radius vector of length r drawn from O to any point P on the surface makes an angle θ with OC, and the radius CP makes an angle θ' with OC produced, dS is an element of area of the surface containing the point P; evaluate the integral

$$\int \frac{\cos\theta\cos\theta'}{r^2}\,dS$$

taken over the larger of the portions into which the surface is divided by a plane, through O, at right angles to OC.

[OXFORD I. P., 1901.]

28. Prove the formula $\frac{2}{3}\pi\int r^3\sin\theta\,d\theta$ for the volume of the surface formed by the revolution of a closed plane curve about the initial line.

The outer loop of $r^{\frac{1}{3}}=a^{\frac{1}{3}}\cos\frac{1}{3}\theta$ revolves about the initial line. Show that the volume of the surface generated is

$$\frac{1}{15}\pi a^3\left(7+\frac{1}{2^{11}}\right).$$ [OXF. I. P., 1911.]

29. Find the area of the finite portion of the surface $2z=x^2+y^2$ cut off by the plane $z=h$. [OXF. I. P., 1913.]

30. Show how to find the volume of the solid formed by revolving the curve $r=f(\theta)$ about the line $\theta=\alpha$, it being assumed that the curve passes through the origin.

Prove that the volume of the solid formed by revolving one loop of the curve $r^2=a^2\cos 2\theta$ about one of the inflexional tangents is

$$\tfrac{1}{8}\pi^2a^3.$$ [OXF. I. P., 1915.]

Show also that the distance of the centroid of this solid from the origin is $\dfrac{4a}{3\pi}$.

CHAPTER XXII.

SURFACES AND VOLUMES IN GENERAL, AND THEIR CENTROIDS, ETC. DOUBLE AND TRIPLE INTEGRATION.

760. Let the equation of a surface be $\phi(x, y, z)=0$ referred to three mutually perpendicular coordinate axes Ox, Oy, Oz. Let us discuss the volume contained between the boundaries

$$z=0,\ \phi(x, y, z)=0;\quad y=0,\ y=F(x);\quad x=0,\ x=a.$$

Let planes
$$\begin{aligned} X&=x, & X&=x+\delta x,\\ Y&=y, & Y&=y+\delta y,\\ Z&=z, & Z&=z+\delta z, \end{aligned}$$

be drawn.

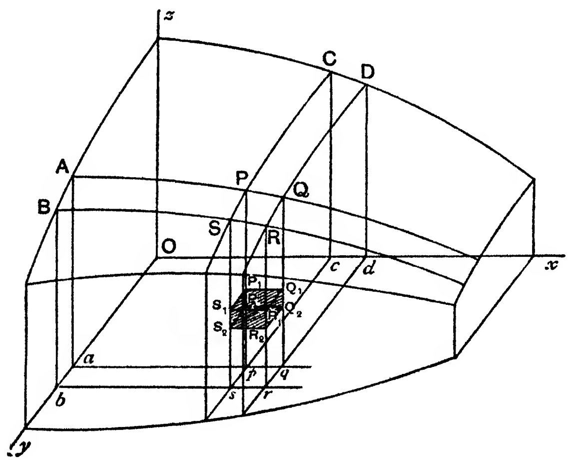


Fig. 262.

Planes $X=x$, $X=x+\delta x$ intercept between them a thin slice or lamina of thickness δx.

Planes $Y=y$, $Y=y+\delta y$ cut from this lamina a prism or tube on rectangular base $\delta x\,\delta y$.

Planes $Z=z$, $Z=z+\delta z$ cut from this prism an elementary rectangular box or "cuboid" of volume $\delta x\,\delta y\,\delta z$, represented in the figure as $P_1Q_1R_1S_1P_2Q_2R_2S_2$. Regarding δx, δy, δz as infinitesimals of the first order, the volume of the slice is a first order infinitesimal, the volume of the prism is a second order infinitesimal, and the volume of the cuboid is a third order infinitesimal. Let the prism intercept on the surface a curvilineal quadrilateral figure $PQRS$, and on the plane x-y the elementary rectangle $pqrs$, viz. $\delta x\,\delta y$. These areas are both infinitesimals of the second order.

If we add up all the complete cuboids on base $\delta x\,\delta y$ from $z=0$ to $z=$the smallest of the values of z of the surface within the quadrilateral $PQRS$, we get the volume of the prism, less by a third order infinitesimal, viz. the portion of a cuboid bounded by a base $\delta x\,\delta y$ for its lower surface, by the curvilinear quadrilateral $PQRS$ for its upper surface, and by four plane faces parallel to the y-z or z-x planes. We may regard the infinitesimal δz as having been taken not less than the difference of the greatest and the least values of z for points on the quadrilateral $PQRS$. This remnant of the prism is therefore less than one of the elementary cuboids forming the whole prism, and is therefore an infinitesimal of not less than the third order.

Next let us add up all the prisms which lie between the planes $X=x$ and $X=x+\delta x$, and bounded on its upper side by the specified surface from the plane $Y=0$ to any definite value of Y. The sum of these second order complete prisms differs from the volume of the lamina between the planes $X=x$ and $X=x+\delta x$ by the sum of the third order infinitesimal remnants of the prisms, and by a second order tubular element on a base less than $\delta x\,\delta y$ at the end of the slice, that is by a second order infinitesimal, the sum of the complete prisms being of the first order.

Finally, let us add up all the slices or laminae from $X=0$ to any definite value of X. The sum of the portions of these laminae made up of complete prisms is a finite quantity. The sum of the remnants of the laminae is the sum of a set

of second order infinitesimals, and forms a first order infinitesimal. Hence it appears that the sum of all the complete cuboids within the figure bounded by the coordinate planes, the planes $X=x_1$, $Y=y_1$, say, and the surface, differs from the whole volume of that figure by a first order infinitesimal at most, and in the limit when δx, δy, δz are diminished without limit, we have the volume given by

$$V=\iiint dx\,dy\,dz.$$

The limits for z are from $z=0$ to $z=$the value found from $\phi(x, y, z)=0$ in terms of x and y, say $z=f(x, y)$.

The limits for y will be from $y=0$ to the value of y specified in any particular manner, say $y=F(x)$.

The limits for x will be such as to go from $x=0$ to $x=a$.

761. Ex. Consider the volume of an **octant of an ellipsoid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Here the limits for z are $z=0$ to $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ for the elementary prism, to add up all the cuboids in the prism.

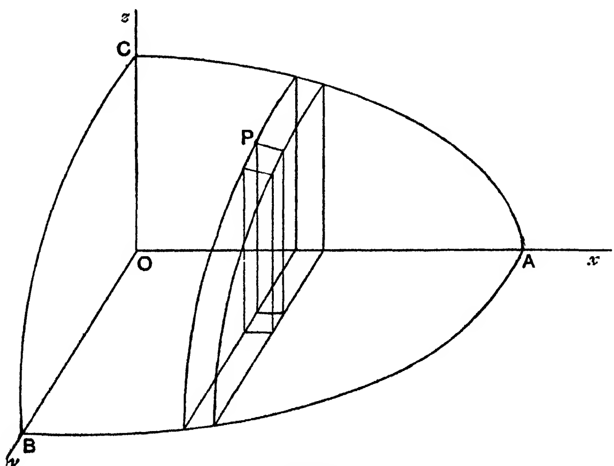


Fig. 263.

For y; $y=0$ to $y=b\sqrt{1-\frac{x^2}{a^2}}$ for the slice, to add up all the prisms in the slice.

For x; from $x=0$ to $x=a$, to add up all the slices.

And $V=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dx\,dy\,dz=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}[z]\,dx\,dy,$

and taking $[z]$ between its limits, this integral

$$=c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy.\quad \text{Write } \frac{\eta^2}{b^2} \text{ for } 1-\frac{x^2}{a^2}.$$

Now $$\int_0^\eta\sqrt{\frac{\eta^2}{b^2}-\frac{y^2}{b^2}}\,dy=\frac{1}{b}\left[\eta\frac{\sqrt{\eta^2-y^2}}{2}+\frac{\eta^2}{2}\sin^{-1}\frac{y}{\eta}\right]_0^\eta=\frac{\eta^2}{2b}\cdot\frac{\pi}{2};$$

$$\therefore\ V=\frac{c}{b}\frac{\pi}{4}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$=\frac{c}{b}\cdot\frac{\pi}{4}b^2\left(a-\frac{a^3}{3a^2}\right)=\frac{c}{b}\cdot\frac{\pi}{4}\cdot b^2\frac{2a}{3}=\frac{\pi abc}{6}.$$

And the volume of the whole ellipsoid is $8V=\frac{4}{3}\pi abc$.

762. Obviously in cases where the volume of a slice can be written down at once, the labour of computation may be saved.

In the case just considered, for instance, the section at distance $X=x$ from the plane of yz is an ellipse, viz.

$$\frac{y^2}{b^2\left(1-\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{x^2}{a^2}\right)}=1,$$

whose semiaxes are $b\sqrt{1-\frac{x^2}{a^2}}$, $c\sqrt{1-\frac{x^2}{a^2}}$;

and the area of the quarter ellipse in the first octant is

$$\tfrac{1}{4}\pi bc\left(1-\frac{x^2}{a^2}\right).$$

Hence the volume of the slice in the first octant is

$$\tfrac{1}{4}\pi bc\left(1-\frac{x^2}{a^2}\right)\delta x,$$

to the first order.

And the sum of the slices is

$$\int_0^a\tfrac{1}{4}\pi bc\left(1-\frac{x^2}{a^2}\right)dx=\frac{\pi bc}{4}\left(a-\frac{a^3}{3a^2}\right)=\frac{\pi abc}{6},$$

as before.

763. When the volume contained is all that is required, we may, in general, start with

$$V=\iint z\,dx\,dy,$$

i.e. we may use the elementary prism on $\delta x\,\delta y$ for base as our element of volume. This amounts of course to integrating with

regard to z in the triple integral formula $\iiint dx\,dy\,dz$ between limits $z=0$ and $z=$ the ordinate of the surface under consideration.

If the upper surface of the region whose volume is required is $z=f_1(x, y)$, and the lower surface be $z=f_2(x, y)$, instead of $z=0$, as taken in Art. 760, we have

$$V=\iint\{f_1(x, y)-f_2(x, y)\}\,dx\,dy.$$

764. Illustrative Examples.

1. The curve $z(a^2+x^2)^{\frac{3}{2}}=a^4$ lying in the plane z-x revolves about the axis of z. Find the volume in the positive octant included between this surface and the planes $x=0$, $x=a$, $y=0$, $y=a$. [COLLEGES ϵ, 1883.]

The equation of the surface generated is

$$z=\frac{a^4}{(a^2+x^2+y^2)^{\frac{3}{2}}},$$

and $$V=\iint z\,dx\,dy=a^4\int_0^a\int_0^a\frac{dx\,dy}{(a^2+x^2+y^2)^{\frac{3}{2}}}.$$ Write b^2 for a^2+x^2.

Then $$\int_0^a\frac{dy}{(b^2+y^2)^{\frac{3}{2}}}=\int_0^{\tan^{-1}\frac{a}{b}}\frac{b\sec^2\theta\,d\theta}{b^3\sec^3\theta},$$ where $y=b\tan\theta$,

$$=\frac{1}{b^2}\int_0^{\tan^{-1}\frac{a}{b}}\cos\theta\,d\theta$$

$$=\frac{1}{b^2}\cdot\frac{a}{\sqrt{a^2+b^2}}.$$

Hence $$\int_0^a\frac{dy}{(a^2+x^2+y^2)^{\frac{3}{2}}}=\frac{a}{(a^2+x^2)\sqrt{2a^2+x^2}},$$

and we have to evaluate $$I=\int_0^a\frac{a^5}{(a^2+x^2)\sqrt{2a^2+x^2}}\,dx.$$

Let $x=a\sqrt{2}\tan\phi$.

Then $$I=\int_0^{\tan^{-1}\frac{1}{\sqrt{2}}}\frac{a^6\,.\,a\sqrt{2}\sec^2\phi\,d\phi}{(a^2+2a^2\tan^2\phi)\,a\sqrt{2}\sec\phi\,\cancel{d\phi}}$$

$$=a^3\int\frac{\cos\phi\,d\phi}{\cos^2\phi+2\sin^2\phi}$$

$$=a^3\int\frac{d\sin\phi}{1+\sin^2\phi}$$

$$=a^3\Big[\tan^{-1}(\sin\phi)\Big]_0^{\tan^{-1}\frac{1}{\sqrt{2}}}$$

$$=a^3\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi a^3}{6};$$

$$\therefore\ V=\frac{\pi a^3}{6}.$$

2. Express the volume contained between the surfaces whose equations are $x^2+y^2+z^2=a^2$, $x^2+y^2=a^2$, $z=a$ and the coordinate planes in the forms $V=\iint z\,dx\,dy$, $V=\iint x\,dz\,dy$; investigating the limits of the integrations and determining the value of V.

(i) For the portion of the elementary prism on $\delta x\,\delta y$ for base lying between the sphere and the plane $z=a$, the length is

$$a-\sqrt{a^2-x^2-y^2}.$$

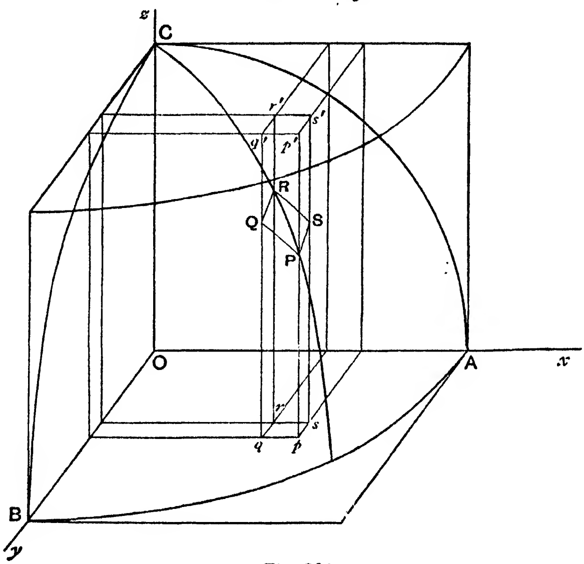


Fig. 264.

This is to be multiplied by $\delta x\,\delta y$ and summed for values of y from $y=0$ to $y=\sqrt{a^2-x^2}$, and afterwards the result is to be summed from $x=0$ to $x=a$.

Then $V=\int_0^a\int_0^{\sqrt{a^2-x^2}}(a-\sqrt{a^2-x^2-y^2})\,dx\,dy$

$$=\int_0^a\left[ay-\left(\frac{y\sqrt{a^2-x^2-y^2}}{2}+\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}}dx$$

$$=\int_0^a\left\{a\sqrt{a^2-x^2}-\frac{\pi}{4}(a^2-x^2)\right\}dx$$

$$=\left[a\left\{x\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}-\frac{\pi}{4}\left(a^2x-\frac{x^3}{3}\right)\right]_0^a$$

$$=\frac{a^3}{2}\cdot\frac{\pi}{2}-\frac{\pi}{4}\cdot\frac{2a^3}{3}$$

$$=\frac{\pi a^3}{12}.$$

(ii) If we use the formula $V=\iint x\,dz\,dy$, integrating with regard to y first, we have for the length of the prism on base $\delta y\,\delta z$ intercepted between the cylinder and the sphere $\sqrt{a^2-y^2}-\sqrt{a^2-z^2-y^2}$, until the prism ceases to cut the sphere, *i.e.* from $y=0$ to $y=\sqrt{a^2-z^2}$, and afterwards the length of this prism is $\sqrt{a^2-y^2}$ from $y=\sqrt{a^2-z^2}$ to $y=a$, and the limits for z are from 0 to a.

Hence

$$V=\int_0^a\int_0^{\sqrt{a^2-z^2}}(\sqrt{a^2-y^2}-\sqrt{a^2-z^2-y^2})\,dz\,dy+\int_0^a\int_{\sqrt{a^2-z^2}}^a\sqrt{a^2-y^2}\,dz\,dy$$

$$=\int_0^a\left[\int_0^a\sqrt{a^2-y^2}\,dy-\int_0^{\sqrt{a^2-z^2}}\sqrt{a^2-z^2-y^2}\,dy\right]dz$$

$$=\int_0^a\left\{\left[y\frac{\sqrt{a^2-y^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a-\left[y\frac{\sqrt{a^2-z^2-y^2}}{2}\right.\right.$$

$$\left.\left.+\frac{a^2-z^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-z^2}}\right]_0^{\sqrt{a^2-z^2}}\right\}dz$$

$$=\int_0^a\left\{\frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{\pi}{4}(a^2-z^2)\right\}dz$$

$$=\frac{\pi}{4}\int_0^a z^2dz=\frac{\pi}{4}\frac{a^3}{3}=\frac{\pi a^3}{12}.$$

(iii) If we use the formula $V=\iint x\,dy\,dz$, integrating with regard to z before we integrate with regard to y, we have the same peculiarity as before, viz. that the prism is of length $\sqrt{a^2-y^2}-\sqrt{a^2-y^2-z^2}$ from $z=0$ to $z=\sqrt{a^2-y^2}$, and of length $\sqrt{a^2-y^2}$ from $z=\sqrt{a^2-y^2}$ to $z=a$, and

$$V=\int_0^a\int_0^{\sqrt{a^2-y^2}}(\sqrt{a^2-y^2}-\sqrt{a^2-y^2-z^2})\,dy\,dz+\int_0^a\int_{\sqrt{a^2-y^2}}^a\sqrt{a^2-y^2}\,dy\,dz,$$

which, as before, $=\dfrac{\pi a^3}{12}$.

765. Mass, Moment, Centroid, etc.

If we regard the space bounded as described in Art. 760 to be filled with matter of specific density ρ at each point, the **Mass** of the elementary cuboid $\delta x\,\delta y\,\delta z$ is $\rho\,\delta x\,\delta y\,\delta z$, where ρ may be either a constant or a variable. And following the same argument as in finding the volume, we have for the mass of the body thus enclosed,

$$M=\iiint\rho\,dx\,dy\,dz.$$

766. In the same way, if the **Moment** of this mass be required about any line whose equations are known, say

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n},$$

l, m, n, being direction cosines; then, if p be the perpendicular from x, y, z upon this line, viz.

$$p^2=(x-a)^2+(y-b)^2+(z-c)^2-[l(x-a)+m(y-b)+n(z-c)]^2,$$

the moment of the solid about this line is

$$\iiint \rho p\, dx\, dy\, dz.$$

767. To determine the coordinates of the **Centroid**, we have only to translate the expressions

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \quad \bar{y}=\frac{\Sigma my}{\Sigma m}, \quad \bar{z}=\frac{\Sigma mz}{\Sigma m}$$

into the language of the *Integral Calculus*. And m being $\rho\, \delta x\, \delta y\, \delta z$, we have

$$\bar{x}=\frac{\iiint \rho x\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}, \quad \bar{y}=\frac{\iiint \rho y\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}, \quad \bar{z}=\frac{\iiint \rho z\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}.$$

768. If the **Moment of Inertia** about a straight line be required, and if p be the perpendicular from (x, y, z) upon the line, we have Moment of inertia $=\Sigma mp^2$,

i.e. in the language of the Calculus,

$$\iiint \rho p^2\, dx\, dy\, dz.$$

Thus, if A, B, C be the moments of inertia about the coordinate axes Ox, Oy, Oz respectively,

$$A=\iiint \rho(y^2+z^2)\, dx\, dy\, dz,$$

$$B=\iiint \rho(z^2+x^2)\, dx\, dy\, dz,$$

$$C=\iiint \rho(x^2+y^2)\, dx\, dy\, dz.$$

769. Similarly for "**Products of Inertia**," *i.e.* for quantities such as $D=\Sigma myz$, $E=\Sigma mzx$, $F=\Sigma mxy$,

we have

$$D=\iiint \rho yz\, dx\, dy\, dz, \quad E=\iiint \rho zx\, dx\, dy\, dz, \quad F=\iiint \rho xy\, dx\, dy\, dz.$$

770. The integration in all such cases takes the same course as in the finding of a volume, first as regards the proper assignment of limits, and second as regards the successive integrations (1) with regard to z, (2) with regard to y, (3) with regard to x.

The order of integration may be changed to suit circumstances, the several limits being suitably changed to ensure that the elementary cuboids into which the specified region is divided are thereby all added up.

As in the case of finding a volume, in some cases one, or perhaps two, of the integrations may be avoided by taking the elementary prism, or the elementary lamina described above, as the primary element, as was done in Art. 762 in the evaluation of the volume of the octant of an ellipsoid.

771. Ex. In the case of a sphere, viz. $x^2+y^2+z^2=a^2$, let us find the **mass of an octant of the sphere**, the density at any point being $\rho=kxyz$.

Here $$M=k\iiint xyz\,dx\,dy\,dz.$$

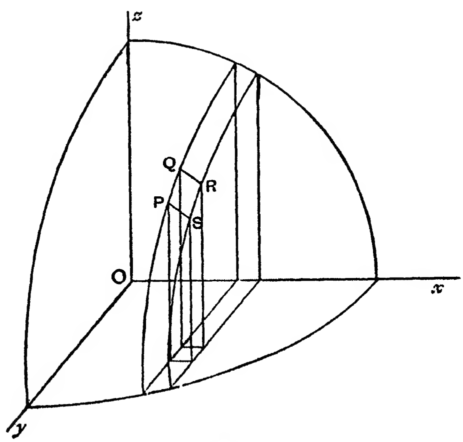


Fig. 265.

The limits for z in the positive octant are

$$z=0 \quad \text{to} \quad z=\sqrt{a^2-x^2-y^2};$$

for y, from $y=0$ to $y=\sqrt{a^2-x^2}$;

for x, from $x=0$ to $x=a$.

Hence
$$M=k\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}xyz\,dx\,dy\,dz$$
$$=k\int_0^a\int_0^{\sqrt{a^2-x^2}}xy\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}}dx\,dy$$
$$=\frac{k}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}}xy(a^2-x^2-y^2)\,dx\,dy$$
$$=\frac{k}{2}\int_0^a x\left[(a^2-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}}dx$$
$$=\frac{k}{2}\int_0^a x\left\{\frac{(a^2-x^2)^2}{2}-\frac{(a^2-x^2)^2}{4}\right\}dx$$
$$=\frac{k}{8}\int_0^a x(a^2-x^2)^2\,dx$$
$$=\frac{k}{8}\left[\frac{a^4x^2}{2}-\frac{2a^2x^4}{4}+\frac{x^6}{6}\right]_0^a$$
$$=\frac{ka^6}{48}.$$

If D be the density at a specific point, say the centre of the surface of the octant, *i.e.* where $x=y=z=\frac{a}{\sqrt{3}}$, we have
$$D=k\frac{a^3}{3\sqrt{3}}\quad\text{and}\quad M=\tfrac{1}{16}Da^3\sqrt{3}.$$

Examples.

1. Establish the following moments of inertia for uniform density, M representing the mass in each case:

(1) For an elliptic disc $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

about the x axis, - - - - - - - $\frac{Mb^2}{4}$;

about the y axis, - - - - - - - $\frac{Ma^2}{4}$;

about a line through the centre perpendicular to the plane, - - - - - - $M\frac{a^2+b^2}{4}$.

(2) For a rectangle of sides $2a$, $2b$,

about a line through the mid-points of sides $2b$, $\frac{Mb^2}{3}$;

about a line through the mid-points of sides $2a$, $\frac{Ma^2}{3}$;

about a line through the centre perpendicular to the plane, - - - - - - $M\frac{a^2+b^2}{3}$.

(3) For a sphere about any diameter $\frac{2Ma^2}{5}$, a being the radius.

(4) For an ellipsoid of semiaxes a, b, c, viz.

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

about the axis of length $2a$, - - - - - $M\frac{b^2+c^2}{5}$;

about the axis of length $2b$, - - - - - $M\frac{c^2+a^2}{5}$;

about the axis of length $2c$, - - - - - $M\frac{a^2+b^2}{5}$.

2. Obtain the position of the centroid of

(1) the quadrant of an ellipse,

$$x^2/a^2+y^2/b^2=1\ ;\quad \bar{x}=\frac{4a}{3\pi};\quad \bar{y}=\frac{4b}{3\pi};$$

(2) the positive octant of the sphere,

$$x^2+y^2+z^2=a^2\ ;\quad \bar{x}=\bar{y}=\bar{z}=\frac{3a}{8};$$

(3) the positive octant of the ellipsoid,

$$x^2/a^2+y^2/b^2+z^2/c^2=1\ ;\quad \bar{x}=\frac{3a}{8}\ ;\quad \bar{y}=\frac{3b}{8}\ ;\quad z=\frac{3c}{8}.$$

3. Show that in all the above cases for the whole elliptic disc, rectangle, sphere or ellipsoid, the products of inertia with regard to two axes of symmetry are zero.

Dr. Routh gave the following useful **mnemonic rule for the moment of inertia** of the circular or elliptic disc, rectangle and sphere or ellipsoid ; viz.

Moment of inertia about an axis of symmetry

$$=\text{Mass}\times\frac{\text{sum of squares of perpendicular semi-axes}}{3,\ 4\text{ or }5},$$

according as the body is rectangular, elliptical or ellipsoidal.

772. Element of Surface.

In estimating the element of surface δS cut from the surface S by the elementary prism on base $\delta x\,\delta y$, we may note that if γ be the angle the normal at P makes with the z-axis,

$$\delta x\,\delta y=\cos\gamma\,\delta S \text{ to the second order of infinitesimals,}$$

for $\delta x\,\delta y$ is the projection of δS upon the x-y plane.

The equations of the normal are:

$$\frac{X-x}{\phi_x}=\frac{Y-y}{\phi_y}=\frac{Z-z}{\phi_z},$$

where $\phi_x\equiv\frac{\partial\phi}{\partial x}$, etc.

Hence $$\cos\gamma=\frac{\phi_z}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}.$$

Then $$S=\iint\frac{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}{\phi_z}\,dx\,dy,$$

when we proceed to the limit and sum the elements by integration.

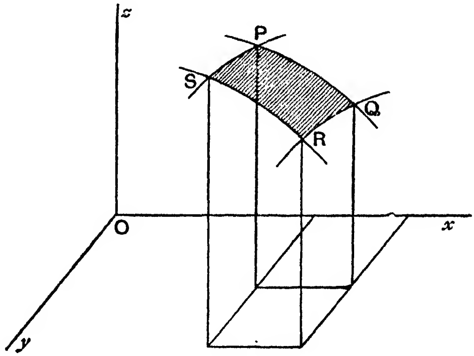


Fig. 266.

If the equation of the surface be thrown into the form
$$z=f(x, y),$$
and if we use the ordinary notation
$$p\equiv\frac{\partial z}{\partial x},\quad q\equiv\frac{\partial z}{\partial y},$$
this equation becomes $S=\iint\sqrt{1+p^2+q^2}\,dx\,dy.$

We may note in passing that the equation $\delta x\,\delta y=\delta S\cos\gamma$ also gives another expression for the volume, viz.
$$V=\iint z\,dx\,dy=\int z\cos\gamma\,dS.$$

We have taken, as is ordinarily the case, x, y as the independent variables.

If this be inconvenient, we should have
$$S=\iint\sqrt{1+\left(\frac{\partial x}{\partial z}\right)^2+\left(\frac{\partial x}{\partial y}\right)^2}\,dy\,dz,$$
or
$$S=\iint\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dz\,dx,$$
according as y, z or z, x be chosen as the independent variables.

773. We may note that the coordinates of P, Q, S and R, the coordinates of the curvilinear "parallelogram" bounding δS are:

$$\left.\begin{array}{llll} \text{for } P, & x, & y, & z; \\ \text{for } Q, & x+\delta x, & y, & z+\dfrac{\partial z}{\partial x}\delta x; \\ \text{for } S, & x, & y+\delta y, & z+\dfrac{\partial z}{\partial y}\delta y; \\ \text{for } R, & x+\delta x, & y+\delta y, & z+\dfrac{\partial z}{\partial x}\delta x+\dfrac{\partial z}{\partial y}\delta y, \end{array}\right\} \text{to the first order;}$$

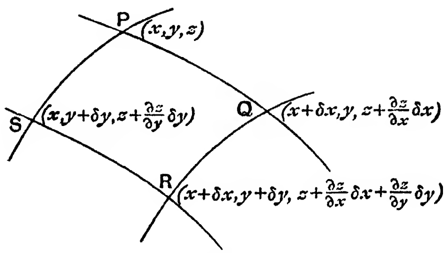


Fig. 267.

and the projections of this curvilinear parallelogram upon the coordinate planes are parallelograms of areas:

(1) upon the x-y plane, $\delta x\,\delta y$;

(2) upon the y-z plane,

$$\pm\begin{vmatrix} y, & z, & 1 \\ y, & z+\dfrac{\partial z}{\partial x}\delta x, & 1 \\ y+\delta y, & z+\dfrac{\partial z}{\partial y}\delta y, & 1 \end{vmatrix} = \pm\begin{vmatrix} y, & z, & 1 \\ 0, & \dfrac{\partial z}{\partial x}\delta x, & 0 \\ \delta y, & \dfrac{\partial z}{\partial y}\delta y, & 0 \end{vmatrix} = \frac{\partial z}{\partial x}\delta x\,\delta y;$$

(3) upon the z-x plane,

$$\pm\begin{vmatrix} x, & z, & 1 \\ x+\delta x, & z+\dfrac{\partial z}{\partial x}\delta x, & 1 \\ x, & z+\dfrac{\partial z}{\partial y}\delta y, & 1 \end{vmatrix} = \pm\begin{vmatrix} x, & z, & 1 \\ \delta x, & \dfrac{\partial z}{\partial x}\delta x, & 0 \\ 0, & \dfrac{\partial z}{\partial y}\delta y, & 0 \end{vmatrix} = \frac{\partial z}{\partial y}\delta x\,\delta y;$$

and the area δS is the square root of the sum of the squares of its projections upon any three mutually perpendicular planes (C. Smith, *Solid Geom.*, Art. 33).

Hence $$\delta S^2 = \delta x^2 \delta y^2 \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right],$$

giving $$S = \iint \sqrt{1+p^2+q^2}\,dx\,dy, \text{ as before.}$$

774. **Element of Volume for Cylindrical Coordinates.**

Instead of taking as our elementary volume one defined as bounded by planes parallel to three coordinate planes, other choices may be made. In some investigations it may be desirable to employ cylindrical coordinates, viz. ordinary polar coordinates r, θ in the x-y plane, retaining the Cartesian

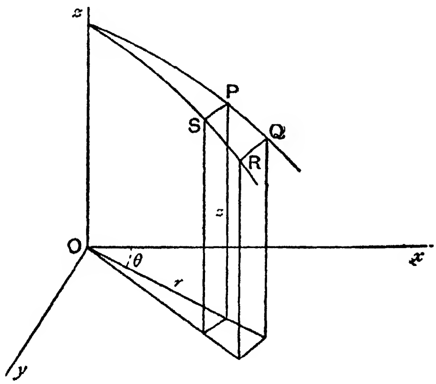


Fig. 268.

z-coordinate. An elementary prism, with this system, will be on a base $r\,\delta\theta\,\delta r$ with a height z, and to the second order its volume is $r\,\delta\theta\,\delta r \times z$, and the volume will be $\iint rz\,d\theta\,dr$, taken between suitable limits. If for any reason it be desirable to subdivide this elementary prism by planes perpendicular to the z-axis, our expression for the volume will be

$$\iiint r\,d\theta\,dr\,dz.$$

Such a necessity would arise, for instance, if the mass of the solid be required and the density be not a constant, but a known function of r, θ, z, when the mass of the elementary prism is $r\delta\theta\,\delta r\int\rho\,dz$, r and θ being regarded as constants during this integration, so as to add up all the elements of varying density through the elementary prism before summing the

masses of the several prisms themselves. We should then write the integral as

$$\text{Mass}=\iiint \rho r\, d\theta\, dr\, dz.$$

775. Spherical Polar Element of Volume.

Again, a spherical polar element of volume may be employed, using r the radius vector, θ the co-latitude and ϕ the azimuthal angle as coordinates.

Here the element of volume has three of its edges, mutually at right angles, δr, $r\,\delta\theta$ and $r \sin\theta\, \delta\phi$, and to the third order of infinitesimals its volume is $r^2 \sin\theta\, \delta\theta\, \delta\phi\, \delta r$, the difference

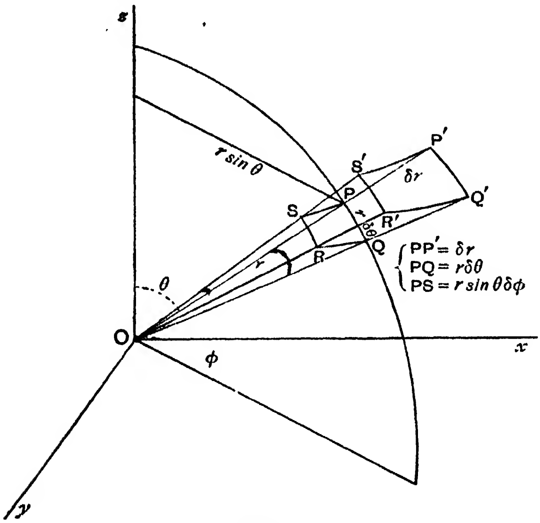


Fig. 269.

between this and the actual volume being at least of the fourth order of infinitesimals.

Upon integrating successively with regard to r, θ and ϕ in any order, the accumulated difference after the three integrations between the volume of any space required and the sum of these elements will be a first order infinitesimal at most, and therefore vanishes when the limit is taken.

Hence we have for the volume required

$$\iiint r^2 \sin\theta\, d\theta\, d\phi\, dr.$$

Further, if it be required to integrate any function of (r, θ, ϕ) throughout the volume, say $f(r, \theta, \phi)$, that is to add up all such elements as $f(r, \theta, \phi)\, r^2 \sin\theta\, \delta\theta\, \delta\phi\, \delta r$, the expression for the result will be

$$\iiint f(r, \theta, \phi)\, r^2 \sin\theta\, d\theta\, d\phi\, dr,$$

the limits being such as to include in the summation all the elements

$$f(r, \theta, \phi)\, r^2 \sin\theta\, \delta\theta\, \delta\phi\, \delta r,$$

which are included in the region under discussion, and no more.

776. Ex. If we apply this formula to find the volume of a sphere whose centre is at the origin,

the limits for r are from 0 to a, the radius of the sphere;
for θ are from 0 to π;
for ϕ are from 0 to 2π;

and

$$\begin{aligned} V &= \int_0^a \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta\, d\theta\, d\phi\, dr \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{a^3}{3} \sin\theta\, d\theta\, d\phi \\ &= \frac{2\pi a^3}{3} \int_0^{\pi} \sin\theta\, d\theta \\ &= \frac{2\pi a^3}{3} \Big[-\cos\theta \Big]_0^{\pi} = \tfrac{4}{3}\pi a^3. \end{aligned}$$

777. Elements of Surface. Cylindrical System.

In the cylindrical system of coordinates the element of surface δS, viz. the curvilinear parallelogram *PQRS*, Fig. 270, has for its projection upon the x-y plane the polar element $r\,\delta\theta\,\delta r$. Its projection upon the meridian plane through P is to the first order, an oblique parallelogram of area $\delta r \cdot \dfrac{\partial z}{\partial \theta}\delta\theta$, for one of its sides is the change in z due to increase of $\delta\theta$ in the independent variable θ, *i.e.* $\dfrac{\partial z}{\partial \theta}\delta\theta$, and the perpendicular between this side and the parallel side is δr.

And the projection upon a plane through P parallel to the z-axis and at right angles to the meridian plane, is similarly $r\,\delta\theta \dfrac{\partial z}{\partial r}\delta r$, for $r\,\delta\theta$ is the height of this parallelogram, and

$\dfrac{\partial z}{\partial r}\delta r$ is the change in z due to an increase δr in r, keeping θ constant, viz. the difference of the ordinates parallel to the z-axis of the points P and Q.

Hence

$$\delta S^2 = \delta r^2 (r\,\delta\theta)^2 + \left(\delta r \cdot \frac{\partial z}{\partial \theta}\delta\theta\right)^2 + (r\,\delta\theta\,\delta r)^2 \left(\frac{\partial z}{\partial r}\right)^2,$$

and taking the square root, proceeding to the limit and integrating,

$$S = \iint \sqrt{r^2 + r^2\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2}\, d\theta\, dr. \quad \ldots\ldots\ldots\ldots(1)$$

$z = f(x, y)$

θ r

z

$y = f(r, \theta)$

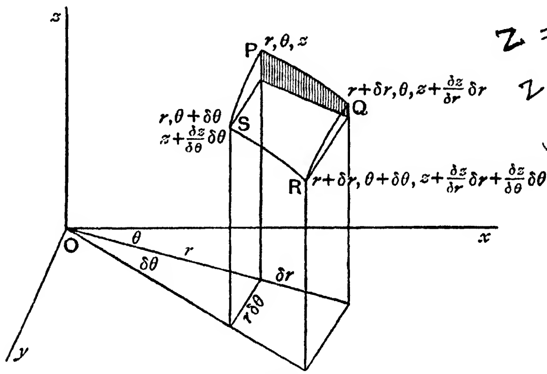


Fig. 270.

Similarly, if it were found preferable to take the pair z and θ for the independent variables, or the pair r and z, we should have in these respective cases,

$$S = \iint \sqrt{r^2 + r^2\left(\frac{\partial r}{\partial z}\right)^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\, dz\, d\theta \quad \ldots\ldots\ldots\ldots(2)$$

and

$$S = \iint \sqrt{1 + r^2\left(\frac{\partial \theta}{\partial z}\right)^2 + r^2\left(\frac{\partial \theta}{\partial r}\right)^2}\, dr\, dz. \quad \ldots\ldots\ldots\ldots(3)$$

To establish (2) an element is taken on the surface bounded by lines on the surface along which z is constant and θ const., viz. z, $z+\delta z$, θ, $\theta+\delta\theta$, and projected upon the same planes as in Case (1), the areas of the projections being

$$r\,\delta\theta\,\delta z, \quad r\,\delta\theta\left(\frac{\partial r}{\partial z}\delta z\right) \quad \text{and} \quad \delta z\left(\frac{\partial r}{\partial \theta}\delta\theta\right).$$

And to establish (3) an element is taken on the surface bounded by lines on the surface along which r=const. and

z=const., viz. r, $r+\delta r$, z, $z+\delta z$, and projection is made upon the same planes as in Case (1), the areas of the projections being

$$\delta r\,\delta z,\quad \left(r\frac{\partial\theta}{\partial z}\delta z\right)\delta r \quad\text{and}\quad \left(r\frac{\partial\theta}{\partial r}\delta r\right)\delta z.$$

The figures are, however, somewhat troublesome, and we shall deduce these formulae from a more general result later.

778. **In the spherical polar system of coordinates** let the meridian planes ϕ and $\phi+\delta\phi$ cut the surface in the curves

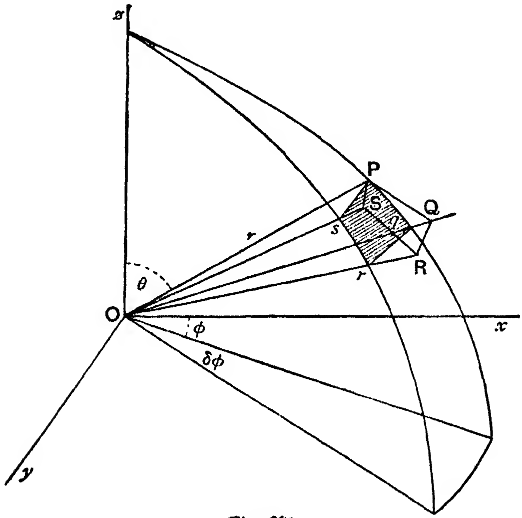


Fig. 271.

PQ, SR, and let the cones θ, $\theta+\delta\theta$ cut the surface in curves PS, QR. Then $PQRS$ is our element of surface. Let the coordinates of the points P, Q, R, S be respectively:

$$\begin{array}{llll}
\text{for } P, & r, & \theta, & \phi, \\[2ex]
\text{for } Q, & r+\dfrac{\partial r}{\partial\theta}\delta\theta, & \theta+\delta\theta, & \phi, \\[2ex]
\text{for } S, & r+\dfrac{\partial r}{\partial\phi}\delta\phi, & \theta, & \phi+\delta\phi, \\[2ex]
\text{for } R, & r+\dfrac{\partial r}{\partial\theta}\delta\theta+\dfrac{\partial r}{\partial\phi}\delta\phi, & \theta+\delta\theta, & \phi+\delta\phi.
\end{array}$$

The projections of this elementary area upon

(1) a plane through P at right angles to the radius vector;

(2) the meridian plane through P;

(3) a plane through P perpendicular to these two planes are respectively, to the second order,

$$r\,\delta\theta\,.\,r\sin\theta\,\delta\phi,\quad r\,\delta\theta\,.\left(\frac{\partial r}{\partial\phi}\,\delta\phi\right)\quad\text{and}\quad r\sin\theta\,\delta\phi\,.\left(\frac{\partial r}{\partial\theta}\,\delta\theta\right),$$

and to the fourth order we have for δS_r the element of area

$$\delta S_r^{\,2}=\left[r^4\sin^2\theta+r^2\left(\frac{\partial r}{\partial\phi}\right)^2+r^2\sin^2\theta\left(\frac{\partial r}{\partial\theta}\right)^2\right]\delta\theta^2\,\delta\phi^2\,;$$

whence, extracting the root, taking the limit and integrating,

$$S=\iint\sqrt{\left[r^4\sin^2\theta+r^2\left(\frac{\partial r}{\partial\phi}\right)^2+r^2\sin^2\theta\left(\frac{\partial r}{\partial\theta}\right)^2\right]}\,d\theta\,d\phi.\ \ldots\ldots(1)$$

779. If it be more convenient to take r and θ as the independent variables and ϕ dependent, elements must be chosen on the surface bounded by r, $r+\delta r$ and θ, $\theta+\delta\theta$, and the resultant expression for the elements will be

$$\delta S_\phi^{\,2}=\left[r^4\sin^2\theta\left(\frac{\partial\phi}{\partial r}\right)^2+r^2+r^2\sin^2\theta\left(\frac{\partial\phi}{\partial\theta}\right)^2\right]\delta\theta^2\,\delta r^2,$$

the areas of the projections on the same planes, as in Case (1), being

$$r\,\delta\theta\,.\,\delta r,\quad\left(r\sin\theta\frac{\partial\phi}{\partial r}\,\delta r\right).\,r\,\delta\theta\quad\text{and}\quad\left(r\sin\theta\,\frac{\partial\phi}{\partial\theta}\,\partial\theta\right).\,\delta r,$$

and the formula for S being

$$S=\iint\sqrt{\left[r^4\sin^2\theta\left(\frac{\partial\phi}{\partial r}\right)^2+r^2+r^2\sin^2\theta\left(\frac{\partial\phi}{\partial\theta}\right)^2\right]}\,d\theta\,dr.\ \ldots\ldots..(2)$$

And in the same way, if we wish to regard r and ϕ as the independent variables and θ dependent, an element of surface is to be chosen bounded by r, $r+\delta r$, ϕ, $\phi+\delta\phi$, and its projections upon the same planes, as in Case (1), being

$$(r\sin\theta\,\delta\phi)\,.\left(r\,\frac{\partial\theta}{\partial r}\,\delta r\right),\quad\left(r\,\frac{\partial\theta}{\partial\phi}\,\delta\phi\right).\,\delta r,\quad(r\sin\theta\,\delta\phi)\,.\,\delta r,$$

we have

$$\delta S_\theta^{\,2}=\left[r^4\sin^2\theta\left(\frac{\partial\theta}{\partial r}\right)^2+r^2\left(\frac{\partial\theta}{\partial\phi}\right)^2+r^2\sin^2\theta\right]\delta\phi^2\,\delta r^2$$

and
$$S=\iint\sqrt{\left[r^4\sin^2\theta\left(\frac{\partial\theta}{\partial r}\right)^2+r^2\left(\frac{\partial\theta}{\partial\phi}\right)^2+r^2\sin^2\theta\right]}\,d\phi\,dr.\ldots(3)$$

But the figures required are, as in the Cases (2) and (3), for cylindrical coordinates somewhat troublesome, and we propose to deduce these formulae from the more general result of Art. 790.

780. Areas on a Spherical Surface, the Origin being at the Centre.

Let a be the radius of the sphere. Then, putting $r=a$, the general formula

$$S=\iint\sqrt{r^4\sin^2\theta+r^2\sin^2\theta\left(\frac{\partial r}{\partial\theta}\right)^2+r^2\left(\frac{\partial r}{\partial\phi}\right)^2}\,d\theta\,d\phi$$

reduces to
$$S=a^2\iint\sin\theta\,d\theta\,d\phi$$
$$=a^2\int[-\cos\theta]\,d\phi.$$

If we apply the result to find the area bounded by two meridian arcs and some specified curve, $\theta=f(\phi)$, the limits for θ are from $\theta=0$ to $\theta=f(\phi)$, and

$$S=a^2\int[1-\cos f(\phi)]\,d\phi,$$

the result of Art. 734.

Cor. For the whole sphere $f(\phi)=\pi$, and

$$S=2a^2\int_0^{2\pi}d\phi=4\pi a^2.$$

781. Spherical Triangle.

Ex. Let us apply the formula obtained to the case of the area bounded by a great circle and two meridian arcs, the radius of the sphere being a.

Take as the plane of xz that through the centre which cuts the great circle perpendicularly, and let p be the spherical perpendicular from the pole upon the great circle arc. The equation of the great circle is then

$$\cos\phi=\frac{\cot\theta}{\cot p}.$$

Then
$$\sin\phi\,d\phi=\frac{\operatorname{cosec}^2\theta\,d\theta}{\cot p},$$

and
$$\text{Area}=a^2\int(1-\cos\theta)\,d\phi=a^2\int\frac{(1-\cos\theta)\operatorname{cosec}^2\theta}{\sqrt{\cot^2 p-\cot^2\theta}}\,d\theta;$$

$$\therefore\ \frac{\text{Area}}{a^2}=\cos^{-1}\frac{\cot\theta}{\cot p}+\sin^{-1}\frac{\sin p}{\sin\theta}$$
$$=[\phi]+[\chi],$$

where χ is the angle a meridian makes with the great circle and ϕ is the azimuthal angle.

If we take limits $\phi=\alpha$ to $\phi=\alpha+A$, the limits for χ will be $\pi-C$ to B where ABC is the spherical triangle formed by the meridians AB, AC and the arc BC.

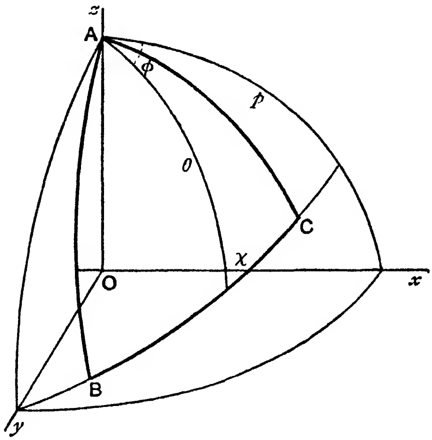


Fig. 272.

This area is therefore $a^2[A+B-(\pi-C)]$

$$=a^2[A+B+C-\pi]$$
$$=a^2E,$$

where E is the spherical excess, a result readily established in an elementary manner. (GIRARD'S THEOREM. See Todhunter and Leathem, *Sph. Trig.*, Art. 127.) Other illustrations have been given earlier. (See Art. 734.)

782. Case of a Solid of Revolution.

In the case of any solid of revolution about the z-axis ϕ varies, but r is independent of ϕ and depends only upon the revolving curve generating the solid.

The general formula

$$S=\iint\sqrt{\left\{r^4\sin^2\theta+r^2\sin^2\theta\left(\frac{\partial r}{\partial\theta}\right)^2+r^2\left(\frac{\partial r}{\partial\phi}\right)^2\right\}}\,d\theta\,d\phi$$

now reduces to

$$S=\iint r\sin\theta\sqrt{r^2+\left(\frac{\partial r}{\partial\theta}\right)^2}\,d\theta\,d\phi$$

$$=2\pi\int r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=2\pi\int r\sin\theta\,ds,$$

in conformity with the result of Art. 748.

783. In the case of **solids formed by the revolution about the z-axis of circles whose planes pass through the z-axis, centred at the origin**, but of varying radius, r is a function of ϕ alone, and

780.

$$S=\iint r\sqrt{r^2\sin^2\theta+\left(\frac{\partial r}{\partial\phi}\right)^2}\,d\theta\,d\phi.$$

entre. ape of the surface may be pictured as somewhat resembling the hermit-crab shell.

Ex. Let the surface be $r=ae^{\phi}$.

$$S=a^2\iint e^{2\phi}\sqrt{1+\sin^2\theta}\,d\theta\,d\phi,$$

and θ, ϕ are independent,

$$=\frac{a^2}{2}\Big[e^{2\phi}\Big]_{\phi_1}^{\phi_2}\sqrt{2}\int_0^{\pi}\sqrt{1-\tfrac{1}{2}\cos^2\theta}\,d\theta.$$

Let
$$\theta=\frac{\pi}{2}-\chi.$$

$$S=\frac{a^2}{\sqrt{2}}(e^{2\phi_2}-e^{2\phi_1})\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\tfrac{1}{2}\sin^2\chi}\,d\chi$$

$$=a^2\sqrt{2}(e^{2\phi_1}-e^{2\phi_2})\int_0^{\frac{\pi}{2}}\sqrt{1-\tfrac{1}{2}\sin^2\chi}\,d\chi$$

$$=a^2\sqrt{2}(e^{2\phi_1}-e^{2\phi_2})\,E_1;\quad \text{mod. }\frac{1}{\sqrt{2}};$$

and if the area be taken from $r=0$, *i.e.* $\phi_2=-\infty$ to any value of r,

$$S=r^2\sqrt{2}\,E_1;\quad \text{mod. }\frac{1}{\sqrt{2}}.$$

784. In the case of **an area of a portion of a right circular cone**, vertex at the origin, axis the z-axis and semivertical angle α, the general formula

$$S=\iint\sqrt{\left\{r^4\sin^2\theta\left(\frac{\partial\theta}{\partial r}\right)^2+r^2\left(\frac{\partial\theta}{\partial\phi}\right)^2+r^2\sin^2\theta\right\}}\,d\phi\,dr$$

reduces to
$$\iint r\sin\alpha\,d\phi\,dr=\frac{\sin\alpha}{2}\int[r^2]\,d\phi.$$

And supposing the area in question to be bounded by some curve drawn upon the cone, say $r=f(\phi)$, and two generators, we have $[r^2]=\{f(\phi)\}^2$, the lower limit being $r=0$, and

$$\therefore\ S=\frac{\sin\alpha}{2}\int\{f(\phi)\}^2\,d\phi.$$

785. The formula is obviously the same thing as

$$\frac{1}{2}\int r^2 d(\phi \sin a),$$

which is the area of the portion of the cone developed upon a plane, the angle between two generators so developed and corresponding to azimuthal angles ϕ and $\phi+\delta\phi$ on the cone, being $\delta\phi \sin a$.

786. Or again it is the same thing as

$$\frac{1}{2}\int (r \sin a)^2\, d\phi = S \sin a,$$

i.e. the area of the projection upon the x-y plane, all elements of the cone making an angle $\frac{\pi}{2}-a$ with the x-y plane.

As a particular and elementary case, the area cut off by a plane perpendicular to the axis and intercepting generators of length l is

$$S=\frac{\sin a}{2}\, l^2 \int_0^{2\pi} d\phi = \pi a l,$$

where a is the radius of the base $= l \sin a$ and l the "slant height," the ordinary mensuration formula.

787. In the case of **any cone with vertex at the origin**, the equation is of the form $\phi=f(\theta)$, r being absent from the equation. Hence $\frac{\partial\phi}{\partial r}=0$. The general expression

$$S=\iint\sqrt{\left\{r^4 \sin^2\theta\left(\frac{\partial\phi}{\partial r}\right)^2+r^2+r^2\sin^2\theta\left(\frac{\partial\phi}{\partial\theta}\right)\right\}^2}\, d\theta\, dr$$

in this case reduces to

$$\iint r\sqrt{1+\sin^2\theta\left(\frac{\partial\phi}{\partial\theta}\right)^2}\, d\theta\, dr,$$

i.e.

$$S=\frac{1}{2}\int [r^2]\sqrt{1+\sin^2\theta\left(\frac{\partial\phi}{\partial\theta}\right)^2}\, d\theta.$$

Hence, if a surface cut a cone whose vertex is the origin, viz. $\phi=f(\theta)$, the area of the cone between two of its generators and the curve in which it meets the surface is

$$\frac{1}{2}\int r^2\{(1+\sin^2\theta f'^2(\theta)\}^{\frac{1}{2}}\, d\theta.$$

788. Ex. The equations of a cylinder and a cone are

$$r\sin\theta = a \quad \text{and} \quad \cot\theta = \sinh\phi.$$

If A_1, A_2, A_3 be the areas of the cone from $\phi=0$ to $\phi=\beta-\alpha$, β and $\beta+\alpha$ respectively, then will

$$A_1 + A_3 = 2A_2\cosh\alpha. \qquad \text{[MATH. TRIPOS, 1875.]}$$

In this case $-\operatorname{cosec}^2\theta = \cosh\phi\dfrac{d\phi}{d\theta}$.

Hence
$$r^2\sqrt{1+\sin^2\theta\left(\frac{\partial\phi}{\partial\theta}\right)^2}\,d\theta = \frac{a^2}{\sin^2\theta}\sqrt{1+\frac{\operatorname{cosec}^2\theta}{\cosh^2\phi}}\,d\theta$$
$$= -a^2\cosh\phi\,d\phi\sqrt{1+\frac{1+\sinh^2\phi}{\cosh^2\phi}}$$
$$= -a^2\sqrt{2}\cosh\phi\,d\phi,$$

and
$$S = -a^2\sqrt{2}\,[\sinh\phi].$$

Hence
$$\frac{A_1+A_3}{A_2} = \frac{\sinh\overline{\beta-\alpha}+\sinh\overline{\beta+\alpha}}{\sinh\beta} = 2\cosh\alpha.$$

789. **Generalised Results. Orthogonal Coordinates.**

If $f(x, y, z) = \lambda$ be any surface, it is required to find the normal distance between the surface and the contiguous surface $\lambda+\delta\lambda$ at the point (x, y, z). Let the normal at P to the surface λ cut the surface $\lambda+\delta\lambda$ at Q, whose coordinates are $x+\delta x$, $y+\delta y$, $z+\delta z$.

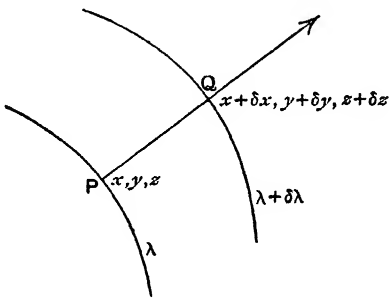


Fig. 273.

The direction cosines of the normal are $\dfrac{\lambda_x}{h}, \dfrac{\lambda_y}{h}, \dfrac{\lambda_z}{h}$, where suffixes represent partial differentiations and $h^2 = \lambda_x^2+\lambda_y^2+\lambda_z^2$.

Then projecting the broken line $\delta x, \delta y, \delta z$ upon PQ, we have

$$PQ = \delta x\frac{\lambda_x}{h} + \delta y\frac{\lambda_y}{h} + \delta z\frac{\lambda_z}{h} = \frac{\delta\lambda}{h}.$$

Let $\quad f_1(x, y, z)=\lambda, \quad f_2(x, y, z)=\mu, \quad f_3(x, y, z)=\nu$
be three mutually orthogonal surfaces. Consider the small element of space whose faces are the three surfaces λ, μ, ν and the contiguous surfaces $\lambda+\delta\lambda$, $\mu+\delta\mu$, $\nu+\delta\nu$.

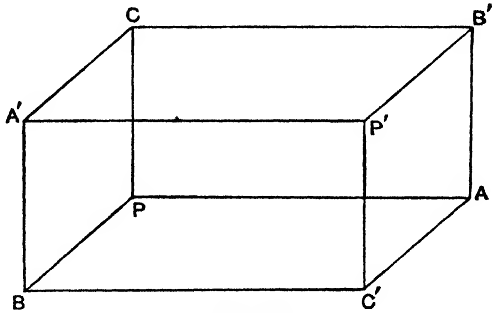


Fig. 274.

Let P be the point (λ, μ, ν), PP' the diagonal through P of the element and $\lambda+\delta\lambda$, $\mu+\delta\mu$, $\nu+\delta\nu$ the coordinates of P'. Let the edges of this element be PA, PB, PC, $P'A'$, $P'B'$, $P'C'$ etc., PA being an element of the normal to λ, etc. This elementary space is ultimately an infinitesimal rectangular parallelepiped or 'cuboid.' Its edges are $\dfrac{\delta\lambda}{h_1}, \dfrac{\delta\mu}{h_2}, \dfrac{\delta\nu}{h_3}$, where

$$h_1^2=\lambda_x^2+\lambda_y^2+\lambda_z^2, \quad h_2^2=\mu_x^2+\mu_y^2+\mu_z^2, \quad h_3^2=\nu_x^2+\nu_y^2+\nu_z^2.$$

Its volume is $$\frac{\delta\lambda\,\delta\mu\,\delta\nu}{h_1 h_2 h_3}.$$

Moreover, if (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3), be the direction cosines of the elements

$$\frac{\delta\lambda}{h_1}, \quad \frac{\delta\mu}{h_2}, \quad \frac{\delta\nu}{h_3},$$

$\dfrac{\delta\lambda}{h_1} l_1=$the projection of PA upon the x-axis

$=$the small change in x due to increase of λ to $\lambda+\delta\lambda$, μ and ν remaining unaltered,

$$=\frac{\partial x}{\partial\lambda}\delta\lambda\,;$$

hence $$l_1=h_1\frac{\partial x}{\partial\lambda}.$$

Similarly $$\frac{\delta\lambda}{h_1}m_1=\frac{\partial y}{\partial\lambda}\delta\lambda,\quad \frac{\delta\lambda}{h_1}n_1=\frac{\partial z}{\partial\lambda}\delta\lambda,$$

$$\frac{\delta\mu}{h_2}l_2=\frac{\partial x}{\partial\mu}\delta\mu,\ \text{etc.};$$

hence we have

$$l_1=h_1\frac{\partial x}{\partial\lambda},\quad m_1=h_1\frac{\partial y}{\partial\lambda},\quad n_1=h_1\frac{\partial z}{\partial\lambda},$$

$$l_2=h_2\frac{\partial x}{\partial\mu},\quad m_2=h_2\frac{\partial y}{\partial\mu},\quad n_2=h_2\frac{\partial z}{\partial\mu},$$

$$l_3=h_3\frac{\partial x}{\partial\nu},\quad m_3=h_3\frac{\partial y}{\partial\nu},\quad n_3=h_3\frac{\partial z}{\partial\nu}.$$

Thus J or $\dfrac{\partial(x, y, z)}{\partial(\lambda, \mu, \nu)}$, the Jacobian* of x, y, z with regard to λ, μ, ν,

$$=\frac{1}{h_1h_2h_3}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=\pm\frac{1}{h_1h_2h_3}.$$

(See C. Smith, *Solid Geometry*, Art. 46.)

Thus the volume of the elementary cuboid is $\pm J\,\delta\lambda\,\delta\mu\,\delta\nu$, and V, the volume of any region which is divided up into elements by this system, is given by

$$V=\iiint J\,d\lambda\,d\mu\,d\nu.$$

The ambiguity of sign disappears when the limits have been suitably assigned for the evaluation of the whole volume under consideration.

COR. (1). In the Cartesian system

$$\lambda=x,\quad \mu=y,\quad \nu=z,\quad h_1=h_2=h_3=1,$$

and the formula reduces to

$$V=\iiint dx\,dy\,dz\,;$$

the formula of Art. 760.

(2) In the cylindrical system $\lambda=r$, $\mu=\theta$, $\nu=z$, $x=r\cos\theta$, $y=r\sin\theta$, $z=z$, and the elements are δr, $r\,\delta\theta$, δz,

$$h_1=1,\quad h_2=\frac{1}{r},\quad h_3=1,$$

* See *Diff. Calc.*, Art. 534.

and the formula reduces to

$$V=\iiint r\,d\theta\,dr\,dz;$$

the formula of Art. 774.

(3) In the spherical polar system $\lambda=r$, $\mu=\theta$, $\nu=\phi$,

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta,$$

and the elements are δr, $r\,\delta\theta$, $r\sin\theta\,\delta\phi$, and

$$h_1=1,\quad h_2=\frac{1}{r},\quad h_3=\frac{1}{r\sin\theta},$$

and the formula reduces to

$$V=\iiint r^2\sin\theta\,d\theta\,d\phi\,dr,$$

the formula established in Art. 775.

790. **Element of Surface.**

Suppose the region bounded by any surface S to have been divided up in the manner described by three families of ortho-

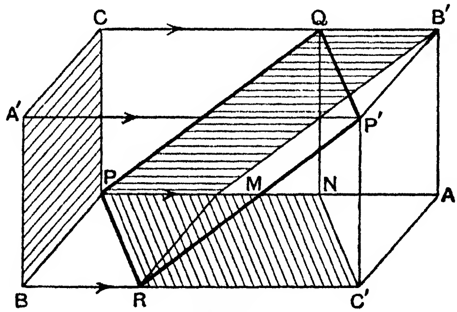


Fig. 275.

gonal surfaces whose distinctive parameters are λ, μ, ν; any pair, say μ, ν, with their contiguous surfaces $\mu+\delta\mu$, $\nu+\delta\nu$, form a tubular region within S. Suppose this tube to cut the tangent plane at P to the surface in the plane $P'RPQ$, which may in the limit be regarded as an indefinitely small parallelogram element of the surface. Its area is an infinitesimal of the second order. We may take it as axiomatic that the difference between the area of the intercepted portion δS_λ of the surface, and the area of this parallelogram is at least of the third order, on the supposition that the curvature is finite and

continuous over the portion considered. The area of the parallelogram $P'RPQ$ is readily found from the fact that the square of any plane area is the sum of the squares of its projections upon any three mutually perpendicular planes (C. Smith, *Solid Geom.*, Art. 33). Let the cuboid element of the μ-ν tube, for which PP' is a diagonal, be constructed as in Art. 789, with PA, PB. PC for adjacent edges through P and $P'A'$, $P'B'$, $P'C'$ for opposite edges through P' (Fig. 275). Let QN and RM be drawn at right angles to PA. Join $C'N$ and $B'M$. Thus the parallelograms $PBA'C$, $PQB'M$, $PRC'N$ are the projections of $PRP'Q$ upon three mutually perpendicular planes. The areas of these figures are respectively

$$PB.PC, \quad PC.PM, \quad PB.PN,$$

and it will be observed that $PN=RC'=MA$, *i.e.*

$$PM+PN=PA.$$

Now, as we have taken $f_1(x, y, z)=\lambda$, $f_2(x, y, z)=\mu$ and $f_3(x, y, z)=\nu$, we can express x, y, z in terms of λ, μ, ν, and the equation of the surface S may be expressed in the form $F(\lambda, \mu, \nu)=0$ by substituting for x, y and z these values. In fact λ, μ, ν form a new system of coordinates; and of these we are regarding μ and ν as independent and λ depending upon them. When μ and ν change to $\mu+\delta\mu$ and $\nu+\delta\nu$, the total change of λ is $\delta\lambda=\dfrac{\partial\lambda}{\partial\mu}\delta\mu+\dfrac{\partial\lambda}{\partial\nu}\delta\nu$ to the first order.

Now, in our Fig. 275, PM represents that part of PA which depends upon $\delta\mu$, and MA, that is, PN represents that part of PA which depends upon $\delta\nu$, *i.e.*

$$PM=\frac{1}{h_1}\frac{\partial\lambda}{\partial\mu}\delta\mu \quad \text{and} \quad PN=\frac{1}{h_1}\frac{\partial\lambda}{\partial\nu}\delta\nu,$$

the two making up the total length of PA, *i.e.* $\dfrac{\delta\lambda}{h_1}$.

We thus have, to the fourth order,

$$\delta S_\lambda^2=(PB.PC)^2+(PC.PM)^2+(PB.PN)^2$$

$$=\left(\frac{\delta\mu}{h_2}\cdot\frac{\delta\nu}{h_3}\right)^2+\left(\frac{\delta\nu}{h_3}\cdot\frac{1}{h_1}\frac{\partial\lambda}{\partial\mu}\delta\mu\right)^2+\left(\frac{\delta\mu}{h_2}\cdot\frac{1}{h_1}\frac{\partial\lambda}{\partial\nu}\delta\nu\right)^2$$

or
$$\delta S_\lambda^2=\left[h_1^2+h_2^2\left(\frac{\partial\lambda}{\partial\mu}\right)^2+h_3^2\left(\frac{\partial\lambda}{\partial\nu}\right)^2\right]\frac{\delta\mu^2\delta\nu^2}{h_1^2h_2^2h_3^2}.$$

Similarly, if we had taken ν, λ or λ, μ as the independent pair of parameters and constructed the corresponding tubes, we should have had

$$\delta S_\mu{}^2=\left[h_2{}^2+h_3{}^2\left(\frac{\partial\mu}{\partial\nu}\right)^2+h_1{}^2\left(\frac{\partial\mu}{\partial\lambda}\right)^2\right]\frac{\delta\nu^2\,\delta\lambda^2}{h_1{}^2h_2{}^2h_3{}^2},$$

$$\delta S_\nu{}^2=\left[h_3{}^2+h_1{}^2\left(\frac{\partial\nu}{\partial\lambda}\right)^2+h_2{}^2\left(\frac{\partial\nu}{\partial\mu}\right)^2\right]\frac{\delta\lambda^2\,\delta\mu^2}{h_1{}^2h_2{}^2h_3{}^2},$$

and any of the three surface elements δS_λ, δS_μ, δS_ν intercepted by μ-ν tubes, ν-λ tubes or λ-μ tubes respectively, may be taken as an element of the surface for integration for the whole.

Thus we obtain, when we proceed to take the square root and integrate,

$$S=\iint\sqrt{h_1{}^2+h_2{}^2\left(\frac{\partial\lambda}{\partial\mu}\right)^2+h_3{}^2\left(\frac{\partial\lambda}{\partial\nu}\right)^2}\,\frac{d\mu\,d\nu}{h_1h_2h_3}$$

$$=\iint\sqrt{h_2{}^2+h_3{}^2\left(\frac{\partial\mu}{\partial\nu}\right)^2+h_1{}^2\left(\frac{\partial\mu}{\partial\lambda}\right)^2}\,\frac{d\nu\,d\lambda}{h_1h_2h_3}$$

$$=\iint\sqrt{h_3{}^2+h_1{}^2\left(\frac{\partial\nu}{\partial\lambda}\right)^2+h_2{}^2\left(\frac{\partial\nu}{\partial\mu}\right)^2}\,\frac{d\lambda\,d\mu}{h_1h_2h_3}.$$

791. Cor. 1. If the Cartesian system be taken,

$$\lambda=x,\quad \mu=y,\quad \nu=z,\quad h_1=h_2=h_3=1,$$

and the elements are δx, δy, δz, and

$$S=\iint\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz$$

$$=\iint\sqrt{1+\left(\frac{\partial y}{\partial z}\right)^2+\left(\frac{\partial y}{\partial x}\right)^2}\,dz\,dx$$

$$=\iint\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy,$$

viz. the formulae of Art. 772.

Cor. 2. If the cylindrical system be taken,

$$\lambda=r,\quad \mu=\theta,\quad \nu=z,$$

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z,$$

and r, θ, z form an orthogonal system, the elements being

$$\delta r, \quad r\,\delta\theta, \quad \delta z \quad \text{and} \quad h_1=1, \quad h_2=\frac{1}{r}, \quad h_3=1;$$

$$\delta S_r^2=(r\,\delta\theta\,.\,\delta z)^2+(\delta z)^2\left(\frac{\partial r}{\partial \vartheta}\,\delta\theta\right)^2+(r\,\delta\theta)^2\left(\frac{\partial r}{\partial z}\,\delta z\right)^2,$$

$$\delta S_\theta^2=(\delta z\,.\,\delta r)^2+(\delta r)^2\left(r\frac{\partial \vartheta}{\partial z}\,\delta z\right)^2+(\delta z)^2\left(r\frac{\partial \theta}{\partial r}\,\delta r\right)^2,$$

$$\delta S_z^2=(\delta r\,.\,r\,\delta\theta)^2+(r\,\delta\theta)^2\left(\frac{\partial z}{\partial r}\,\delta r\right)^2+(\delta r)^2\left(\frac{\partial z}{\partial \vartheta}\,\delta\theta\right)^2,$$

according as r, θ or z is the dependent variable, giving the formulae

$$S=\iint\sqrt{r^2+\left(\frac{\partial r}{\partial\vartheta}\right)^2+r^2\left(\frac{\partial r}{\partial z}\right)^2}\,d\vartheta\,dz$$

$$=\iint\sqrt{1+r^2\left(\frac{\partial\vartheta}{\partial z}\right)^2+r^2\left(\frac{\partial\theta}{\partial r}\right)^2}\,dz\,dr$$

$$=\iint\sqrt{r^2+r^2\left(\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial\vartheta}\right)^2}\,dr\,d\theta,$$

which are in agreement with those of Art. 777.

COR. 3. In the spherical polar system,

$$\lambda=r, \quad \mu=\theta, \quad \nu=\phi$$

and $\quad x=r\sin\theta\cos\phi, \quad y=r\sin\theta\sin\phi, \quad z=r\cos\theta,$

and r, θ, ϕ form an orthogonal system, the elements being

$$\delta r, \quad r\,\delta\vartheta, \quad r\sin\theta\,\delta\phi \quad \text{and} \quad h_1=1, \quad h_2=\frac{1}{r}, \quad h_3=\frac{1}{r\sin\theta};$$

whence

$$\delta S_r^2=r^2\,\delta\theta^2\,.\,r^2\sin^2\vartheta\,\delta\phi^2+r^2\sin^2\theta\,\delta\phi^2\left(\frac{\partial r}{\partial\vartheta}\,\delta\theta\right)^2$$
$$+r^2\,\delta\theta^2\left(\frac{\partial r}{\partial\phi}\,\delta\phi\right)^2,$$

$$\delta S_\theta^2=r^2\sin^2\theta\,\delta\phi^2\,.\,\delta r^2\quad+\delta r^2\left(r\frac{\partial\theta}{\partial\phi}\,\delta\phi\right)^2$$
$$+r^2\sin^2\theta\,\delta\phi^2\left(r\frac{\partial\theta}{\partial r}\,\delta r\right)^2,$$

$$\delta S_\phi^2=\delta r^2\,r^2\,\delta\theta^2\quad+r^2\,\delta\theta^2\left(r\sin\theta\frac{\partial\phi}{\partial r}\,\delta r\right)^2$$
$$+\delta r^2\left(r\sin\theta\frac{\partial\phi}{\partial\theta}\,\delta\theta\right)^2,$$

giving the formulae

$$S=\iint\sqrt{r^4\sin^2\theta+r^2\sin^2\theta\left(\frac{\partial r}{\partial\theta}\right)^2+r^2\left(\frac{\partial r}{\partial\phi}\right)^2}\,d\theta\,d\phi$$
$$=\iint\sqrt{r^2\sin^2\theta+r^2\left(\frac{\partial\theta}{\partial\phi}\right)^2+r^4\sin^2\theta\left(\frac{\partial\theta}{\partial r}\right)^2}\,d\phi\,dr$$
$$=\iint\sqrt{r^2+r^4\sin^2\theta\left(\frac{\partial\phi}{\partial r}\right)^2+r^2\sin^2\theta\left(\frac{\partial\phi}{\partial\theta}\right)^2}\,dr\,d\theta,$$

according as r, θ or ϕ is taken as the dependent variable, formulae in agreement with those of Arts. 778 and 779.

792. Change of the Variables. **Form of Element of Area.**

Supposing the coordinates x, y of any point in the plane of x-y to be expressed in terms of two new variables u, v, let us consider the nature of the figure bounded by the four curves obtained by assigned values of u, v, viz.

$$u,\quad u+\delta u,\quad v,\quad v+\delta v.$$

Let the figure thus bounded be $PQRS$,

δu being zero along PS,

δv being zero along PQ.

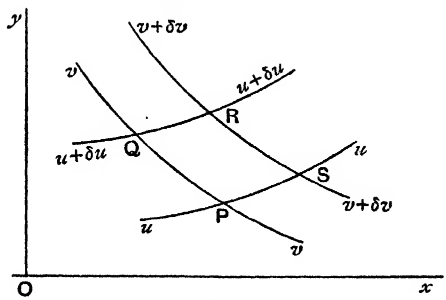


Fig. 276.

The several Cartesian coordinates of the four corners are, to the first order,

for P, $\quad x,\qquad y$;

for Q, $\quad x+\dfrac{\partial x}{\partial u}\delta u,\qquad y+\dfrac{\partial y}{\partial u}\delta u$;

for S, $\quad x+\dfrac{\partial x}{\partial v}\delta v,\qquad y+\dfrac{\partial y}{\partial v}\delta v$;

for R, $\quad x+\dfrac{\partial x}{\partial u}\delta u+\dfrac{\partial x}{\partial v}\delta v,\qquad y+\dfrac{\partial y}{\partial u}\delta u+\dfrac{\partial y}{\partial v}\delta v.$

The direction ratios of PQ and SR are $\frac{\partial x}{\partial u}\delta u,\ \frac{\partial y}{\partial u}\delta u,$

and of PS and QR $\quad \frac{\partial x}{\partial v}\delta v,\ \frac{\partial y}{\partial v}\delta v.$

Hence the chords joining the corresponding points are such as, to the first order, to form the four sides of a parallelogram whose area is

$$\begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v}, & \frac{\partial y}{\partial v} \end{vmatrix} \delta u\, \delta v \quad \text{or} \quad \frac{\partial(x, y)}{\partial(u, v)}\delta u\, \delta v.$$

This then is, to the second order, the area of the elementary curvilineal "parallelogram" PR, the difference between this area and that of the chordal parallelogram being at least of the third order of infinitesimals. Hence, taking the limit and integrating between any assigned limits, for u and v, we have

$$\text{Area} = \iint \frac{\partial(x, y)}{\partial(u, v)} du\, dv = \iint J\, du\, dv,$$

where J is the Jacobian of x, y with regard to u and v.

It will be remembered that if J' be the Jacobian of u, v with regard to x, y, we have $JJ'=1$ (*Diff. Calc.*, Art. 540).

And in cases where u and v are already expressed in terms of x and y, instead of x, y in terms of u and v, this rule will often facilitate the calculation of J.

Similarly, if we wish to integrate any function of x and y, say $f(x, y)$, over the area considered, *i.e.* to find $\Sigma f(x, y)\,\delta A$ where δA is an infinitesimal element of the area, it is only necessary to express x and y in terms of u and v, and then to transform the function $f(x, y)$ so as to express it as a function of u and v, say $F(u, v)$, then to multiply it by $J\,\delta u\,\delta v$, and integrate, the result being

$$\iint F(u, v)\, J\, du\, dv.$$

793. **Illustrative Examples.**

1. Find the area of the Carnot's cycle bounded by the isothermals $xy=\alpha_1$, $xy=\alpha_2$, and the adiabatics $xy^\gamma=\beta_1$, $xy^\gamma=\beta_2$.

Putting $xy=u$, $xy^\gamma=v$, take an element of the area bounded by the curves u, v, $u+\delta u$, $v+\delta v$.

Here
$$J'=\begin{vmatrix} \frac{\partial u}{\partial x}, & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y}, & \frac{\partial v}{\partial y} \end{vmatrix}=\begin{vmatrix} y, & y^\gamma \\ x, & \gamma x y^{\gamma-1} \end{vmatrix}$$
$$=(\gamma-1)\,xy^\gamma=(\gamma-1)\,v;$$
$$\therefore\ J=\frac{1}{\gamma-1}\cdot\frac{1}{v};$$

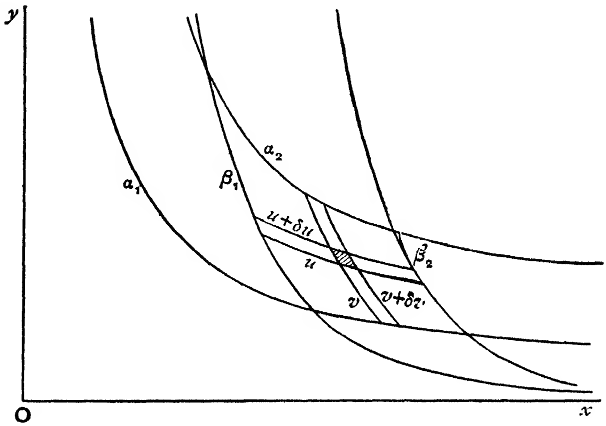


Fig. 277.

and
$$\text{Area required}=\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}\frac{1}{\gamma-1}\cdot\frac{1}{v}\,du\,dv$$
$$=\frac{\alpha_2-\alpha_1}{\gamma-1}\log\frac{\beta_2}{\beta_1}.$$ (See page 63, Ex. 28.)

2. The portions of the curves $xy=a^2$, $x^2-y^2=b^2$, which lie in the positive quadrant, are drawn intersecting at B. The former intersects the asymptote of the latter in C, and the latter meets OX in A. If every element of the area $OABC$ be multiplied by the square of its distance from the origin O, the sum will be equal to $\frac{1}{2}a^2b^2$. [Colleges α, 1884.]

794. Change of the Variables. Form of Element of Volume.

Again, let the coordinates x, y, z of any point in space be expressed in terms of three new independent variables u, v, w, the surfaces u=const., v=const., w=const., not necessarily as in Art. 789, forming an orthogonal system.

Let us consider the nature of the figure bounded by the six surfaces obtained by assigned values of u, v, w, viz.

$$u, \quad u+\delta u, \quad v, \quad v+\delta v, \quad w, \quad w+\delta w.$$

$$\left.\begin{array}{l}\text{Let the figure thus bounded be } PQS'RP'Q'SR',\\ \delta u \text{ being zero over the surface } PRQ'S,\\ \delta v \text{ being zero over the surface } PQR'S,\\ \delta w \text{ being zero over the surface } PQS'R,\end{array}\right\}\text{Fig. 278.}$$

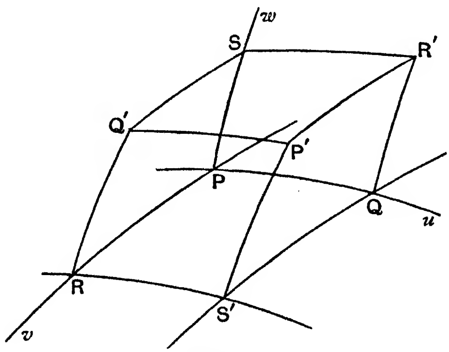


Fig. 278.

The several coordinates of these eight corners are, to the first order,

$$\begin{array}{llll}
\text{for } P, & x, & y, & z,\\[1ex]
\text{for } Q, & x+\dfrac{\partial x}{\partial u}\delta u, & y+\dfrac{\partial y}{\partial u}\delta u, & z+\dfrac{\partial z}{\partial u}\delta u,\\[1ex]
\text{for } R, & x+\dfrac{\partial x}{\partial v}\delta v, & y+\dfrac{\partial y}{\partial v}\delta v, & z+\dfrac{\partial z}{\partial v}\delta v,\\[1ex]
\text{for } S, & x+\dfrac{\partial x}{\partial w}\delta w, & y+\dfrac{\partial y}{\partial w}\delta w, & z+\dfrac{\partial z}{\partial w}\delta w,\\[1ex]
\text{for } Q', & \multicolumn{3}{l}{x+\dfrac{\partial x}{\partial v}\delta v+\dfrac{\partial x}{\partial w}\delta w, \text{ etc.},}\\[1ex]
\text{for } R', & \multicolumn{3}{l}{x+\dfrac{\partial x}{\partial w}\delta w+\dfrac{\partial x}{\partial u}\delta u, \text{ etc.},}\\[1ex]
\text{for } S', & \multicolumn{3}{l}{x+\dfrac{\partial x}{\partial u}\delta u+\dfrac{\partial x}{\partial v}\delta v; \text{ etc.},}\\[1ex]
\text{for } P', & \multicolumn{3}{l}{x+\dfrac{\partial x}{\partial u}\delta u+\dfrac{\partial x}{\partial v}\delta v+\dfrac{\partial x}{\partial w}\delta w, \text{ etc.}}
\end{array}$$

The direction ratios of PQ, RS', $Q'P'$, SR' are

$$\frac{\partial x}{\partial u}\delta u,\quad \frac{\partial y}{\partial u}\delta u,\quad \frac{\partial z}{\partial u}\delta u;$$

those of PR, QS', $R'P'$, SQ' are

$$\frac{\partial x}{\partial v}\delta v,\quad \frac{\partial y}{\partial v}\delta v,\quad \frac{\partial z}{\partial v}\delta v,$$

and those of PS, RQ', $S'P'$, QR' are

$$\frac{\partial x}{\partial w}\delta w,\quad \frac{\partial y}{\partial w}\delta w,\quad \frac{\partial z}{\partial w}\delta w.$$

Hence the chords joining the corresponding angular points are such as, to the first order, to form the eight edges of an oblique parallelepiped, whose volume is

$$\begin{vmatrix} \dfrac{\partial x}{\partial u}, & \dfrac{\partial y}{\partial u}, & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v}, & \dfrac{\partial y}{\partial v}, & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w}, & \dfrac{\partial y}{\partial w}, & \dfrac{\partial z}{\partial w} \end{vmatrix} \delta u\,\delta v\,\delta w = \frac{\partial(x, y, z)}{\partial(u, v, w)}\delta u\,\delta v\,\delta w.$$

This is, to the third order, the volume of the elementary solid PP', the difference between this volume and that of the oblique parallelepiped being at least of the fourth order of infinitesimals. Hence, taking the limit and integrating between any assigned limits for u, v, w, we have

$$V=\iiint \frac{\partial(x, y, z)}{\partial(u, v, w)}\,du\,dv\,dw=\iiint J\,du\,dv\,dw,$$

where J is the Jacobian of x, y, z with regard to u, v, w; and, as noted in Art. 792, it is to be remembered that if J' be the Jacobian of u, v, w with regard to x, y, z, we have $JJ'=1$ (*Diff. Calc.*, Art. 540). And for cases where u, v, w are expressed as functions of x, y, z, instead of x, y, z, in terms of u, v, w, this rule will facilitate the calculation of J.

795. Ex. Find the volume enclosed by the six hyperbolic cylinders

$$\begin{aligned} yz&=a_1^2, & yz&=a_2^2,\\ zx&=b_1^2, & zx&=b_2^2,\\ xy&=c_1^2, & xy&=c_2^2. \end{aligned}$$

Putting $\qquad yz=u, \quad zx=v, \quad xy=w,$

$$J' = \begin{vmatrix} 0, & z, & y \\ z, & 0, & x \\ y, & x, & 0 \end{vmatrix} = zxy + yzx = 2\sqrt{uvw};$$

$$\therefore \quad V = \frac{1}{2}\int_{a_1^2}^{a_2^2}\int_{b_1^2}^{b_2^2}\int_{c_1^2}^{c_2^2} \frac{du\,dv\,dw}{\sqrt{uvw}} = 4(a_2-a_1)(b_2-b_1)(c_2-c_1).$$

796. It follows that if we wish to integrate the function $f(x, y, z)$ throughout the volume bounded by surfaces specified by two specific values of u, two specific values of v and two specific values of w, *i.e.* to add up all quantities of the form

$$f(x, y, z) \times \text{an element of volume at } x, y, z,$$

we have only to express x, y, z in terms of u, v, w, and substitute these values for x, y, z in $f(x, y, z)$, obtaining, say $F(u, v, w)$, as the transformed function. Then taking, as before, the same element of volume, viz. $J\,\delta u\,\delta v\,\delta w$, the integral required will be

$$\iiint F(u, v, w)\, J\, du\, dv\, dw.$$

797. Thus, if we wished to obtain the product of inertia with regard to the y, z axes in the above example (of Art. 795), each element of mass $\rho J\,\delta u\,\delta v\,\delta w$ is to be multiplied by yz, *i.e.* u, and assuming a uniform volume density ρ, the product of inertia required is $\iiint \rho u J\, du\, dv\, dw$, or

$$\frac{\rho}{2}\iiint \frac{\sqrt{u}}{\sqrt{vw}}\, du\, dv\, dw = \tfrac{4}{3}\rho(a_2^3 - a_1^3)(b_2 - b_1)(c_2 - c_1)$$
$$= \frac{M}{3}(a_2^2 + a_1 a_2 + a_2^2),$$

where M is the mass of the solid in question.

798. If we wish for the x-coordinate of the centroid of the solid,

$$\bar{x} = \frac{\Sigma mx}{\Sigma m} = \frac{\iiint \rho x J\, du\, dv\, dw}{\iiint \rho J\, du\, dv\, dw} = \frac{\iiint \sqrt{\frac{vw}{u}} \cdot \frac{du\, dv\, dw}{2\sqrt{uvw}}}{\iiint \frac{du\, dv\, dw}{2\sqrt{uvw}}}$$

$$= \frac{\iiint \frac{du\, dv\, dw}{2u}}{\iiint \frac{du\, dv\, dw}{2\sqrt{uvw}}} = \frac{(\log a_2 - \log a_1)(b_2^2 - b_1^2)(c_2^2 - c_1^2)}{8(a_2 - a_1)(b_2 - b_1)(c_2 - c_1)}$$

$$= \frac{1}{8}\frac{(b_2 + b_1)(c_2 + c_1)}{a_2 - a_1}\log\left(\frac{a_2}{a_1}\right),$$

and similarly for other integrals.

799. We consider next the case in which the three coordinates x, y, z are expressed, or expressible, in terms of two independent parameters u and v, and therefore the point travels upon a definite surface. Consider the four points P, Q, S, R on the surface defined by the values

$$(u, v), \quad (u+\delta u, v), \quad (u, \ +\delta v), \quad (u+\delta u, v+\delta v),$$

i.e.

$$\begin{array}{lll} x, & y, & z; \\ x+\dfrac{\partial x}{\partial u}\delta u, & y+\dfrac{\partial y}{\partial u}\delta u, & z+\dfrac{\partial z}{\partial u}\delta u; \\ x+\dfrac{\partial x}{\partial v}\delta v, & y+\dfrac{\partial y}{\partial v}\delta v, & z+\dfrac{\partial z}{\partial v}\delta v; \\ x+\dfrac{\partial x}{\partial u}\delta u+\dfrac{\partial x}{\partial v}\delta v, & y+\dfrac{\partial y}{\partial u}\delta u+\dfrac{\partial y}{\partial v}\delta v, & z+\dfrac{\partial z}{\partial u}\delta u+\dfrac{\partial z}{\partial v}\delta v. \end{array}$$

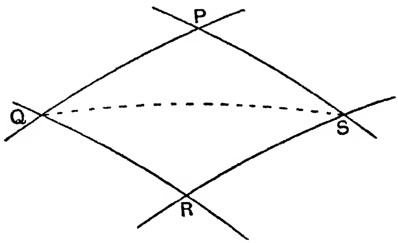


Fig. 279.

The direction ratios of PQ and SR are each

$$\frac{\partial x}{\partial u}\delta u, \quad \frac{\partial y}{\partial u}\delta u, \quad \frac{\partial z}{\partial u}\delta u,$$

and those of PS and QR are each

$$\frac{\partial x}{\partial v}\delta v, \quad \frac{\partial y}{\partial v}\delta v, \quad \frac{\partial z}{\partial v}\delta v,$$

and to the first order $PQRS$ is a parallelogram. Let its area be δS.

The coordinates of the projections of P, Q, S, R on the plane of x-y are

$$(x, y), \quad \left(x+\frac{\partial x}{\partial u}\delta u, \ y+\frac{\partial y}{\partial u}\delta u\right), \text{ etc.,}$$

and the area of this projection is

$$\begin{vmatrix} x+\dfrac{\partial x}{\partial u}\delta u, & y+\dfrac{\partial y}{\partial u}\delta u, & 1 \\ x+\dfrac{\partial x}{\partial v}\delta v, & y+\dfrac{\partial y}{\partial v}\delta v, & 1 \\ x, & y, & 1 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u}, & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v}, & \dfrac{\partial y}{\partial v} \end{vmatrix} \delta u\, \delta v = \frac{\partial(x, y)}{\partial(u, v)}\delta u\, \delta v,$$

and similarly its projections upon the other coordinate planes are
$$\frac{\partial(y, z)}{\partial(u, v)}\delta u\,\delta v;\quad \frac{\partial(z, x)}{\partial(u, v)}\delta u\,\delta v,$$
whence its area δS is given by
$$\delta S^2=\left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2\delta u^2\,\delta v^2+\left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2\delta u^2\,\delta v^2+\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2\delta u^2\,\delta v^2.$$
Hence, proceeding to the limit and integrating,
$$S=\iint\sqrt{\left[\left\{\frac{\partial(y, z)}{\partial(u, v)}\right\}^2+\left\{\frac{\partial(z, x)}{\partial(u, v)}\right\}^2+\left\{\frac{\partial(x, y)}{\partial(u, v)}\right\}^2\right]}\,du\,dv$$
i.e.
$$S=\iint\sqrt{J_1^2+J_2^2+J_3^2}\,du\,dv,$$
where
$$J_1=\frac{\partial(y, z)}{\partial(u, v)},\quad J_2=\text{etc.},\quad J_3=\text{etc.}$$

Also if the surface integral of any function $f(x, y, z)$ be required, $f(x, y, z)$ is to be expressed in terms of u and v, as $\phi(u, v)$, and the surface integral required is
$$\iint\phi(u, v)\sqrt{J_1^2+J_2^2+J_3^2}\,du\,dv.$$

If we write
$$E\equiv\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2,\quad F\equiv\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$
$$G=\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2;$$
we have, from the algebraic identity,
$$(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2+(ll'+mm'+nn')^2$$
$$=(l^2+m^2+n^2)(l'^2+m'^2+n'^2),$$
$$J_1^2+J_2^2+J_3^2=EG-F^2;$$
$\therefore$ the surface integral may be written
$$\iint\phi(u, v)\sqrt{EG-F^2}\,du\,dv,$$
as shown otherwise in Art. 744.

800. **Results connecting δV and δS.**

If δS be an element of the area S of a surface, and P be the perpendicular from the origin on the corresponding tangent plane, we have for the volume of the cone whose vertex is at the origin and base δS, $\quad \frac{1}{3}P\,\delta S.$

Hence the volume of any region bounded by a given surface and a cone with vertex at the origin, and generators passing through the perimeter of any closed curve drawn upon the surface, is

$$V=\frac{1}{3}\int P\,dS;$$

or, which is the same thing, if l, m, n be the direction cosines of the normal to the element δS, so that

$$P=lx+my+nz,$$

is the equation of the tangent plane, we have

$$V=\frac{1}{3}\int(lx+my+nz)\,dS.$$

801. If the equation of the surface be written as $z=f(x, y)$, the equation of the tangent plane at x, y, z is

$$Z-z=p(X-x)+q(Y-y),$$

where $$p=\frac{\partial z}{\partial x}, \quad q=\frac{\partial z}{\partial y},$$

and the perpendicular P from the origin upon it is

$$P=\frac{px+qy-z}{\sqrt{1+p^2+q^2}}.$$

Hence the formula for the volume, viz. $\frac{1}{3}\int P\,dS$, becomes

$$\frac{1}{3}\iint(px+qy-z)\,dx\,dy,$$

for $$\delta x\,\delta y=\delta S\cos\gamma=\frac{\delta S}{\sqrt{1+p^2+q^2}},$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal,

i.e. $$V=\frac{1}{3}\iint[px+qy-f(x, y)]\,dx\,dy.$$

802. Let the inward drawn normal at a point P on a surface make an angle χ with the radius vector from the origin, and let p be the perpendicular from the origin upon the tangent plane at P, r the radius vector from the origin to P, and δS an element of the surface about P

Then $\frac{p}{r}=\cos\chi$, and the formula for an element of volume forming an elementary cone with vertex O and base δS, viz. $\frac{1}{3}p\,\delta S$, becomes $\frac{1}{3}r\cos\chi\,\delta S$.

Hence we have another expression for the volume bounded by any curved surface and a cone whose vertex is the origin and passing through the perimeter of the region defined by a given closed curve drawn upon the surface, viz.

$$V=\frac{1}{3}\int r\cos\chi\, dS;$$

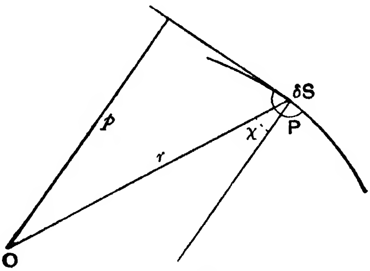


Fig. 280.

or again, seeing that this element of volume is

$$\frac{r^3}{3}\sin\theta\,\delta\theta\,\delta\phi,$$

we have $$\delta S=\frac{r^3}{p}\sin\theta\,\delta\theta\,\delta\phi$$

and $$S=\iint\frac{r^3}{p}\sin\theta\, d\theta\, d\phi.$$

803. Ex. Find the surface and the volume of the solid formed by the revolution of the cardioide $r=a(1+\cos\theta)$ about the initial line.

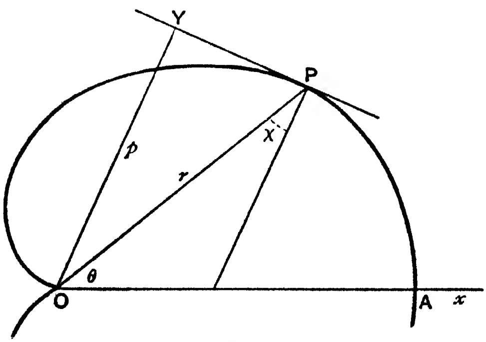


Fig. 281.

Here $$\chi=\frac{\theta}{2},\quad p=r\cos\chi=2a\cos^3\frac{\theta}{2}.$$

$$S=\int_0^{\pi}\int_0^{2\pi}\frac{r^3}{p}\sin\theta\,d\theta\,d\phi$$

$$=2\pi.\frac{(2a)^3}{(2a)}\int_0^{\pi}\cos^3\frac{\theta}{2}\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

$$=16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta$$

$$=16\pi a^2\left[-\frac{2}{5}\cos^5\frac{\theta}{2}\right]_0^{\pi}=\tfrac{32}{5}\pi a^2.$$

Also $$V=\iiint r^2\sin\theta\,d\theta\,d\phi\,dr,$$

the limits for r being 0 to $a(1+\cos\theta)$,
ϕ from 0 to 2π,
θ from 0 to π.

Hence $$V=\frac{2\pi a^3}{3}\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta$$

$$=\frac{2\pi a^3}{3}\left[-\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}=\tfrac{8}{3}\pi a^3.$$ (See Art. 751, Ex. 3.)

804. **Tetrahedral Volume.**

An expression for the evaluation of a volume for a surface given by a tetrahedral equation may be obtained in the same

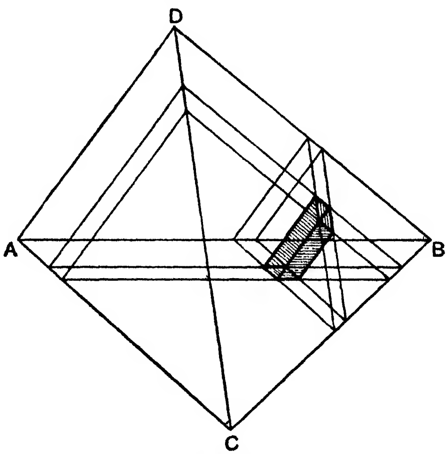


Fig. 282.

way as that adopted for an area in areal coordinates (Art. 461).

For let V_0 be the volume of the tetrahedron of reference, and let $\alpha, \beta, \gamma, \delta$ be the tetrahedral coordinates of a point P,

and x, y, z be their Cartesian equivalents with reference to some given rectangular system of axes; then x, y and z are linear functions of α, β and γ, for we have $\alpha+\beta+\gamma+\delta=1$.

Hence $$V=\iiint dx\,dy\,dz=K\iiint d\alpha\,d\beta\,d\gamma,$$

where K is some determinate constant (Art. 794).

To determine K, apply the formula to the fundamental tetrahedron itself. If we integrate first with regard to α for the tube bounded by two given planes β and $\beta+\delta\beta$, and two planes γ and $\gamma+\delta\gamma$, keeping β and γ constant, the limits for α will be from the point at which this tube cuts the plane $\alpha=0$ to the point in which it cuts $\delta=0$, *i.e.* from $\alpha=0$ to $\alpha=1-\beta-\gamma$. Then we have

$$V_0=K\iint(1-\beta-\gamma)\,d\beta\,d\gamma.$$

Next, integrating this with respect to β, keeping γ constant, the limits for β will be from $\beta=0$ to the point where $\alpha=0$ and $\delta=0$, *i.e.* where $\beta=1-\gamma$, and

$$V_0=K\int\left[\beta-\frac{\beta^2}{2}-\gamma\beta\right]_0^{1-\gamma}d\gamma=K\int\frac{(1-\gamma)^2}{2}d\gamma.$$

Lastly, integrating from $\gamma=0$ to $\gamma=1$, $V_0=\dfrac{K}{6}$.

Hence $K=6V_0$; therefore the formula is

$$V=6V_0\iiint d\alpha\,d\beta\,d\gamma.$$

805. **Surface generated by the Revolution of a Tortuous Curve about an Axis.**

Let a curve of double curvature revolve round the z-axis; it is required to find the surface generated.

Let PP' be the element ds of the curve.

Let revolution about the z-axis be made through the angle $d\theta$, and let the perpendiculars $PN, P'N'$ turn into the positions P_1N, $P_1'N'$.

Then $$PP_1=NP\,d\theta,$$
$$P'P_1'=N'P'\,d\theta=NP\,d\theta$$

to the first order, and $NP=\sqrt{x^2+y^2}$, and the area of the element $PP_1P_1'P'$ is $NP\,d\theta\,.\,ds\sin\chi$ to the second order, where χ

is the angle between P_1P and P_1P_1', *i.e.* between directions whose direction cosines are

$$\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds} \quad\text{and}\quad \frac{-y}{\sqrt{x^2+y^2}},\ \frac{x}{\sqrt{x^2+y^2}},\ 0.$$

Hence $$\cos\chi = \left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)\Big/\sqrt{x^2+y^2}$$

and $$\sin\chi = \sqrt{(x^2+y^2)-\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)^2}\Big/\sqrt{x^2+y^2}.$$

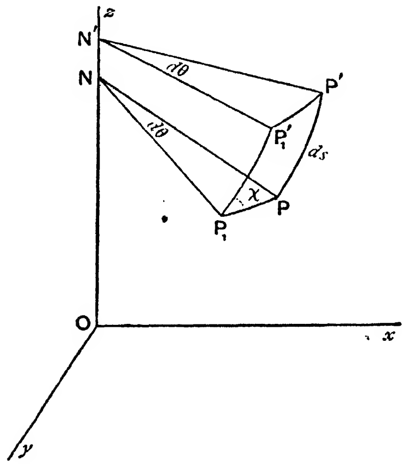


Fig. 283.

Hence Area of element $PP_1P_1'P'$

$$=\sqrt{x^2+y^2}\,d\theta\sqrt{dx^2+dy^2+dz^2}\sqrt{(x^2+y^2)-\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)^2}\Big/\sqrt{x^2+y^2}$$

$$=d\theta\sqrt{(x^2+y^2)(dx^2+dy^2+dz^2)-(x\,dy-y\,dx)^2}$$

$$=d\theta\sqrt{(x\,dx+y\,dy)^2+(x^2+y^2)\,dz^2}.$$

Hence, for a complete revolution the area traced out is

$$2\pi\int\sqrt{\{(x\,dx+y\,dy)^2+(x^2+y^2)\,dz^2\}},$$

or in cylindricals, (ρ, ϕ, z),

$$=2\pi\int\rho\sqrt{d\rho^2+dz^2}.$$

That is the area of the surface described is the same as would be traced out by a rotation about the z-axis through the same angle, of a new plane curve constructed by first swinging back each point of the tortuous curve from its actual position without alteration of its distance from the axis of rotation into a corresponding position upon the initial plane.

And if ds' be an elementary arc of this new curve,

$$ds'^2 = d\rho^2 + dz^2,$$

and therefore $$\text{Area} = 2\pi \int \rho \, ds'.$$

806. Ex. Let us employ this formula to find the surface of a hyperboloid of revolution included between two planes perpendicular to the

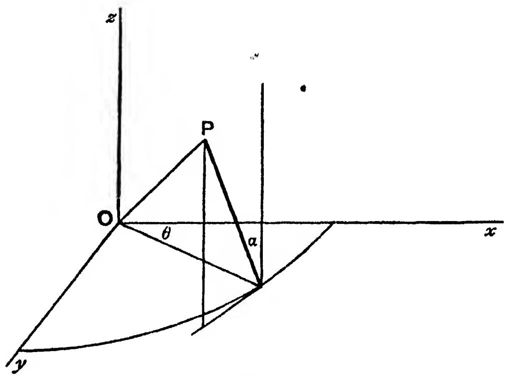


Fig. 284.

axis, the surface being regarded as generated by the revolution of a straight line about the axis, which we take as the z-axis, the line making a constant angle with the z-axis and not cutting it. The equations of the line are

$$x = a\cos\theta - z\tan\alpha\sin\theta,$$
$$y = a\sin\theta + z\tan\alpha\cos\theta.$$

Hence $$x^2 + y^2 = a^2 + z^2\tan^2\alpha$$

and $$x\,dx + y\,dy = z\,dz\tan^2\alpha;$$

$$\therefore \quad S = 2\pi\int\sqrt{z^2\,dz^2\tan^4\alpha + (a^2 + z^2\tan^2\alpha)\,dz^2}$$

$$= 2\pi\int\sqrt{a^2 + z^2\tan^2\alpha\sec^2\alpha}\,dz$$

$$= 2\pi\tan\alpha\sec\alpha\int\sqrt{z^2 + a^2\frac{\cos^4\alpha}{\sin^2\alpha}}\,dz.$$

Hence

$$S=\pi \tan\alpha \sec\alpha\left[z\sqrt{z^2+a^2\frac{\cos^4\alpha}{\sin^2\alpha}}+\frac{a^2\cos^4\alpha}{\sin^2\alpha}\sinh^{-1}\frac{z\sin\alpha}{a\cos^2\alpha}\right]_{z_1}^{z_2}.$$

807. **Case of an Annular Element of Surface. Surface of the Ellipsoid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad (a>b>c).$$

Legendre's Formula.

The equations of the normal at x, y, z are

$$\frac{X-x}{\frac{x}{a^2}}=\frac{Y-y}{\frac{y}{b^2}}=\frac{Z-z}{\frac{z}{c^2}},$$

and its direction cosines are $\frac{px}{a^2}, \frac{py}{b^2}, \frac{pz}{c^2}$, where p is the central perpendicular upon the tangent planes at x, y, z, viz. such that

$$\frac{1}{p^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}.$$

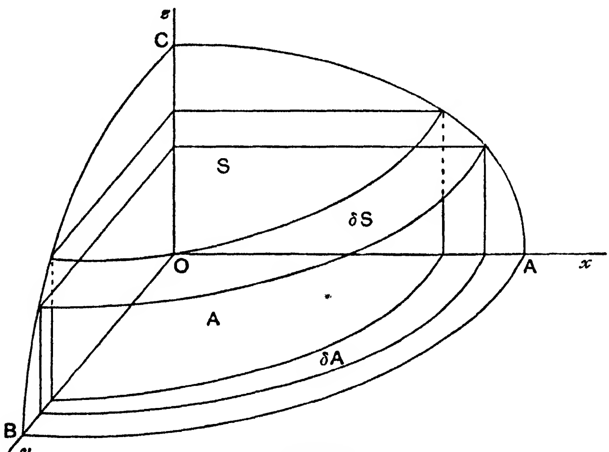


Fig. 285.

Let a cone be drawn whose vertex is at the origin O, and cutting the ellipsoid at all those points at which the normal makes a constant angle θ with the z-axis. Its equation is

$$\frac{pz}{c^2}=\cos\theta \quad \text{or} \quad \frac{z^2}{c^4\cos^2\theta}=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}.$$

Let S be the area of the ellipsoidal cap cut off by this cone.

If we eliminate z between the equation of the cone and the equation of the ellipsoid, we obtain the projection of this curve of intersection upon the plane of xy, viz.

$$\frac{\sec^2\theta}{c^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{1}{c^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

or $\dfrac{x^2}{a^4\sin^2\theta}(a^2\sin^2\theta+c^2\cos^2\theta)+\dfrac{y^2}{b^4\sin^2\theta}(b^2\sin^2\theta+c^2\cos^2\theta)=1,$

viz. an ellipse of area

$$A=\pi a^2b^2\frac{\sin^2\theta}{\sqrt{a^2\sin^2\theta+c^2\cos^2\theta}\sqrt{b^2\sin^2\theta+c^2\cos^2\theta}}.$$

If we increase θ to $\theta+\delta\theta$, we increase S and A respectively to $S+\delta S$ and $A+\delta A$. Now δA, the difference between the areas of two ellipses, is the projection of δS upon the x-y plane. And when $\delta\theta$ is indefinitely small, all elements of δS cut off by contiguous meridian planes make the same angle θ with their projections, which are the corresponding elements of δA. Hence

$$\delta A=\delta S\cos\theta \quad \text{and} \quad \delta S=\frac{\delta A}{\cos\theta},$$

and taking the limit and integrating

$$S=\int\frac{dA}{\cos\theta}.$$

To effect the integration of $\dfrac{dA}{\cos\theta}$, we shall change the variable. We have

$$A=\pi a^2b^2\frac{\sin^2\theta}{\sqrt{a^2-(a^2-c^2)\cos^2\theta}\sqrt{b^2-(b^2-c^2)\cos^2\theta}}.$$

Put $\cos\theta=\dfrac{a}{\sqrt{a^2-c^2}}\sin\phi=\dfrac{\sin\phi}{\sin\gamma}$, where $c=a\cos\gamma$.

Then
$$A=\pi a^2b^2\frac{1-\dfrac{\sin^2\phi}{\sin^2\gamma}}{a\cos\phi\,.\,b\sqrt{1-\dfrac{a^2}{b^2}\dfrac{b^2-c^2}{a^2-c^2}\sin^2\phi}}$$

$$=\frac{\pi ab}{\sin^2\gamma}\frac{\sin^2\gamma-\sin^2\phi}{\cos\phi\sqrt{1-k^2\sin^2\phi}}$$

$$=\frac{\pi ab}{\sin^2\gamma}\frac{\sin^2\gamma-\sin^2\phi}{\cos\phi\Delta}, \text{ where } k^2=\frac{a^2(b^2-c^2)}{b^2(a^2-c^2)}$$

which is <1, and $\Delta^2=1-k^2\sin^2\phi$.

And $$dS = \sin\gamma \frac{dA}{\sin\phi}$$

$$= \sin\gamma\left[d\,\frac{A}{\sin\phi} + \frac{A\cos\phi}{\sin^2\phi}\,d\phi\right]$$

$$= \sin\gamma\left[d\,\frac{A}{\sin\phi} + \frac{\pi ab}{\sin^2\gamma}\,\frac{\sin^2\gamma - \sin^2\phi}{\Delta\cos\phi}\cdot\frac{\cos\phi}{\sin^2\phi}\,d\phi\right]$$

$$= \frac{\pi ab}{\sin\gamma}\left[d\left(\frac{\sin^2\gamma-\sin^2\phi}{\Delta\sin\phi\cos\phi}\right) + \frac{\sin^2\gamma-\sin^2\phi}{\Delta\sin^2\phi}\,d\phi\right]$$

$$= \frac{\pi ab}{\sin\gamma}\left[d\left(\frac{\sin^2\gamma-\sin^2\phi}{\Delta\sin\phi\cos\phi}\right) + \frac{\sin^2\gamma}{\Delta\sin^2\phi}\,d\phi - \frac{d\phi}{\Delta}\right]. \quad \ldots(1)$$

Now $$\frac{d}{d\phi}(\Delta\cot\phi) = -\frac{k^2\cos^2\phi}{\Delta} - \frac{\Delta}{\sin^2\phi}$$

$$= \frac{-k^2+1-\Delta^2}{\Delta} - \frac{1-k^2\sin^2\phi}{\Delta\sin^2\phi}$$

$$= \frac{1}{\Delta} - \Delta - \frac{1}{\Delta\sin^2\phi}.$$

Hence

$$dS = \frac{\pi ab}{\sin\gamma}\left[d\left(\frac{\sin^2\gamma-\sin^2\phi}{\Delta\sin\phi\cos\phi}\right)\right.$$
$$\left. + \sin^2\gamma\left\{\left(\frac{1}{\Delta}-\Delta\right)d\phi - d(\Delta\cot\phi)\right\} - \frac{d\phi}{\Delta}\right]$$

$$= \frac{\pi ab}{\sin\gamma}\left[d\left(\frac{\sin^2\gamma-\sin^2\phi}{\Delta\sin\phi\cos\phi} - \Delta\sin^2\gamma\cot\phi\right)\right.$$
$$\left. - \sin^2\gamma\,\Delta\,d\phi - \cos^2\gamma\,\frac{d\phi}{\Delta}\right]$$

$$= \frac{\pi ab}{\sin\gamma}\left[d\left\{\frac{\tan\phi}{\Delta}\left(\overline{1-k^2\sin^2\phi}\,\sin^2\gamma - \overline{1-k^2\sin^2\gamma}\right)\right.\right.$$
$$\left.\left. - \sin^2\gamma\,\Delta\,d\phi - \cos^2\gamma\,\frac{d\phi}{\Delta}\right\}\right],$$

where $$1-k^2\sin^2\gamma = 1 - \frac{a^2(b^2-c^2)}{b^2(a^2-c^2)}\cdot\frac{a^2-c^2}{a^2} = \frac{c^2}{b^2},$$

and the limits for θ are 0 to $\frac{\pi}{2}$ for the upper half of the ellipsoid, and the consequent limits for ϕ are γ to 0, and double to take in the lower half of the surface.

Thus for the whole surface

$$S=\frac{2\pi ab}{\sin\gamma}\left[\frac{\tan\phi}{\Delta}\left(\overline{1-k^2\sin^2\phi}\sin^2\gamma-\frac{c^2}{b^2}\right)\right]_\gamma^0$$

$$+\frac{2\pi ab}{\sin\gamma}\left[\sin^2\gamma\int_0^\gamma\Delta\,d\phi+\cos^2\gamma\int_0^\gamma\frac{d\phi}{\Delta}\right]$$

$$=2\pi c^2+\frac{2\pi ab}{\sin\gamma}[\sin^2\gamma\,E(\gamma,\,k)+\cos^2\gamma\,F(\gamma,\,k)],$$

where $\cos\gamma=\frac{c}{a}$, a form due to Legendre.*

808. **Cases.**

In the case of the oblate spheroid, $a=b$, $k=1$, and the elliptic functions degenerate,

$$E \text{ becoming } \int_0^\gamma\sqrt{1-\sin^2\phi}\,d\phi=\sin\gamma$$

and $$F \text{ becoming } \int_0^\gamma\frac{d\phi}{\cos\phi}=\log\tan\left(\frac{\gamma}{2}+\frac{\pi}{4}\right),$$

giving $$S=2\pi c^2+\frac{2\pi a^2}{\sin\gamma}\left[\sin^3\gamma+\cos^2\gamma\log\tan\left(\frac{\gamma}{2}+\frac{\pi}{4}\right)\right]$$

$$=2\pi a^2+\frac{2\pi c^2}{\sin\gamma}\log\tan\left(\frac{\gamma}{2}+\frac{\pi}{4}\right),$$

and for the prolate spheroid $b=c$, $k=0$, $E=\gamma$ and $F=\gamma$, giving

$$S=2\pi c^2+\frac{2\pi ab}{\sin\gamma}\cdot\gamma$$

$$=2\pi c^2\left(1+\frac{\gamma}{\sin\gamma\cos\gamma}\right)$$

or $$\frac{2\pi ac}{\sin\gamma}(\gamma+\sin\gamma\cos\gamma).$$

809. **Another Method for the Surface of an Ellipsoid.**

From the formula $$S=\int\frac{1}{\cos\theta}\,dA$$

we may deduce another form of expression for the area of an ellipsoid. Substituting the value of dA, we have

$$dS=\pi a^2b^2\,\frac{1}{\cos\theta}\,d\,\frac{1}{\sqrt{(a^2+c^2\cot^2\theta)(b^2+c^2\cot^2\theta)}}.$$

Put $\cot\theta=\frac{\sqrt{\lambda}}{c}$.

* See Serret, *Calcul Intégral*, pages 338-342; Legendre, *Exercices du Calcul Intégral*, p. 193.

Then

$$\frac{dS}{\pi a^2 b^2}=\frac{\sqrt{\lambda+c^2}}{\sqrt{\lambda}}\,d\,\frac{1}{\sqrt{(a^2+\lambda)(b^2+\lambda)}}$$

$$=-\frac{\sqrt{\lambda+c^2}}{\sqrt{\lambda}}\left[\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\right]\frac{d\lambda}{2\sqrt{(a^2+\lambda)(b^2+\lambda)}}$$

$$=-(\lambda+c^2)\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\right)\frac{d\lambda}{2\sqrt{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}$$

$$=-\frac{c^2}{2}\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)\frac{d\lambda}{\sqrt{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}$$

$$-\frac{1}{2}\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}-\frac{1}{\lambda}\right)\frac{\lambda\,d\lambda}{\sqrt{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

and the limits of integration for the upper half of the ellipsoid are $\theta=0$ to $\theta=\frac{\pi}{2}$, *i.e.* $\lambda=\infty$ to $\lambda=0$. The result must be doubled to include the lower half of the surface.

Now $$\int_0^\infty\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}-\frac{1}{\lambda}\right)\frac{\sqrt{\lambda}\,d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}$$

$$=-2\int_0^\infty\frac{d}{d\lambda}\frac{\sqrt{\lambda}}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}\,d\lambda$$

$$=-2\left[\frac{\sqrt{\lambda}}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}\right]_0^\infty=0.$$

(See Art. 363, Ex. 5.)

Hence

$$S=\pi a^2b^2c^2\int_0^\infty\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)\frac{d\lambda}{\sqrt{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},\text{*}$$

for the whole area of the surface of the ellipsoid.

810. We now revert to the consideration of the generalised system of orthogonal coordinates discussed in Art. 789.

It will be remembered that we there obtained expressions for the direction cosines of the elements $\frac{\delta\lambda}{h_1}, \frac{\delta\mu}{h_2}, \frac{\delta\nu}{h_3}$ in terms of partial differential coefficients of x, y, z with regard to λ, μ, ν.

We may also readily express the same direction cosines in terms of partial differential coefficients of λ, μ, ν with regard to x, y, z.

* Mathematical Tripos, 1896.

Regard $\frac{\delta\lambda}{h_1}, \frac{\delta\mu}{h_2}, \frac{\delta\nu}{h_3}$ as the directions of a new set of three coordinate axes OA, OB, OC.

Referred to such axes the direction cosines of the original axes are:

$$\text{for } Ox;\ l_1,\ l_2,\ l_3,$$
$$\text{for } Oy;\ m_1,\ m_2,\ m_3,$$
$$\text{for } Oz;\ n_1,\ n_2,\ n_3;$$

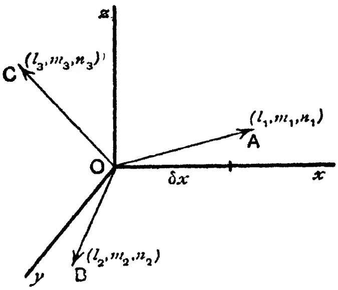


Fig. 286.

then $l_1\delta x$ is the projection of δx upon OA

= a small element on OA due to an increase of x to $x+\delta x$, y and z remaining unaltered,

$$=\frac{1}{h_1}\frac{\partial\lambda}{\partial x}\delta x.$$

Similarly $\qquad l_2\delta x=\frac{1}{h_2}\frac{\partial\mu}{\partial x}\delta x$ and $l_3\delta x=\frac{1}{h_3}\frac{\partial\nu}{\partial x}\delta x,$

$$m_1\delta y=\frac{1}{h_1}\frac{\partial\lambda}{\partial y}\delta y, \text{ etc.};$$

and we have the system of equations

$$l_1=\frac{1}{h_1}\frac{\partial\lambda}{\partial x},\quad m_1=\frac{1}{h_1}\frac{\partial\lambda}{\partial y},\quad n_1=\frac{1}{h_1}\frac{\partial\lambda}{\partial z},$$
$$l_2=\frac{1}{h_2}\frac{\partial\mu}{\partial x},\quad m_2=\frac{1}{h_2}\frac{\partial\mu}{\partial y},\quad n_2=\frac{1}{h_2}\frac{\partial\mu}{\partial z},$$
$$l_3=\frac{1}{h_3}\frac{\partial\nu}{\partial x},\quad m_3=\frac{1}{h_3}\frac{\partial\nu}{\partial y},\quad n_3=\frac{1}{h_3}\frac{\partial\nu}{\partial z};$$

whence it follows that J', *i.e.* $\frac{\partial(\lambda, \mu, \nu)}{\partial(x, y, z)}$,

$$=h_1h_2h_3\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=\pm h_1h_2h_3,$$

which might have been anticipated from the theorem $JJ'=1$ (*Diff. Calc.*, Art. 540).

We thus have the following relations between the several partial differential coefficients, by comparing with Art. 789, viz.

$$h_1^2\frac{\partial x}{\partial\lambda}=\frac{\partial\lambda}{\partial x},\quad h_1^2\frac{\partial y}{\partial\lambda}=\frac{\partial\lambda}{\partial y},\quad h_1^2\frac{\partial z}{\partial\lambda}=\frac{\partial\lambda}{\partial z},$$

$$h_2^2\frac{\partial x}{\partial\mu}=\frac{\partial\mu}{\partial x},\quad h_2^2\frac{\partial y}{\partial\mu}=\frac{\partial\mu}{\partial y},\quad h_2^2\frac{\partial z}{\partial\mu}=\frac{\partial\mu}{\partial z},$$

$$h_3^2\frac{\partial x}{\partial\nu}=\frac{\partial\nu}{\partial x},\quad h_3^2\frac{\partial y}{\partial\nu}=\frac{\partial\nu}{\partial y},\quad h_3^2\frac{\partial z}{\partial\nu}=\frac{\partial\nu}{\partial z},$$

$$h_1^2=\lambda_x^2+\lambda_y^2+\lambda_z^2=h_1^4(x_\lambda^2+y_\lambda^2+z_\lambda^2),$$

and
$$\frac{1}{h_1^2}=x_\lambda^2+y_\lambda^2+z_\lambda^2.$$

Similarly
$$\frac{1}{h_2^2}=x_\mu^2+y_\mu^2+z_\mu^2,$$

$$\frac{1}{h_3^2}=x_\nu^2+y_\nu^2+z_\nu^2.$$

811. It is plain that the areas of the three faces of the elementary cuboid which lie on the surfaces $\lambda=\text{const.}$, $\mu=\text{const.}$, $\nu=\text{const.}$, are respectively

$$\frac{\delta\mu\,\delta\nu}{h_2h_3},\quad \frac{\delta\nu\,\delta\lambda}{h_3h_1},\quad \frac{\delta\lambda\,\delta\mu}{h_1h_2},$$

and that the infinitesimal distance between x, y, z and $x+\delta x$, $y+\delta y$, $z+\delta z$, viz. the diagonal through P of the elementary cuboid, is

$$\delta s^2=\delta x^2+\delta y^2+\delta z^2=\frac{\delta\lambda^2}{h_1^2}+\frac{\delta\mu^2}{h_2^2}+\frac{\delta\nu^2}{h_3^2}.$$

[See Todhunter, *Functions of Laplace, Lamé and Bessel*, pages 210-233; also E. J. Routh, *Anal. Statics*, vol. ii., Arts. 109, 110.]

812. **Elliptic Coordinates.**

The most remarkable case of these orthogonal surfaces is that of the three confocal conicoids, $(a>b>c)$,

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,\quad \frac{x^2}{a^2+\mu}++=1,\quad \frac{x^2}{a^2+\nu}++=1,$$

viz. an ellipsoid, a hyperboloid of one sheet and a hyperboloid of two sheets respectively, so that λ is $\nless -c^2$, μ between $-c^2$ and $-b^2$, and ν between $-b^2$ and $-a^2$.

To express x, y and z in terms of the parameters λ, μ, ν, we resort to a well-known algebraical device, viz.

Consider the equality

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1+\frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a^2+\theta)(b^2+\theta)(c^2+\theta)}, \quad \text{....(A)}$$

where x, y, z have the values obtained from the above equations. This is either an equation to find θ, or it is an identity true for all values of θ.

If an equation, it is of quadratic nature; for θ^3 disappears upon multiplying up by $(a^2+\theta)(b^2+\theta)(c^2+\theta)$. Hence it could not be satisfied by more values of θ than two. This equality, however, is obviously satisfied by $\theta=\lambda$, $\theta=\mu$ and $\theta=\nu$, *i.e.* more than two values. Hence it is not an equation, but an identity and true for all values of θ.

Multiply then by $\theta+a^2$.

$$x^2 \equiv \left(1-\frac{y^2}{b^2+\theta}-\frac{z^2}{c^2+\theta}\right)(a^2+\theta)+\frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(b^2+\theta)(c^2+\theta)}.$$

In this identity put $\theta=-a^2$; hence

$$x^2=\frac{(\lambda+a^2)(\mu+a^2)(\nu+a^2)}{(a^2-b^2)(a^2-c^2)}.$$

Similarly $$y^2=\frac{(\lambda+b^2)(\mu+b^2)(\nu+b^2)}{(b^2-c^2)(b^2-a^2)}$$

and $$z^2=\frac{(\lambda+c^2)(\mu+c^2)(\nu+c^2)}{(c^2-a^2)(c^2-b^2)}.$$

Hence $$2x\frac{\partial x}{\partial\lambda}=\frac{(\mu+a^2)(\nu+a^2)}{(a^2-b^2)(a^2-c^2)}=\frac{x^2}{a^2+\lambda},$$

that is $$2\frac{\partial x}{\partial\lambda}=\frac{x}{a^2+\lambda},$$

and similarly $$2\frac{\partial y}{\partial\lambda}=\frac{y}{b^2+\lambda}, \quad 2\frac{\partial z}{\partial\lambda}=\frac{z}{c^2+\lambda}.$$

Again, if we differentiate the identity (A) with regard to θ, we obtain another identity, viz.

$$\frac{x^2}{(a^2+\theta)^2}+\frac{y^2}{(b^2+\theta)^2}+\frac{z^2}{(c^2+\theta)^2}=\frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a^2+\theta)(b^2+\theta)(c^2+\theta)}$$
$$\times\left[\frac{1}{\lambda-\theta}+\frac{1}{\mu-\theta}+\frac{1}{\nu-\theta}+\frac{1}{a^2+\theta}+\frac{1}{b^2+\theta}+\frac{1}{c^2+\theta}\right],$$

and putting $\theta=\lambda$ in this result,

$$4\left[\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2+\left(\frac{\partial z}{\partial\lambda}\right)^2\right]=\frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)},$$

i.e.
$$h_1^2=4\frac{\Delta_\lambda}{(\lambda-\mu)(\lambda-\nu)},$$

where $\Delta_\lambda\equiv(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$, $\Delta_\mu\equiv$ etc., $\Delta_\nu\equiv$ etc.

Hence

$$h_1=\frac{2}{\sqrt{(\nu-\lambda)(\lambda-\mu)}}\sqrt{-\Delta_\lambda},\quad h_2=\frac{2}{\sqrt{(\lambda-\mu)(\mu-\nu)}}\sqrt{-\Delta_\mu},$$

$$h_3=\frac{2}{\sqrt{(\mu-\nu)(\nu-\lambda)}}\sqrt{-\Delta_\nu}.$$

We thus have for an expression for a volume divided up into elementary cuboids defined by the faces of the three confocals λ, μ, ν, and the three contiguous confocals

$$\lambda+\delta\lambda,\quad \mu+\delta\mu,\quad \nu+\delta\nu,$$

$$V=\iiint\frac{d\lambda\,d\mu\,d\nu}{h_1h_2h_3}=\frac{1}{8}\iiint\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{\sqrt{-\Delta_\lambda\Delta_\mu\Delta_\nu}}\,d\lambda\,d\mu\,d\nu$$

813. In case of integration throughout the volume contained by the ellipsoid,
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the limits are: for λ, from $\lambda=0$ to $\lambda=-c^2$;
for μ, from $\mu=-c^2$ to $\mu=-b^2$;
for ν, from $\nu=-b^2$ to $\nu=-a^2$.

814. If any function $F(x, y, z)$ is to be integrated through any specific region bounded, say, by confocals λ_1, λ_2, μ_1, μ_2, ν_1, ν_2, we must convert F into a function of λ, μ, ν by substituting for x, y, z their values, obtaining, say, $F_1(\lambda, \mu, \nu)$, and then the required summation will be

$$\frac{1}{8}\int_{\lambda_1}^{\lambda_2}\int_{\mu_1}^{\mu_2}\int_{\nu_1}^{\nu_2}F_1(\lambda,\mu,\nu)\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{\sqrt{-\Delta_\lambda\Delta_\mu\Delta_\nu}}\,d\lambda\,d\mu\,d\nu.$$

815. For instance, if the function to be integrated be

$$F_1(\lambda,\mu,\nu)\equiv\frac{\sqrt{-\Delta_\lambda\Delta_\mu\Delta_\nu}}{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}\lambda\mu\nu,$$

we have
$$I=\frac{1}{8}\int_{\lambda_1}^{\lambda_2}\int_{\mu_1}^{\mu_2}\int_{\nu_1}^{\nu_2}\lambda\mu\nu\,d\lambda\,d\mu\,d\nu$$
$$=\tfrac{1}{64}(\lambda_2^2-\lambda_1^2)(\mu_2^2-\mu_1^2)(\nu_2^2-\nu_1^2).$$

816. In particular we may gather from the known volume of an ellipsoid, viz. $\frac{4}{3}\pi abc$, that the value of the definite integral

$$\int_0^{-c^2}\int_{-c^2}^{-b^2}\int_{-b^2}^{-a^2}\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{\sqrt{-\Delta_\lambda\Delta_\mu\Delta_\nu}}\,d\lambda\,d\mu\,d\nu \text{ is } \tfrac{32}{3}\pi abc.$$

817. The elements of surface of the three confocals at a point of intersection are respectively

$$dS_1=\frac{\delta\mu\,\delta\nu}{h_2h_3}=\tfrac{1}{4}(\mu-\nu)\frac{\sqrt{(\nu-\lambda)(\lambda-\mu)}}{\sqrt{\Delta_\mu\Delta_\nu}},$$

$$dS_2=\frac{\delta\nu\,\delta\lambda}{h_3h_1}=\tfrac{1}{4}(\nu-\lambda)\frac{\sqrt{(\lambda-\mu)(\mu-\nu)}}{\sqrt{\Delta_\nu\Delta_\lambda}},$$

$$dS_3=\frac{\delta\lambda\,\delta\mu}{h_1h_2}=\tfrac{1}{4}(\lambda-\mu)\frac{\sqrt{(\mu-\nu)(\nu-\lambda)}}{\sqrt{\Delta_\lambda\Delta_\mu}}.$$

818. We may thus, for instance, express the area of any portion of the ellipsoid $\lambda=0$, bounded by confocals μ_1, μ_2, ν_1, ν_2, as

$$S=\frac{1}{4}\int_{\mu_1}^{\mu_2}\int_{\nu_1}^{\nu_2}(\mu-\nu)\sqrt{-\frac{\mu\nu}{\Delta_\mu\Delta_\nu}}\,d\mu\,d\nu.$$

819. The distance δs from λ, μ, ν to $\lambda+\delta\lambda$, $\mu+\delta\mu$, $\nu+\delta\nu$ is given by

$$\begin{aligned}\delta s^2&=\delta x^2+\delta y^2+\delta z^2\\&=\frac{\delta\lambda^2}{h_1^2}+\frac{\delta\mu^2}{h_2^2}+\frac{\delta\nu^2}{h_3^2}\\&=\frac{1}{4}\left[\frac{(\nu-\lambda)(\lambda-\mu)}{-\Delta_\lambda}\delta\lambda^2+\,+\right].\end{aligned}$$

And

$$s=\frac{1}{2}\int\left[\frac{(\lambda-\mu)(\lambda-\nu)}{\Delta_\lambda}d\lambda^2+\frac{(\mu-\nu)(\mu-\lambda)}{\Delta_\mu}d\mu^2+\frac{(\nu-\lambda)(\nu-\mu)}{\Delta_\nu}d\nu^2\right]^{\frac{1}{2}}.$$

In the case where the line lies on the ellipsoid $\lambda=0$,

$$s=\frac{1}{2}\int\left\{\frac{\mu(\mu-\nu)}{\Delta_\mu}d\mu^2+\frac{\nu(\nu-\mu)}{\Delta_\nu}d\nu^2\right\}^{\frac{1}{2}}.$$

And when the curve on the ellipsoid is further defined by a relation between μ and ν, further reduction may be effected. For instance, along the line of curvature which is the intersection of the intersection of $\lambda=0$ with $\mu=\text{const.}=\mu_0$, say,

$$s=\frac{1}{2}\int_{\nu_1}^{\nu_2}\sqrt{\frac{\nu(\nu-\mu_0)}{\Delta_\nu}}\,d\nu,$$

or writing

$$\nu+a^2=\omega^2, \quad \mu_0+a^2=d^2, \quad a^2-b^2=b_1^2, \quad a^2-c^2=c_1^2,$$

we have $$s=\int_{\omega_1}^{\omega_2}\sqrt{\frac{(\omega^2-a^2)(\omega^2-d^2)}{(\omega^2-b_1^2)(\omega^2-c_1^2)}}\,d\omega,$$

for the length of a specified arc of a specified line of curvature upon the ellipsoid.

820. If we write

$$\begin{array}{lll} \lambda+a^2=\lambda_1^2, & \mu+a^2=\mu_1^2, & \nu+a^2=\nu_1^2, \\ \lambda+b^2=\lambda_1^2-b_1^2, & \mu+b^2=\mu_1^2-b_1^2, & \nu+b^2=\nu_1^2-b_1^2, \\ \lambda+c^2=\lambda_1^2-c_1^2, & \mu+c^2=\mu_1^2-c_1^2, & \nu+c^2=\nu_1^2-c_1^2, \end{array}$$

the conicoids become

$$\frac{x^2}{\lambda_1^2}+\frac{y^2}{\lambda_1^2-b_1^2}+\frac{z^2}{\lambda_1^2-c_1^2}=1, \quad \frac{x^2}{\mu_1^2}+\frac{y^2}{\mu_1^2-b_1^2}+\frac{z^2}{\mu_1^2-c_1^2}=1,$$
$$\frac{x^2}{\nu_1^2}+\frac{y^2}{\nu_1^2-b_1^2}+\frac{z^2}{\nu_1^2-c_1^2}=1,$$

and we have a certain amount of simplification of the formulae, but with a loss of symmetry.*

Thus we obtain

$$x^2=\frac{\lambda_1^2\mu_1^2\nu_1^2}{b_1^2c_1^2}, \quad y^2=\frac{(\lambda_1^2-b_1^2)(\mu_1^2-b_1^2)(\nu_1^2-c_1^2)}{(b_1^2-c_1^2)\,b_1^2},$$
$$z^2=\frac{(\lambda_1^2-c_1^2)(\mu_1^2-c_1^2)(\nu_1^2-c_1^2)}{(c_1^2-b_1^2)\,c_1^2},$$

$$V=\iiint\frac{(\mu_1^2-\nu_1^2)(\nu_1^2-\lambda_1^2)(\lambda_1^2-\mu_1^2)\,d\lambda_1\,d\mu_1\,d\nu_1}{\sqrt{(\lambda_1^2-b_1^2)(\lambda_1^2-c_1^2)(\mu_1^2-b_1^2)(c_1^2-\mu_1^2)(b_1^2-\nu_1^2)(c_1^2-\nu_1^2)}};$$

and for the volume of the ellipsoid

$$\frac{x^2}{\lambda_1^2}+\frac{y^2}{\lambda_1^2-b_1^2}+\frac{z^2}{\lambda_1^2-c_1^2}=1,$$

the limits are: for λ_1, from c_1 to λ_1;
for μ_1, from b_1 to c_1;
for ν_1, from 0 to b_1.

Hence it follows that the value of the definite integral

$$\int_{c_1}^{\lambda_1}\int_{b_1}^{c_1}\int_0^{b_1}\frac{(\mu_1^2-\nu_1^2)(\nu_1^2-\lambda_1^2)(\lambda_1^2-\mu_1^2)\,d\lambda_1\,d\mu_1\,d\nu_1}{\sqrt{(\lambda_1^2-b_1^2)(\lambda_1^2-c_1^2)(\mu_1^2-b_1^2)(c_1^2-\mu_1^2)(b_1^2-\nu_1^2)(c_1^2-\nu_1^2)}}$$

is $$\tfrac{1}{8}\cdot\tfrac{4}{3}\pi\lambda_1\sqrt{\lambda_1^2-b_1^2}\sqrt{\lambda_1^2-c_1^2},$$

being an octant of the ellipsoid.

*This is the notation adopted by Todhunter, *Functions of Laplace, Lamé and Bessel*; Bertrand, *Calc. Int.*

The suffix has been retained to prevent misconception as to the meanings of the several letters, but may now be dropped. For this and the values of other definite integrals of similar nature, see Todhunter, *Functions of Laplace, Lamé and Bessel*, Chapter XXI.

821. **Solid Angle.**

Let C be any closed curve, plane or twisted, bounding any region upon a surface, O a fixed point, and S a sphere of unit radius, with centre O. Let a cone with vertex O and generators passing through the perimeter of C, isolate on the unit sphere an area ω. Then ω is called the "solid angle" subtended at O by the portion of surface bounded by C.

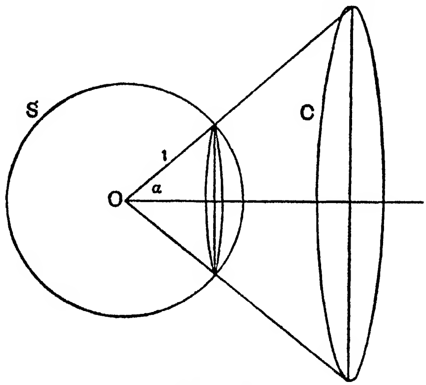


Fig. 287.

The area of a sphere being $4\pi \times (\text{radius})^2$, it follows that the solid angle subtended by any closed surface at a point within it is 4π; at a point upon it which is not a singularity, 2π; at a point outside, 0. The solid angle subtended at a corner of a cube by the rest of the cube is $\frac{4\pi}{8} = \frac{\pi}{2}$. At a point on the line of intersection of two planes cutting at right angles, each of the regions into which space is divided by the two planes subtends a solid angle $\frac{4\pi}{4} = \pi$. At the vertex of a right circular cone of semivertical angle a, the solid angle is the area of the portion of unit sphere, centre at the vertex, cut off by the cone, *i.e.* $2\pi \,.\, 1 \,.\, (1 - \cos a)$, *i.e.* $2\pi \operatorname{vers} a$.

A circular disc of radius a subtends at a point O on the axis whose distance from the plane of the disc is h, a solid angle

$$2\pi\left[1-\cos\left(\tan^{-1}\frac{a}{h}\right)\right]=2\pi\left(1-\frac{h}{\sqrt{a^2+h^2}}\right).$$

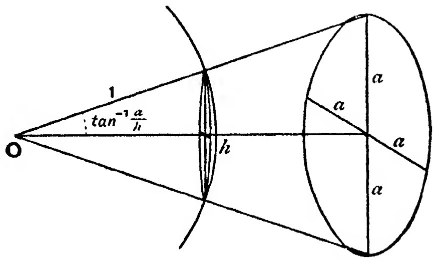


Fig. 288.

822. In the spherical polar system of coordinates, the face of the elementary cuboid $r^2\sin\theta\,\delta\theta\,\delta\phi\,\delta r$, which is at right

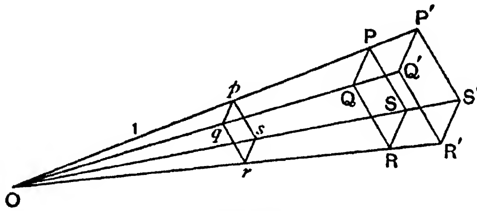


Fig. 289.

angles to the radius vector, is $r^2\sin\theta\,\delta\theta\,\delta\phi$, and if $\delta\omega$ be the solid angle subtended at the origin O, we have

$$\frac{r^2\sin\theta\,\delta\theta\,\delta\phi}{\delta\omega}=\frac{r^2}{1^2},$$

i.e. the area $pqrs$, viz. $\delta\omega$, intercepted upon unit sphere by radii vectores to the boundary of the element whose face is $PQRS$, viz. $r^2\sin\theta\,\delta\theta\,\delta\phi$, is given by

$$\delta\omega=\sin\theta\,\delta\theta\,\delta\phi.$$

The element of volume $r^2\sin\theta\,\delta\theta\,\delta\phi\,\delta r$ may therefore be written as $r^2\,\delta\omega\,\delta r$, and

$$V=\iint r^2\,d\omega\,dr=\frac{1}{3}\int r^3\,d\omega.$$

In the case of the sphere r is constant, and

$$=\tfrac{1}{3}r^3\,.\,4\pi=\tfrac{4}{3}\pi r^3.$$

823. Let the inward drawn normal at any point of a closed surface make an angle χ with the radius vector r to the point, and let δS be an element of the surface about the

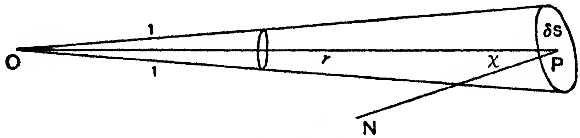


Fig. 290.

point; then the projection of δS upon a plane cutting the radius vector perpendicularly is $\delta S \cos\chi$, and in the limit when δS is infinitesimal, we have

$$\frac{\delta S \cos\chi}{\delta\omega} = \frac{r^2}{1^2} \quad \text{or} \quad \delta S = r^2 \sec\chi\, \delta\omega,$$

to the second order; whence

$$S = \int r^2 \sec\chi\, d\omega.$$

Also, if p be the perpendicular upon the tangent plane at the point r, θ, ϕ, we have

$$p = r\cos\chi \quad \text{and} \quad S = \int \frac{r^3}{p}\, d\omega.$$

Obviously it follows also that

$$\int \frac{\cos\chi}{r^2}\, dS = \int d\omega = \omega,$$

and if the closed surface surrounds the pole O, this gives

$$\int \frac{\cos\chi}{r^2}\, dS = 4\pi.$$

If O lies at a point on the surface where there is no singularity,

$$\int \frac{\cos\chi}{r^2}\, dS = 2\pi.$$

If O lies outside the closed surface,

$$\int \frac{\cos\chi}{r^2}\, dS = 0.$$

If O lies at a conical point of solid angle ω,

$$\int \frac{\cos\chi}{r^2}\, dS = \omega.$$

These theorems are of great importance in the theory of attractions, and are due to Gauss. (See E. J. Routh, *Anal. Statics*, vol. ii., Art. 106.)

824. Solid angle subtended by a triangle at a point not in its plane.

Let ABC be a triangle of sides a, b, c lying anywhere in a given plane XY, let O be a point not in this plane, and let OA, OB, OC be respectively p, q, r. Let the planes OBC, OCA, OAB intercept on the unit sphere, centre O, the spherical triangle $A'B'C'$ of sides a', b', c', and let p' be the great circle perpendicular from A' on $B'C'$, and let ω be the solid angle subtended by ABC at O, and E' the spherical excess of the triangle $A'B'C'$.

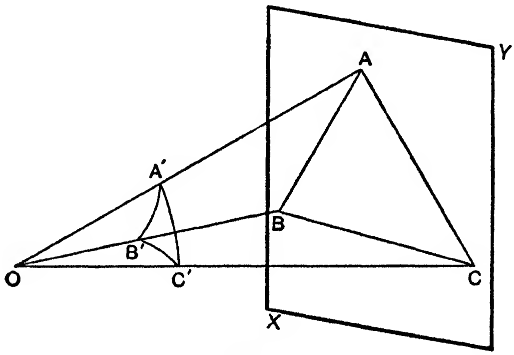


Fig. 291.

Then ω is measured by the area of $A'B'C'$, *i.e.*

$$\omega = E' = A' + B' + C' - \pi.$$

Hence it appears that triangles bounded by planes such that the sum of the angles between them is constant subtend the same solid angle at O.

Cagnoli's theorem gives

$$\sin\frac{E'}{2} = \frac{\sqrt{\sin s' \sin(s'-a') \sin(s'-b') \sin(s'-c')}}{2\cos\frac{a'}{2}\cos\frac{b'}{2}\cos\frac{C'}{2}},$$

or, which is the same thing,

$$= \frac{\sin a' \sin b' \sin C'}{4\cos\frac{a'}{2}\cos\frac{b'}{2}\cos\frac{C'}{2}}.$$

[Todhunter and Leathem, *Spherical Trigonometry*, Art. 132.]

Now let the volume of the tetrahedron $OABC$ be called V; then

$$\tfrac{1}{3} \cdot \tfrac{1}{2} qr \sin a' \,.\, p \sin p' = V,$$

i.e. $$pqr \sin a' \sin b' \sin C' = \text{constant} = 6V.$$

Again, $$q^2+r^2-a^2=2qr\cos a',$$

i.e. $$(q+r)^2-a^2=4qr\cos^2\frac{a'}{2},$$

and if Π^2 represent $[(q+r)^2-a^2][(r+p)^2-b^2][(p+q)^2-c^2]$, we have

$$\Pi^2=64p^2q^2r^2\cos^2\frac{a'}{2}\cos^2\frac{b'}{2}\cos^2\frac{c'}{2} \quad \text{and} \quad \Pi=8pqr\cos\frac{a'}{2}\cos\frac{b'}{2}\cos\frac{c'}{2}.$$

Hence $$\sin\frac{\omega}{2}=12\frac{V}{\Pi}.$$

Also, if h be the distance of O from the plane of ABC and Δ the area of the triangle,

$$V=\tfrac{1}{3}h\Delta \quad \text{and} \quad \sin\frac{\omega}{2}=4h\frac{\Delta}{\Pi}.$$

If then the triangle moves in its own plane in such manner as to make

$$[(q+r)^2-a^2][(r+p)^2-b^2][(p+q)^2-c^2]=\text{constant},$$

the solid angle at O will remain constant.

If the triangle ABC be a fixed non-conducting lamina uniformly electrified, this equation will determine the lines of equal density of electricity induced upon an infinite parallel plane conducting and uninsulated.

825. Illustrative Examples.

1. To find the volume of the portion of the paraboloid

$$\frac{x^2}{a}+\frac{y^2}{b}=2z$$

cut off by the plane $lx+my+nz=p$.

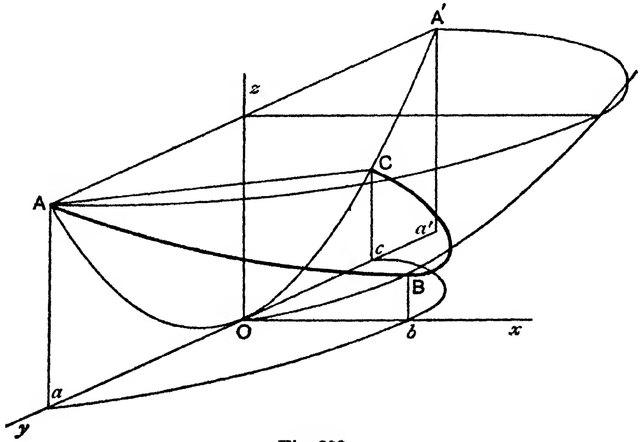


Fig. 292.

The difference of the z-ordinates of the plane and the paraboloid is

$$\zeta=\frac{p-lx-my}{n}-\frac{1}{2}\left(\frac{x^2}{a}+\frac{y^2}{b}\right).$$

The projection of the curve of intersection upon the x-y plane is

$$\frac{x^2}{a}+\frac{y^2}{b}+2\frac{lx+my-p}{n}=0, \quad \textit{i.e.} \text{ an ellipse.}$$

The problem of finding the volume required is that of finding the mass of this elliptic lamina with a surface density ζ. We have to evaluate

$$\iint \zeta\, dx\, dy$$

over the area of the ellipse.

Keeping x constant, we have

$$V=\int\left[\left(\frac{p-lx}{n}-\frac{x^2}{2a}\right)(y_2-y_1)-\frac{m}{2n}(y_2^2-y_1^2)-\frac{1}{6b}(y_2^3-y_1^3)\right]dx,$$

where y_1, y_2 are the ordinates of the ellipse on the x-y plane for any given value of x.

Now, the quadratic for y being

$$\frac{y^2}{b}+\frac{2m}{n}y+\left(\frac{x^2}{a}+\frac{2lx}{n}-\frac{2p}{n}\right)=0,$$

we have $$y_1+y_2=-\frac{2mb}{n}, \quad y_1y_2=b\left(\frac{x^2}{a}+\frac{2lx}{n}-\frac{2p}{n}\right).$$

Hence the subject of integration is

$$(y_2-y_1)\left[-\frac{y_1y_2}{2b}+\frac{(y_1+y_2)^2}{4b}-\frac{y_2^2+y_2y_1+y_1^2}{6b}\right]=\frac{(y_1-y_2)^3}{12b}.$$

Also $$(y_1\sim y_2)^2=\left(\frac{2mb}{n}\right)^2-4b\left(\frac{x^2}{a}+\frac{2lx}{n}-\frac{2p}{n}\right)=\frac{4b}{a}(c^2-\xi^2),$$

where $$c^2=a\frac{al^2+bm^2+2np}{n^2} \quad\text{and}\quad \xi=x+a\frac{l}{n},$$

and the limits for ξ are $-c$ and $+c$.

Hence $$V=\frac{1}{12b}\left(\frac{4b}{a}\right)^{\frac{3}{2}}\int_{-c}^{c}(c^2-\xi^2)^{\frac{3}{2}}\,d\xi.$$

To effect the final integration, let $\xi=c\sin\theta$.

Then the limits are 0 to $\frac{\pi}{2}$ and double.

Hence $$V=2\cdot\frac{1}{12b}\left(\frac{4b}{a}\right)^{\frac{3}{2}}c^4\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$

$$=\frac{1}{6b}\cdot\frac{8b^{\frac{3}{2}}}{a^{\frac{3}{2}}}\cdot c^4\frac{3}{4}\frac{1}{2}\frac{\pi}{2}$$

$$=\frac{\pi}{4}\cdot\frac{b^{\frac{1}{2}}}{a^{\frac{3}{2}}}\cdot a^2\left\{\frac{al^2+bm^2+2np}{n^2}\right\}^2$$

$$=\frac{\pi}{4}\cdot\frac{\sqrt{ab}}{n^4}(al^2+bm^2+2pn)^2.$$

We might elect to do the same thing by taking laminae parallel to the plane $lx+my+nz=p$.

The area of such a section is

$$\frac{\pi\sqrt{ab}}{n^3}(al^2+bm^2+2pn)$$

[C. Smith, *Solid Geometry*, p. 99.]

The thickness of a slice is δp.

The slice of zero area is such that

$$al^2+bm^2+2p_1n=0,$$

p_1 being the corresponding value of p.

The limits of integration with respect to p are from p_1 to p.

Hence
$$V=\frac{\pi\sqrt{ab}}{n^3}\int_{p_1}^{p}(al^2+bn^2+2pn)\,dp$$
$$=\frac{\pi\sqrt{ab}}{n^3}\Big[(al^2+bn^2)(p-p_1)+n(p^2-p_1^2)\Big]$$
$$=\frac{\pi\sqrt{ab}}{n^3}[-2p_1n(p-p_1)+n(p^2-p_1^2)]$$
$$=\frac{\pi\sqrt{ab}}{n^3}[p-p_1]^2n=\frac{\pi\sqrt{ab}}{n^2}(p-p_1)^2$$
$$=\frac{\pi\sqrt{ab}}{4n^4}(2pn+al^2+bn^2)^2,$$

as before.

We may note that frusta of finite thickness whose bases are parallel to a given plane are such that their volumes vary as the squares of their thicknesses; also that frusta of given thickness are such that their volumes vary as the squares of the secants of the angles which the normals to their bases make with the axis of the paraboloid.

2. To calculate the value of $\iiint\phi(lx+my+nz)\,dx\,dy\,dz$, the integrations being conducted through the volume of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

l, m, n being such that $l^2+m^2+n^2=1$.

Let
$$lx+my+nz=\delta.$$

The area of this section of the ellipsoid is

$$A=\frac{\pi abc}{p}\left(1-\frac{\delta^2}{p^2}\right),$$

where $p^2=a^2l^2+b^2m^2+c^2n^2$.

Consider the ellipsoid divided into thin slices parallel to this plane. The volume of such a slice is $A\,d\delta$ to the first order, $d\delta$ being the thickness of the slice, and $\phi(\delta)$ is, to the first order, constant through the slice.

Hence

$$\iiint\phi(lx+my+nz)\,dx\,dy\,dz=\frac{\pi abc}{p}\int_{-p}^{p}\phi(\delta)\left(1-\frac{\delta^2}{p^2}\right)d\delta.$$

3. To calculate the value of $\iiint \phi\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dx\,dy\,dz$, the integrations being conducted through the volume of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1.$$

Take
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\delta.$$

The volume of the ellipsoidal shell bounded by the similar ellipsoids δ and $\delta+d\delta$ is
$$d(\tfrac{4}{3}\pi abc\delta^{\frac{3}{2}})=2\pi abc\delta^{\frac{1}{2}}\,d\delta,$$
and $\phi(\delta)$ is constant throughout this shell.

Hence
$$\iiint \phi\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dx\,dy\,dz=2\pi abc\int_0^1\phi(\delta)\sqrt{\delta}\,d\delta$$

4. Find the mass of a thick focaloid,* *i.e.* a shell bounded by confocal ellipsoids, the layers of equal density being confocal surfaces, and the density at each point inversely proportional to the volume contained by the confocal through the point.

Let $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$ be the confocal through the point, and let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1$$

be the outer and inner surfaces of the shell.

The volume contained by the ellipsoid λ is
$$V=\tfrac{4}{3}\pi\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}.$$

The volume of the layer between the surfaces λ and $\lambda+d\lambda$ is
$$dV=\tfrac{2}{3}\pi\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)d\lambda.$$

The law of density is
$$\rho=k/\tfrac{4}{3}\pi\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)},\ \ k \text{ being a constant.}$$

Hence the mass of the layer is
$$\rho\,dV=\frac{k}{2}\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)d\lambda,$$

and the mass of the thick shell is
$$M=\int_{a'^2-a^2}^{0}\rho\,dV=\left[\frac{k}{2}\log(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\right]_{a'^2-a^2}^{0}$$
$$=\frac{k}{2}\log a^2b^2c^2-\frac{k}{2}\log a'^2(b^2+a'^2-a^2)(c^2+a'^2-a^2)$$
$$=k\log\frac{abc}{a'b'c'},\quad\text{for } a^2-a'^2=b^2-b'^2=c^2-c'^2;$$

and if D be the density of the outer layer,
$$D=\frac{k}{\frac{4}{3}\pi abc}.$$

Hence
$$M=\tfrac{4}{3}\pi abcD\log\frac{abc}{a'b'c'}.$$

*For this term see remarks by E. J. Routh, *Anal. Statics*, vol. ii., p. 97, and Thomson and Tait's *Natural Philosophy*.

5. Consider the region bounded by

(1) a sphere $x^2+y^2+z^2=a^2$;

(2) a right circular cylinder $x^2+y^2=bx \quad (a \not< b)$;

(3) the two planes $y=\pm x \tan \alpha$.

We shall first find the volume enclosed by these surfaces in the positive octant of space.

Take cylindrical coordinates r, θ, z.

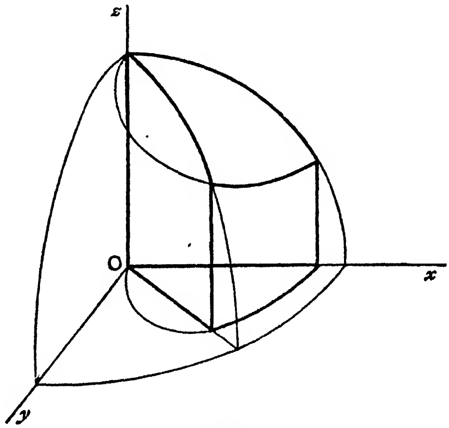


Fig. 293.

The elementary prism on base $r\,\delta\theta\,\delta r$ has volume $rz\,\delta\theta\,\delta r$ to the second order, and

$$V=\iint rz\,d\theta\,dr$$

$$=\iint r\sqrt{a^2-r^2}\,d\theta\,dr$$

$$=-\frac{1}{3}\int[(a^2-r^2)^{\frac{3}{2}}]\,d\theta,$$

and the equation of the trace of the cylinder upon the xy plane being $r=b\cos\theta$, the limits for r are 0 to $b\cos\theta$, whilst the limits for θ are from $\theta=0$ to $\theta=\alpha$.

Hence
$$V=\frac{1}{3}\int_0^{\alpha}\{a^3-(a^2-b^2\cos^2\theta)^{\frac{3}{2}}\}\,d\theta$$

$$=\frac{1}{3}a^3\alpha-\frac{a^3}{3}\int_0^{\alpha}\left(1-\frac{b^2}{a^2}\cos^2\theta\right)^{\frac{3}{2}}d\theta.$$

Writing $\theta=\frac{\pi}{2}-\phi$ and $\alpha=\frac{\pi}{2}-\beta$ in the integral,

$$V=\frac{1}{3}a^3\left(\frac{\pi}{2}-\beta\right)-\frac{a^3}{3}\int_{\beta}^{\frac{\pi}{2}}\Delta^3\,d\phi;$$

where $$\Delta^2=\left(1-\frac{b^2}{a^2}\sin^2\phi\right),$$

$$V=\frac{a^3}{3}\left(\frac{\pi}{2}-\beta\right)-\frac{a^3}{3}\left\{\left(\int_0^{\frac{\pi}{2}}-\int_0^{\beta}\right)\Delta^3\phi\,d\phi\right\},$$

and by Legendre's formula (No. 10, p. 399),

$$\int_0^{\phi}\Delta^3 d\phi=\frac{k^2}{3}\Delta\sin\phi\cos\phi+\frac{4-2k^2}{3}E-\frac{k'^2}{3}F;$$

and if E_1, F_1 be the real quarter periods, we have

$$V=\frac{a^3}{3}\left(\frac{\pi}{2}-\beta\right)-\frac{a^3}{3}\left\{-\frac{1}{3}\frac{b^2}{a^2}\sin\beta\cos\beta\sqrt{1-\frac{b^2}{a^2}\sin^2\beta}\right.$$
$$\left.+\frac{4a^2-2b^2}{3a^2}(E_1-E)-\frac{a^2-b^2}{3a^2}(F_1-F)\right\},$$

where $E=\int_0^{\beta}\sqrt{1-\frac{b^2}{a^2}\sin^2\phi}\,d\phi$ and $F=\int_0^{\beta}\frac{d\phi}{\sqrt{1-\frac{b^2}{a^2}\sin^2\phi}}$.

And for the *whole* volume of the sphere included between the specified boundaries, we have four times this quantity.

When the cylinder just touches the sphere, *i.e.* $b=a$, the elliptic functions degenerate.

We then have for the volume in the positive octant

$$V=\frac{a^3}{3}\int_0^{\alpha}(1-\sin^3\theta)\,d\theta$$
$$=\frac{a^3}{3}\int_0^{\alpha}\left(1-\frac{3\sin\theta-\sin 3\theta}{4}\right)d\theta$$
$$=\frac{a^3}{12}[4\alpha-3(1-\cos\alpha)+\tfrac{1}{3}(1-\cos 3\alpha)]$$
$$=\frac{a^3}{36}(12\alpha-9\operatorname{vers}\alpha+\operatorname{vers}3\alpha);$$

and in the case where the planes $y=\pm x\tan\alpha$ coincide with the y-z plane, *i.e.* $\alpha=\frac{\pi}{2}$, the whole volume cut out of the sphere by the cylinder $r=a\cos\theta$ is

$$4V=\frac{a^3}{9}(6\pi-8)=\frac{2a^3}{9}(3\pi-4).$$

To find the surface of the sphere thus bounded in the positive octant, we have

$$S=\iint\sec\gamma\,.\,r\,d\theta\,dr,$$

γ being as usual the angle the normal to the sphere at r, θ, z makes with the z-axis; that is $\cos\gamma=\frac{z}{a}=\frac{\sqrt{a^2-r^2}}{a}$.

Hence
$$S=\iint \frac{ar}{\sqrt{a^2-r^2}}\,d\theta\,dr$$
$$=-a\int_0^{a}\left[\sqrt{a^2-r^2}\right]_0^{b\cos\theta}d\theta$$
$$=a\int_0^{a}\{a-\sqrt{a^2-b^2\cos^2\theta}\}\,d\theta\;;$$

and putting as before $\theta=\frac{\pi}{2}-\phi$ and $a=\frac{\pi}{2}-\beta$,
$$S=a^2\left(\frac{\pi}{2}-\beta\right)+a^2\int_{\frac{\pi}{2}}^{\beta}\sqrt{1-\frac{b^2}{a^2}\sin^2\phi}\,d\phi$$
$$=a^2\left(\frac{\pi}{2}-\beta\right)+a^2\left(\int_0^{\beta}-\int_0^{\frac{\pi}{2}}\right)\sqrt{1-\frac{b^2}{a^2}\sin^2\phi}\,d\phi$$
$$=a^2\left(\frac{\pi}{2}-\beta\right)+a^2\left\{E\left(\beta,\frac{b}{a}\right)-E_1\left(\frac{\pi}{2},\frac{b}{a}\right)\right\}\;;$$

and when $b=a$, we have
$$S=a^2\int_0^{a}(1-\sin\theta)\,d\theta$$
$$=a^2(a-\text{vers}\,a)\;;$$

and for the further particular case when $a=\frac{\pi}{2}$,
$$S=a^2\left(\frac{\pi}{2}-1\right).$$

And in each case the *whole* of the surface of the sphere intercepted in this manner is four times the portion which has been found.

6. At every point of an elliptic lamina a straight line is drawn perpendicular to the plane of the lamina and of such length that the volume (μ, say) of the rectangular parallelepiped formed by this length and the distances of the point from the foci of the elliptic boundary is constant. Given that a and b are the semiaxes of the elliptic boundary, show that the volume of the solid thus formed is
$$\frac{\pi\mu}{4}\log\frac{a+b}{a-b}.$$
[COLLEGES, 1891.]

Taking $x+\iota y=c\cos(\theta+\iota\phi)$, we have
$$x=c\cos\theta\cosh\phi,\quad y=-c\sin\theta\sinh\phi,$$

and the loci $\phi=\text{constant}$, $\theta=\text{constant}$ are the confocal conics
$$\frac{x^2}{c^2\cosh^2\phi}+\frac{y^2}{c^2\sinh^2\phi}=1\quad\text{and}\quad\frac{x^2}{c^2\cos^2\theta}-\frac{y^2}{c^2\sin^2\theta}=1,$$

and the focal radii r_1, r_2 are such that $r_1+r_2=2c\cosh\phi$, $r_1-r_2=2c\cos\theta$.

Let the elliptic area be divided up into elements by confocals in this way, taking the element bounded by θ, $\theta+\delta\theta$, ϕ, $\phi+\delta\phi$ as a type.

Now
$$\iint F(x,y)\,dx\,dy=\iint F_1(\theta,\phi)\,J\,d\theta\,d\phi,$$

where F_1 is the equivalent of F in terms of θ, ϕ.

Also $$J=\frac{\partial(x, y)}{\partial(\theta, \phi)}=\begin{vmatrix} -c\sin\theta\cosh\phi, & c\cos\theta\sinh\phi \\ -c\cos\theta\sinh\phi, & -c\sin\theta\cosh\phi \end{vmatrix}$$
$$=c^2(\sin^2\theta\cosh^2\phi+\cos^2\theta\sinh^2\phi)$$
$$=c^2(\cosh^2\phi-\cos^2\theta)$$
$$=\tfrac{1}{4}\{(r_1+r_2)^2-(r_1-r_2)^2\}=r_1r_2$$
and by the condition of the question $\mu=zr_1r_2$.

Thus
$$\text{Volume}=V=\iint z\,dx\,dy=\iint\frac{\mu}{r_1r_2}\,.\,r_1r_2\,d\theta\,d\phi=\mu[\theta][\phi],$$
and the limits for θ are $\theta=0$ to $\theta=\frac{\pi}{2}$, and for ϕ from $\phi=0$ to the value for which $c\cosh\phi=a$ and $c\sinh\phi=b$, that is $\phi=\sinh^{-1}\frac{b}{\sqrt{a^2-b^2}}$.

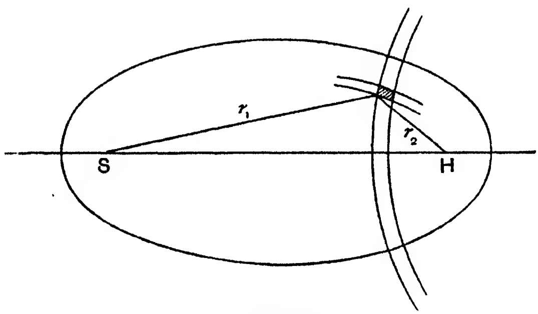


Fig. 294.

Thus
$$V=\mu\frac{\pi}{2}\sinh^{-1}\frac{b}{\sqrt{a^2-b^2}}=\mu\frac{\pi}{2}\log\frac{b+a}{\sqrt{a^2-b^2}}=\mu\frac{\pi}{4}\log\frac{a+b}{a-b}.$$

7. In the evaluation of such integrals as $I_n\equiv\int\frac{dS}{p^n}$ taken over the surface of an ellipsoid of semi-axes a, b, c, where the surface is S and the volume V, p being the central perpendicular upon any tangent plane, consider three points P, Q, R on the surface, which are the extremities of three semi-conjugate diameters. Let δS_1, δS_2, δS_3 be any elements of the surface about the three points and p_1, p_2, p_3 the corresponding perpendiculars.

Then
$$I_n=\int\frac{dS_1}{p_1^{\,n}},\quad\text{or}\quad\int\frac{dS_2}{p_2^{\,n}},\quad\text{or}\quad\int\frac{dS_3}{p_3^{\,n}}$$
$$=\frac{1}{3}\int\left(\frac{dS_1}{p_1^{\,n}}+\frac{dS_2}{p_2^{\,n}}+\frac{dS_3}{p_3^{\,n}}\right).$$

Now suppose these elements of area δS_1, δS_2, δS_3 to have been so chosen that
$$\frac{\delta S_1}{p_1^{\,n-2}}=\frac{\delta S_2}{p_2^{\,n-2}}=\frac{\delta S_3}{p_3^{\,n-2}}=\frac{\delta S}{p^{n-2}},\ \text{say}.$$

Then, since $$\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},$$

we have $$I_n=\frac{1}{3}\int\left(\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}\right)\frac{dS}{p^{n-2}},$$

i.e. $$I_n=\frac{1}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)I_{n-2};$$

we also have $$I_{-1}=\int p\,dS=3V, \quad \text{and} \quad I_0=S;$$

whence we can readily infer the values of I_1, I_2, I_3, etc., viz.

$$\frac{I_{2n}}{S}=\frac{I_{2n-1}}{3V}=\frac{1}{3^n}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^n.$$

PROBLEMS.

1. Find by integration the volume of a frustum of
   (1) a pyramid on a triangular base,
   (2) a pyramid on a square base,
   (3) a cone.

2. Find the volume of the portion of a sphere bounded by planes through the centre which cut the sphere in the sides of a given spherical triangle ABC.

3. Show that the volume cut off from the paraboloid
$$x^2+y^2=4az$$
by the plane $$x+y+z=a$$
is $$18\pi a^3.$$

4. Show that the volume of the solid bounded by
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^{2n}}{c^{2n}}=1$$
is $$\frac{4n}{2n+1}\pi abc.$$

5. Show that the volume bounded by the surface
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\left(\frac{x}{a}+\frac{y}{b}\right)\frac{z^n}{c^n}$$
and the planes $$z=0, \quad z=h$$
is $$\frac{2\pi abh}{2n+1}\left(\frac{h}{c}\right)^{2n}$$

6. Show that the volume of a slice of the ellipsoid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

bounded by the parallel planes

$$lx+my+nz=\delta_1,$$
$$lx+my+nz=\delta_2,$$

is
$$\frac{\pi abc}{3p^3}(\delta_1-\delta_2)(3p^2-\delta_1^2-\delta_1\delta_2-\delta_2^2),$$

where p is the central perpendicular upon a tangent plane parallel to the faces of the slice.

7. If A be the area of a central section of an ellipsoid parallel to the tangent plane at the elementary area δS, show that

$$\int\frac{dS}{A}=4,$$

the integration being taken over the surface of the ellipsoid.

8. Prove that over an ellipsoid of semiaxes a, b, c,

$$\int p\,dS=4\pi abc,$$
$$\int\frac{dS}{p}=\frac{4}{3}\pi\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right),$$
$$\int\frac{dS}{p^2}=\frac{S}{3}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right),$$

dS being an element of surface, and p the central perpendicular upon the tangent plane.

Investigate also the value of $\int\frac{dS}{p^3}$.

9. Apply the formula $V=\frac{1}{3}\iint(lx+my+nz)\,dS$ to find the volume of an ellipsoid, x, y, z being the coordinates of any point on the surface, and l, m, n the direction cosines of the normal there.

[COLLEGES α, 1881.]

10. If the ellipsoid of semiaxes a, b, c be very nearly spherical, then its area is, to the first order (inclusive) of the small quantities, represented by the difference of the axes

$$4\pi a^{\frac{2}{3}}b^{\frac{2}{3}}c^{\frac{2}{3}}.$$

[TRINITY, 1891.]

11. Show that a portion of a spherical surface (radius unity) may be bent into the surface of revolution defined by the equations

$$x=k\cos p\cos\frac{q}{k},\quad y=k\cos p\sin\frac{q}{k},\quad z=E(p,k)\ \left(=\int_0\sqrt{1-k^2\sin^2p}\,dp\right);$$

and explain the geometrical theory, distinguishing the two cases $k<1$, $k>1$.

[MATH. TRIPOS, 1887.]

12. The curve $z=f(x)$, $y=0$ revolves about the axis of x, and the surface thus formed is intersected by the right cylinder $y=\phi(x)$, which is symmetrical with respect to the axis of x: prove that the cylinder cuts off from the first surface a portion the area of which can be determined by the evaluation of the integral

$$\int z\sqrt{1+\left(\frac{dz}{dx}\right)^2}\sin^{-1}\frac{y}{z}\,dx$$

between proper limits. [OXFORD II. P., 1888.]

13. Show that the cylinder $(x-c)^2+y^2=(a-c)^2$ cuts off from the sphere $x^2+y^2+z^2=a^2$ a portion of which the area is

$$8a\left\{a\cos^{-1}(c^{\frac{1}{2}}a^{-\frac{1}{2}})-c^{\frac{1}{2}}(a-c)^{\frac{1}{2}}\right\},$$

a being supposed greater than c. [OXFORD II. P., 1888.]

14. Prove that the volume cut off from the paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$$

by the plane $$z=px+qy+r$$

is $$\frac{\pi abc}{4}\left(\frac{a^2p^2}{c^2}+\frac{b^2q^2}{c^2}+\frac{2r}{c}\right)^2.$$ [OXFORD II. P., 1902.]

15. Show that the volume enclosed between the surface

$$z^2\{(x^2+y^2+c^2)^2-4c^2x^2\}=c^4y^2$$

and the cylinder $$x^2+y^2=c^2$$

is $$(\pi-2)c^3.$$ [OXFORD II. P., 1886.]

16. By application of the formulae $V=\frac{1}{3}\int p\,dS$, $V=\int z\cos\gamma\,dS$ to the evaluation of the volume of an ellipsoid, establish the results

$$(1)\quad \int_b^c\int_0^b\frac{(\mu^2-\nu^2)\,d\mu\,d\nu}{\sqrt{(b^2-\nu^2)(c^2-\nu^2)(c^2-\mu^2)(\mu^2-b^2)}}=\frac{\pi}{2},$$

$$(2)\quad \int_b^c\int_0^b(\mu^2-\nu^2)\frac{\sqrt{(c^2-\mu^2)(c^2-\nu^2)}}{\sqrt{(b^2-\nu^2)(\mu^2-b^2)}}\,d\mu\,d\nu=\frac{\pi}{6}c^2(c^2-b^2).$$

(See Art. 820 for the notation.) [LAMÉ.]

[TODHUNTER, *Functions of Laplace, Lamé and Bessel*, pages 216, 217; BERTRAND, *Calc. Int.*, pages 424, 426.]

17. Show that the volume bounded by the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\left(\frac{x}{a}+\frac{y}{b}\right)\phi(z)$$

and the planes $$z=0,\quad z=z_1$$

is $$2\pi ab\int_0^{z_1}[\phi(z)]^2\,dz.$$

18. A cavity is just large enough to allow of the complete revolution of a circular disc of radius c, whose centre describes a circle of the same radius c, while the plane of the disc is constantly parallel to a fixed plane, and perpendicular to that in which the centre moves. Show that the volume of the cavity is

$$\frac{2c^3}{3}(3\pi+8).$$

19. If O be a point without a sphere of radius a and centre C, and r the distance of any point of the sphere from O, show that, integrating $\frac{1}{r^n}$ over the surface, we have

$$\int\frac{dS}{r^n}=\frac{2\pi}{n-2}\cdot\frac{a}{c}\left[(c-a)^{2-n}-(c+a)^{2-n}\right] \quad \text{if } n\neq 2,$$

and $$2\pi\frac{a}{c}\log\frac{c-a}{c+a} \quad \text{if } n=2.$$

What will be the results if O lies within the sphere?

20. A surface is obtained by making the diameter $2a$ of a semicircle move parallel to itself, the path of the centre being perpendicular to the initial plane of the semicircle, whilst the plane of the semicircle rotates round the diameter; and when the plane has moved through an angle θ the distance which the diameter has moved is $c\sin\theta$. Prove that the volume of the whole surface so generated is

$$\tfrac{4}{3}\pi a^3+\tfrac{1}{2}\pi^2ca^2.$$ [TRINITY, 1890.]

21. Use the theorem

$$V=\iiint dx\,dy\,dz=\iiint J\,du\,dv\,dw$$

to find the volume of the parallelepiped enclosed by the planes

$$ax+by+cz=0,\quad a_1x+b_1y+c_1z=0,\quad a_2x+b_2y+c_2z=0,$$
$$ax+by+cz=d,\quad a_1x+b_1y+c_1z=d_1,\quad a_2x+b_2y+c_2z=d_2.$$

22. Prove that the area of that portion of the surface

$$(m^2-1)(x^2+y^2)=z^2,$$

which is cut out by the surface

$$z=a^{-1}x^2+b^{-1}y^2$$

where a and b are positive, is

$$\frac{\pi}{2}m(m^2-1)a^{\frac{1}{2}}b^{\frac{1}{2}}(a+b).$$ [OXFORD II. P., 1890.]

23. Show that when $f(x)$ is a slowly changing function,

$$\int_a^b f(x)\,dx$$

is approximately equal to

$$\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

Prove that this formula may be used to calculate exactly the volume cut from a hyperboloid of one sheet by parallel planes meeting it in elliptic sections. [COLLEGES α, 1881.]

24. Prove that the volume included in the positive octant between the surface $$z^2(x^2+y^2+a^2)^{2n+2}(x^2+2a^2)^{2n}=a^{8n+4}y^2$$
and the planes $$x=0,\quad x=\infty,\quad y=a,\quad y=\infty$$
is
$$\frac{\pi a^3}{2^{n+\frac{3}{2}}n}\,\frac{1\,.\,3\,.\,5\,\ldots\,(4n-3)}{2\,.\,4\,.\,6\,\ldots\,(4n-2)},$$
n being a positive integer.

25. Show that the area of that part of the sphere $r=1$, enclosed by the cone $\tan\dfrac{\theta}{2}=\sqrt{3}\cos\phi$, is π. [COLLEGES α, 1881.]

26. Show that the volume of the solid, the equation to the surface of which is $$z^2+ax^2+2\delta xy+\beta y^2=2\mu z,$$
is
$$\frac{4\pi}{3}\,\frac{\mu^3}{\sqrt{a\beta-\delta^2}}.$$
[COLLEGES, 1882.]

27. If in the tangent plane at the vertex of a paraboloid two ellipses be described whose axes are in the principal sections and proportional to their parameters, the cylinders whose bases are these ellipses, and whose generators are parallel to the axis of the paraboloid, will intercept on the surface a portion whose area is proportional to the difference between the radii of curvature of either of the principal sections at the points where it intersects the bounding curve. [COLLEGES, 1892.]

28. If the density of a tetrahedron at any point vary as the n^{th} power of the sum of the distances of the point from the faces of the tetrahedron, show that the mass of the tetrahedron

$$=kV\frac{1\,.\,2\,.\,3}{(r+1)(r+2)(r+3)}\sum\frac{p_1^{\,r+3}}{(p_1-p_2)(p_1-p_3)(p_1-p_4)},$$

where V is the volume; p_1, p_2, p_3, p_4 are the perpendiculars from the corners upon the opposite faces, and k the density at the centroid of the volume.

Examine what happens in the case of a regular tetrahedron.

29. Find the volume contained between any two planes perpendicular to the axis of x and the surface whose equation is

$$(y^2+z^2)^2=(a^2+\beta x)y^2+(a'^2+\beta' x)z^2.$$

[ST. JOHN'S, 1884.]

30. Show that the mass contained between a paraboloid of revolution and a sphere, with centre at the vertex and diameter $2a$, equal to the latus rectum of the paraboloid, where the density at any point varies as the square of the latus rectum of the paraboloid containing it and having the same vertex and axis as the bounding paraboloid, is

$$\frac{\pi}{15}(7-4\sqrt{2})a^3\rho,$$

where ρ is the density at the external surface of the paraboloid.

[COLLEGES δ, 1883.]

31. Find the volume between the surfaces

$$x(y^2+z^2)^\gamma=a_1,\quad y^2+z^2=4\beta_1 x,\quad y=b_1 z,$$
$$x(y^2+z^2)^\gamma=a_2,\quad y^2+z^2=4\beta_2 x,\quad y=b_2 z.$$

[COLLEGES δ, 1881.]

32. Prove that if a, b, c be any positive quantities in descending order of magnitude, the solid angle of that part of the cone

$$ax^2z^2+(by^2-cz^2)(x^2+y^2)=0$$

which lies on the positive side of the plane xy is equal to

$$4\sin^{-1}\left(\frac{c}{a}\right)^{\frac{1}{2}}-4\left(\frac{b}{a+b}\right)^{\frac{1}{2}}\sin^{-1}\left\{\frac{c(a+b)}{a(c+b)}\right\}^{\frac{1}{2}}.$$

[COLLEGES β, 1891.]

33. Prove that the volume common to a sphere and a circular cylinder which touches it, and also passes through the centre, is $\frac{1}{2}-\frac{2}{3\pi}$ of the volume of the sphere. [ST. JOHN'S, 1891.]

Also show that the sum of the two spherical caps cut off by the cylinder forms $\frac{1}{2}-\frac{1}{\pi}$ of the area of the sphere.

34. A sphere of radius a is cut by two diametral planes so as to form a lune of angle α, which is itself cut in two by a plane inclined at an angle β to its edge and passing through one end of it, and equally inclined to the two faces of the lune; show that the volume of the pointed part is

$$\frac{2}{3}a^3\sin\beta\left\{(2+\cos^2\beta)\tan^{-1}\left(\sin\beta\tan\frac{\alpha}{2}\right)+\frac{\sin\beta\cos^2\beta\tan\frac{\alpha}{2}}{1+\sin^2\beta\tan^2\frac{\alpha}{2}}\right\}.$$

[ST. JOHN'S, 1881.]

35. Prove that the moment of inertia about the axis of z of the part of the paraboloid $2z = ax^2 + by^2$, cut off by the plane

$$lx + my + nz = p, \quad \text{is}$$

$$\frac{\pi}{24n^6(ab)^{\frac{5}{2}}}(bl^2 + am^2 + 2pnab)^2\{bl^2(a + 7b) + am^2(7a + b) + 2pnab(a + b)\},$$

the density being taken as unity. [MATH. TRIPOS, 1890.]

36. If $A + B + C = 0$ and the coordinate axes be rectangular, prove that

$$\iint \{(A, B, C, D, E, F \between x, y, z)^2 \times (A', B', C', D', E', F' \between x, y, z)^2\} d\omega$$

$$= \frac{8\pi}{15}(AA' + BB' + CC' + 2DD' + 2EE' + 2FF'),$$

where the integration extends over the whole surface of a sphere of unit radius whose centre is the origin of coordinates.

[COLLEGES, 1892.]

Also show that the unconditional result is

$$\frac{4\pi}{15}[A'(3A + B + C) + B'(3B + C + A) + C'(3C + A + B) + 4DD' + 4EE' + 4FF'].$$

37. A flexible envelope is in the form of an oblate spheroid, such that e is the eccentricity of a meridian section: the part between two meridians, the planes of which are inclined to each other at the angle $2\pi(1 - e)$, is cut away, and the edges are then sewn together. Prove that the meridian curve of the new surface is the "curve of sines," and that the volume enclosed is changed in the ratio

$$3\pi e^2 : 8.$$ [ST. JOHN'S, 1889.]

38. A surface is such that $ABCD$ being any rectangle in the plane of x, y, with its sides parallel to Ox, Oy, and AP, BQ, CR, DS being drawn parallel to Oz to meet the surface in P, Q, R, S, the volume of the solid $ABCDPQRS$ is equal to the base $ABCD$, multiplied by the arithmetic mean of AP, BQ, CR, DS. Prove that the surface is a hyperbolic paraboloid. [MATH. TRIPOS, 1876.]

39. Show that the integral

$$\iiint e^{2\frac{x+y+z}{\sqrt{a^2+b^2+c^2}}} dx\, dy\, dz$$

taken over the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $$\frac{\pi abc}{4}(e^2 + 3e^{-2}).$$ [COLLEGES, 1885.]

Prove more generally that

$$\iint e^{k\frac{lx+my+nz}{\sqrt{a^2+b^2+c^2}}}dx\,dy\,dz$$

over the volume of the ellipsoid

$$=\frac{4\pi abc}{k^3}(k\cosh k-\sinh k),$$

and find the values of

$$\iiint e^x dx\,dy\,dz;\quad \iiint e^{x+y}dx\,dy\,dz;\quad \iiint e^{x+y+z}dx\,dy\,dz$$

through the same space.

40. On a closed oval surface of volume V and surface S, whose curvature is everywhere finite, rolls a sphere of radius a; the surface of the envelope of the sphere is S'. Prove that the volume of the envelope is

$$V+a(S'+S)-\tfrac{16}{3}\pi a^3.$$ [MATH. TRIPOS, 1886.]

41. Show that the volume of the pedal of an ellipsoid taken with the centre as origin is less than that taken with regard to any other origin; and that the sum of the volumes of the pedals, taken with regard to the extremities of three semi-conjugate diameters, is six times that taken with regard to the centre. [MATH. TRIPOS, 1887.]

42. Show that the moment of inertia of the ellipsoid

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

about the axis of x is

$$\tfrac{1}{5}M(ca-g^2+ab-h^2)(abc+2fgh-af^2-bg^2-ch^2)^{-1},$$

where M is the mass of the ellipsoid. [TRINITY, 1890.]

43. Find the envelope of the conics $x^2\sec^3\theta-y^2\tan^3\theta=a^2$, where θ is the variable parameter. Show that in addition to certain lines it consists of a curve whose asymptotes are $x=\pm a$. Also, if the area between the axis of x, an asymptote, and the corresponding branch of the curve be A, and the volume generated by the revolution of this branch about the axis of x be V, prove that

$$V=\pi aA=\tfrac{1}{2}\pi a^3\int_0^{\frac{\pi}{2}}(\sin\phi)^{\frac{1}{3}}\,d\phi.$$ [COLLEGES β, 1890.]

44. Show that the value of

$$\iiint\frac{xyz\,dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}},$$

taken throughout the positive octant of the ellipsoid

$$a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 = 1$$

is
$$\frac{a^2b^2c^2}{15}\,\frac{bc+ca+ab}{(b+c)(c+a)(a+b)}.\qquad \text{[Oxford II. P., 1888.]}$$

45. Prove that the mass of a sphere of radius a, whose density at any point P is $\frac{k}{AP}$, where k is a constant and A is a fixed point distant f ($>a$) from the centre of the sphere, is equal to

$$\frac{4}{3}\frac{\pi k a^3}{f}.\qquad \text{[Oxf. I. P., 1914.]}$$

46. Prove that the volume which lies within the sphere

$$x^2 + y^2 + z^2 = a^2$$

and the ellipsoid

$$x^2 \sin^2\alpha \operatorname{cosec}^2\beta + y^2 \cos^2\alpha \sec^2\beta + z^2 = a^2,$$

where $0 < \alpha < \beta < \frac{1}{2}\pi$, is

$$\tfrac{4}{3}a^3(\pi - 2\beta + 2\alpha \sin 2\beta \operatorname{cosec} 2\alpha).\qquad \text{[Oxf. I. P., 1916.]}$$

47. P is a point of abscissa x (>0) on the parabola

$$x^2 = 2ay, \quad z = 0,$$

and Sa^2 is the area of the segment bounded by the arc OP and the radius vector OP; the straight line PQ of length $2Sa$ is drawn parallel to Oz. The locus of Q being a curve which passes through the origin, prove that

(1) the length of the arc OQ is $x + x^3/6a^2$;

(2) the cylindrical area bounded by the arcs OP, OQ and the straight line PQ is

$$a^2/45 + (3x^2 - 2a^2)(x^2 + a^2)^{\frac{3}{2}}/90a^3.\qquad \text{[Oxf. I. P., 1916.]}$$

48. Show that the two cylinders $x^2/a^2 + z^2/c^2 = 1$ and $y^2 = 2b(c - z)$ intercept on the plane $z = k$ (where $k^2 < c^2$), a rectangle of area

$$4a(1 - k/c)\sqrt{2b(c+k)}.$$

Show that the volume cut off from the cylinder $x^2/a^2 + z^2/c^2 = 1$ by the cylinder $y^2 = 2b(c - z)$ is

$$\tfrac{128}{15}ac\sqrt{bc}.\qquad \text{[Oxf. I. P., 1917.]}$$

49. The sphere $x^2 + y^2 + z^2 = a^2$ is intersected by the cylinder

$$x^2 + y^2 = az.$$

Prove that the ratio of the spherical area cut off by the cylinder to the cylindrical area cut off by the sphere is

$$\pi - 2 : 2.\qquad \text{[Oxf. I. P., 1915.]}$$

50. Integrate $$\int_0^1 \left[\int_0^{x^2} \frac{dy}{\sqrt{x^2 - y^2}} \right] dx.$$ [Oxf. I. P., 1915.]

51. Find the value of $\iint \frac{dx\,dy}{(a^2 + x^2 + y^2)^p}$ taken all over the plane x, y; p being greater than unity. [Oxf. I. P., 1915.]

52. Find the four points where any line parallel to the axis of z intersects the surface $(x^2 + y^2 + z^2)^2 = 4(a^2z^2 + x^2y^2)$.

Prove that the volume enclosed by that part of the surface which lies above the plane $z = 0$ is $\frac{16}{3}a^3$. [Oxf. II. P., 1915.]

53. If the coordinates of a point on a certain surface be expressed as

$$x = a \sin u, \quad y = a \sin v, \quad z = a \cos u + a \cos v,$$

prove that the area of the portion of the surface bounded by

$$u = 0, \quad u = \tfrac{1}{2}\pi, \quad v = 0, \quad v = \tfrac{1}{2}\pi,$$

is $$\frac{\pi a^2}{4}\left(1 - \frac{c_2{}^3}{1} - \frac{c_4{}^3}{3} - \frac{c_6{}^3}{5} - \frac{c_8{}^3}{7} - \dots\right),$$

where $$c_{2r} = \frac{(2r-1)(2r-3)\dots 1}{2r(2r-2)\dots 2}.$$ [Oxf. II. P., 1915.]

ANSWERS TO EXAMPLES AND PROBLEMS.

VOLUME I.

CHAPTER I

PAGE 12.

1. $\dfrac{b^2-a^2}{2}$, $\dfrac{b^3-a^3}{3}$. 2. $\frac{8}{3}a^2$. 4. $\frac{1}{3}\pi h^3 \tan^2\alpha$.

5. Gradient at $x=15$, $36° 20'$; slope $=\cdot 735$. Slope at $9\cdot 5$ is $\dfrac{y}{x}$,

$$\int_{11}^{15} y\,dx = 17\cdot 4 \text{ square units.}$$

PAGE 15

1. $\dfrac{1}{2}$, $\dfrac{1}{3}$, $\dfrac{1}{6}$, $\dfrac{1}{n+1}$. 2. 1, 1, 1, $\sqrt{2}-1$. 3. $\dfrac{\pi}{4}$, $\dfrac{\pi}{2}$, $\log 2$, $e-1$.

PAGE 25.

1. $\frac{4}{3}\pi ab^2$. 3. $\bar{x}=\dfrac{m+1}{m+2}a$. 4. $\dfrac{m+1}{m+3}$. Mass . a^2.

6. Using paper ruled to 10ths and 5 inches to represent unity on each of the axes, the area $=\cdot 78500$. As this should be $\dfrac{\pi}{4}$, we have the approximation $\pi=3\cdot 1400$, the true value being $3\cdot 141592\ldots$, showing an error of about $\cdot 05$ per cent.

PAGE 28.

1. Harmonic oscillation. 2. $\displaystyle\int_{x_0}^{x} y\,dx$. 4. $\frac{4}{3}\pi a^2 b$ 5. $2\pi a h^2$. 7. c^t/t.

10. Mean by trapezoidal rule with unit increments $=23\cdot 78$.
True result $=23\cdot 026\ldots$. (Unit increments are, however, too large for a very exact result.)

$$\int_1^{10} 10x^{-0\cdot 9}dx = 25\cdot 9\,; \quad \int_1^{10} 10x^{-1\cdot 1}dx = 20\cdot 6\,;$$

$$\int_1^{10} 10x^{-0\cdot 99}dx = 23\,; \quad \int_1^{10} 10x^{-1\cdot 01}dx = 22.$$

13. About 141,550 cubic yards.

14. (1) $\left.\begin{array}{l}\bar{x}=\frac{2}{3}a,\\ \text{Mom. In.}=M\dfrac{a^2}{2};\end{array}\right\}$ (2) $\left.\begin{array}{l}\bar{x}=\frac{3}{4}a,\\ \text{Mom. In.}=\frac{3}{5}Ma^2,\end{array}\right\}$ where $M\equiv$ mass, $a\equiv$ length.

15. $M=\dfrac{2\pi\rho_0 a^2}{n+2}$ if ρ_0 = density at the edge. Mom. In. $=\dfrac{n+2}{n+4}Ma^2$.

17. About 213 tons.

20. 13,863 foot-lbs., 10,574 foot-lbs.

25. Taking ordinates at 10° intervals and four figure tables, the trapezoidal rule gave $\cdot 2501\pi$, the true value being $\dfrac{\pi}{4}$.

29.
$$\text{Area}=\frac{A}{3}(c^3-a^3)+\frac{B}{2}(c^2-a^2)+C(c-a),$$
where $A=-\sum(b-c)y_1/\Pi$, $B=\sum(b^2-c^2)y_1/\Pi$, $C=-\sum bcy_1/\Pi$, $\Pi=(b-c)(c-a)(a-b)$.

30. True values (1) $=25\pi$ and (2) $100+25\pi$.

33. 59 c.c., q.p.

35. $\dfrac{\pi}{4}a^2c+(b-a)ac+\dfrac{\pi}{8}c(b-a)^2$ cubic inches, 3438·3 cubic inches.

36. Binomial Expansion to 3 terms gives ·1204, q.p.
Graphically with $\frac{1}{10}=1$ linear inch, the trapezoidal rule gave ·1178. When this was corrected for curvature of the arcs by the approximate addition of small squares, the approximation was ·1203.

40. 8465·7

41. Perimeter = 30·1026 cm., q.p.

42. The true value is $\dfrac{\pi}{2}$. This will appear later.

43. When t is large I becomes $\dfrac{V}{R}$ and Q becomes $\dfrac{V}{R}t-\dfrac{VL}{R^2}$.

44. $Q=at+\dfrac{bt^2}{2}-c\dfrac{t^3}{3}$, $V=aR+bL+(bR-2cL)t-cRt^2$.

45. In the 'Otto Cycle' of operations there is one explosion for two revolutions. About 16 H.P.

46. Weddle's rule gives $-1\cdot08873$; true value $-1\cdot08878$.

48. $5\frac{1}{16}$ miles.

53. ·821, q.p.

CHAPTER II.

Page 51.

1. $\dfrac{x^{11}}{11}$, $-\dfrac{x^{-9}}{9}$, x, C, $\dfrac{5}{12}x^{\frac{12}{5}}$, $\dfrac{7}{2}x^{\frac{2}{7}}$, $3x^{\frac{1}{3}}$, $2\sqrt{x}$, $\dfrac{2}{13}x^{\frac{13}{2}}$.

2. $\dfrac{2}{3}ax^{\frac{3}{2}}+2bx^{\frac{1}{2}}$, $\dfrac{p}{p+1}ax^{\frac{p+1}{p}}+\dfrac{p}{p-1}bx^{\frac{p-1}{p}}$,
$$\frac{pqacx^{(p+q+pq)/pq}}{p+q+pq}+\frac{padx^{(p+1)/p}}{p+1}+\frac{qbcx^{(q+1)/q}}{q+1}+bdx,\quad ax+b\log x-\frac{c}{x}.$$

3. $ac\frac{x^2}{2}+b(a+c)x+(a^2+b^2+c^2)\log x-\frac{b}{x}(a+c)-\frac{ac}{2x^2}$,
$-\log(a-x)$, $\frac{1}{a-x}$, $\frac{(a-x)^{1-p}}{p-1}$.

4. $\log\frac{a+x}{a-x}$, $x-a\log(a+x)$, $\frac{2x}{a^2-x^2}$, $x+\frac{a^4}{3x^3}$.

5. $\frac{20}{13}\cdot 2^{\frac{3}{10}}=1\cdot 894\ldots$, $\frac{3}{2}\left(5^{\frac{2}{3}}-3^{\frac{2}{3}}\right)$, $\frac{1}{2}\log\frac{7}{5}$.

6. $832421\frac{2}{3}$.

7. $\frac{a}{2}(7+\log 4)$.

8. In 5 seconds at a distance of 25 feet.

9. $400\log_e 2$. The integration is that of finding the work done in allowing a gas to expand according to Boyle's law from $v=10$ to $v=20$. If p and v be in lbs.-wt. per sq. foot and in cubic feet respectively, the result is in foot-lbs.

10. $8\frac{11}{30}$, $-\frac{18}{30}$, $\frac{11}{30}$, $-\frac{18}{30}$, $8\frac{11}{30}$. The portions are alternately above and below the x-axis.

11. $\frac{(ae^x+b)^{n+1}}{(n+1)a}$, $\frac{c}{a}\log(ae^x+b)$, $\frac{1}{n+1}\left(ax+\frac{b}{x}+c\right)^{n+1}$, $\frac{(ax^p+bc^x)^{n+1}}{n+1}$.

12. $\log(e^{ax}+e^{bx})$, $\frac{1}{2}\log\sin 2x$, $\log\cosh x$, $\frac{(ax^{2n}+bx^n+c)^{1-p}}{(1-p)n}$.

13. $\log\tan^{-1}x$, $-\frac{1}{n-1}\frac{1}{(\tan^{-1}x)^{n-1}}$, $\frac{(\sin^{-1}x)^{n+1}}{n+1}$, $\log\sin^{-1}x$, $\log\operatorname{vers}^{-1}x$.

14. $\log\log x$, $\log\log\log x$, $\frac{(\log\log\log x)^{1-n}}{1-n}$, $\frac{(l^{r+1}x)^{1-n}}{1-n}$.

Page 53.

1. $\log(x+1)$, $x-2a\log(x+a)$, $\frac{1}{2}\log(x^2+a^2)$, $\frac{1}{2}\log(x^2+a^2)+\tan^{-1}\frac{x}{a}$,
$$\frac{1}{3}\log(x^3+a^3),\ \frac{1}{n}\log(x^n+a^n).$$

2. $\frac{2^x}{\log 2}$, x^2, $2\log x$, $\frac{x^3}{3}$, $\frac{x^4}{4}+\frac{3^x}{\log 3}$, $ax+\frac{b^x}{\log b}+\frac{c^{2x}}{2\log c}+\frac{d^{3x}}{3\log d}$.

3. $\frac{x+\sin x}{2}$, $\frac{x-\sin x}{2}$, $\log\tan x$, $\log\sin x-\operatorname{cosec} x$.

4. $\sin^{-1}\frac{x}{3}$, $\sinh^{-1}\frac{x}{3}$, $\cosh^{-1}\frac{x}{3}$, $\frac{1}{3}\tan^{-1}\frac{x}{3}$, $\frac{1}{6}\log\frac{3+x}{3-x}\equiv\frac{1}{3}\tanh^{-1}\frac{x}{3}$, $\frac{1}{6}\log\frac{x-3}{x+3}$.

5. $\frac{1}{2}\sec^{-1}\frac{x}{2}$, $\cosh^{-1}\frac{x}{2}+\frac{1}{2}\operatorname{sech}^{-1}\frac{x}{2}$, $-a\sqrt{c^2-x^2}+b\sin^{-1}\frac{x}{c}$,
$$a\sqrt{x^2-c^2}+b\cosh^{-1}\frac{x}{c},\ a\sqrt{x^2+c^2}+b\sinh^{-1}\frac{x}{c}.$$

6. $\sin^{-1}(2x-1)$, $\frac{1}{3\sqrt{3}}\sec^{-1}\frac{x}{3}$, $\frac{1}{\sqrt{3}}\sin^{-1}\frac{x}{3}$, $x-4\tan^{-1}\frac{x}{2}$,
$$x+2\log\frac{x-2}{x+2}\equiv\log\left\{e^x\left(\frac{x-2}{x+2}\right)^2\right\}.$$

7. (i) $-\frac{1}{2}\operatorname{cosec}^2 x$, (ii) $\log\tan x$, (iii) $\frac{(e^x+a)^{n+1}}{n+1}$,

(iv) $\frac{1}{3(n+1)}(x^3+a^3)^{n+1}$, (v) $\frac{1}{n+1}(ax^3+bx+c)^{n+1}$.

8. (i) $\log\tan^{-1}x$, (ii) $-\frac{1}{\sin^{-1}x}$, (iii) $-\frac{1}{2(\log x)^2}$.

9. (i) $\log\frac{7}{3}$, (ii) $\frac{4}{21}$.

11. (i) $\frac{1}{4}(e^{4x}-1)$, (ii) $\frac{2}{n}(e^{nx}-1)$, (iii) $e-e^{-1}$, (iv) $\frac{b^2-a^2}{4}+\frac{1}{2}\log\frac{b}{a}$.

12. (i) 1, (ii) $\frac{\pi}{4}$, (iii) $\frac{1}{2}$, (iv) $\sinh x+\sin x$.

13. (i) $\frac{1}{n}$, (ii) $\sqrt{2}-1$, (iii) $\frac{\pi}{12}$, (iv) $\frac{\pi}{2}$.

14. (i) $\frac{x^n}{n}+\frac{ax^{n-1}}{n-1}+\frac{a^2x^{n-2}}{n-2}+\ldots+\frac{a^{n-1}x}{1}$, (ii) Last result $+a^n\log(x-a)$,

(iii) $\frac{x^5}{5}+\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}-\frac{x}{1}$, (iv) $\frac{x^5}{5}+\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+\log(x-1)$, (v) $\frac{x^2}{2}-3x$.

Page 56.

1. (1) $\log x-\frac{(a+b+c)}{x}-\frac{ab+bc+ca}{2x^2}-\frac{abc}{3x^3}$.

(2) $\frac{x^3}{3}-(a+b)\frac{x^2}{2}+(a^2-ab+b^2)x$. (3) x.

(4) $\log(a\sin x+b\cos x+c)$. (5) $\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}$.

(6) $\frac{1}{6}\left(\tan^{-1}\frac{x}{3}\right)^2$. (7) $\log\tan x$. (8) $-\operatorname{cosec} x+\log\sin x$.

(9) $-\cot\frac{x}{2}$. (10) $-\cos\left(x+\frac{\pi}{4}\right)$. (11) $\tan x-\tan^{-1}x$.

(12) $\tan x+\log\sec x$. (13) $\sec x+\log\sec x$.

(14) $a\sec x-b\operatorname{cosec} x$. (15) $-2(\operatorname{cosec} x+\sec x)$.

(16) $\frac{1}{3}\tan^3 x+\frac{a+b}{2}\tan^2 x+ab\tan x$.

(17) $\tan^{-1}\log x$ (18) $\sin\log x$.

(19) $\frac{x^5}{5}+\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+2\log(x-1)$. (20) $\frac{1}{a}\tan^{-1}(ae^x)$.

18. $\frac{1}{3}$ of a mile. 19. $\frac{4}{3}a^3b$; about 9 feet. 20. $\frac{dx}{dt}=-ax$, $\frac{dy}{dt}=ax-by$.

22. $\frac{dz}{dx}=$ the ordinate PQ; $\frac{d^2z}{dx^2}=$ tangent of angle the tangent at Q makes with OK; $y=a\sec^2\frac{x}{a}$.

23. $y=ae^{\frac{x-h}{h}}$, $y=14{\cdot}778\ldots$.

24. Approx. value given by formula ·122422. True value ·122416.

26. $-\frac{x^4}{4!}e^{-x}$, $e^x\int_0^x\frac{a^{n-1}}{(n-1)!}e^{-a}da$. 27. True value of integral $=\pi$.

28. (1) $p_1v_1\log\frac{v_2}{v_1}$; $\frac{p_2v_2^{\gamma}}{1-\gamma}(v_3^{1-\gamma}-v_2^{1-\gamma})$; $-p_3v_3\log\frac{v_4}{v_3}$; $-\frac{p_4v_4^{\gamma}}{1-\gamma}(v_1^{1-\gamma}-v_4^{1-\gamma})$.

29. 97·25 units.

33. $y=x+\frac{x^2}{2}+\frac{x^3}{6}+*-\frac{x^5}{40}\dots$

35. $-\left[\frac{z^{2n+1}}{2n+1}+n(1-c)\frac{z^{2n}}{2n}+\frac{n(n-1)}{1.2}(1-c)^2\frac{z^{2n-1}}{2n-1}+\dots+(1-c)^n\frac{z^{n+1}}{n+1}\right]$, where $z=\frac{1-x}{x}$.

37. $\frac{\sin 3\theta}{3}-\frac{\sin 2\theta}{2}$.

38. $f(x)\equiv 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\equiv\cos x$. $F(x)\equiv\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\equiv\sin x$.

42. $\left[-16\frac{\cos^5\theta}{5}-8\frac{\cos^4\theta}{4}-12\frac{\cos^3\theta}{3}+4\frac{\cos^2\theta}{2}+\cos\theta\right]$.

CHAPTER III.

Page 75

1. (i) $\log(1+x^3)$, (ii) $\tan^{-1}\frac{1}{7}$, (iii) $\frac{\pi}{3}$, (iv) $\tan^{-1}\left(\frac{e-1}{e+1}\right)$,
(v) $\tan^{-1}\left(\frac{e^x}{e^x+1}\right)$, (vi) $\frac{1}{3}\tan^3x$, (vii) $\frac{1}{m}\tanh mx$.

2. (i) $\frac{\pi a^2}{4}$, (ii) $\frac{\pi a^2}{2}$.

3. (i) $\frac{a^3}{3}$, (ii) $\frac{\pi a^4}{16}$.

4. $\sin^{-1}\frac{1}{c}\left(ax+\frac{b}{x}\right)$.

5. $\frac{1}{2n+1}\frac{1}{a^2}\left(\frac{x}{\sqrt{x^2+a^2}}\right)^{2n+1}$.

6. $5\tan^{\frac{1}{5}}\theta$.

7. $\frac{2}{3}\tan^{\frac{3}{2}}x$.

8. (i) $\frac{1}{2}\sec^{-1}x^2$, (ii) $-\frac{1}{2}\operatorname{sech}^{-1}x^2$, (iii) $-\frac{1}{2}\operatorname{cosech}^{-1}x^2$.

9. (i) $e^{x+\frac{1}{x}}$, (ii) $\tan^{-1}\left(ax+\frac{b}{x}\right)$, (iii) $\frac{\left(ax+\frac{b}{x}\right)^{n+1}}{n+1}$.
(iv) $\frac{1}{bc-ae}\sin\frac{a+bx}{c+ex}$, (v) $\frac{1}{a}e^{a\tan^{-1}x}$, (vi) $\frac{1}{a}e^{a\sin^{-1}x}$,
(vii) $\frac{\log(a^2\cos^2x+b^2\sin^2x)}{2(b^2-a^2)}$.

10. (i) $\phi(x)\psi(x)$, (ii) $\frac{\psi(x)}{\phi(x)}$, (iii) $\tan^{-1}\phi(x)$,
(iv) $e^{\phi(x)}$, (v) $e^{-\psi(x)}\log\phi(x)$.

Page 98.

1. $\frac{1}{6}\log\frac{3+x}{3-x}$, $\frac{1}{12}\log\frac{3+2x}{3-2x}$, $\frac{1}{4}\log\frac{x-2}{x+2}$, $\frac{1}{12}\log\frac{3x-2}{3x+2}$,
$\frac{x}{2}\sqrt{16-9x^2}+\frac{8}{3}\sin^{-1}\frac{3x}{4}$, $\frac{x}{2}\sqrt{3x^2-5}-\frac{5}{6}\sqrt{3}\cosh^{-1}\frac{x\sqrt{3}}{\sqrt{5}}$,
$\frac{x}{2}\sqrt{3x^2+2}+\frac{\sqrt{3}}{3}\sinh^{-1}\left(\frac{x\sqrt{3}}{2}\right)$.

2. $2\cosh^{-1}\sqrt{\frac{x}{4}}$, $2\sin^{-1}\sqrt{\frac{x}{4}}$, $2\sinh^{-1}\sqrt{\frac{x}{4}}$, $\sin^{-1}\frac{x-1}{\sqrt{3}}$,

$\sinh^{-1}(x-1)$, $\frac{x+a}{2}\sqrt{x^2+2ax}-\frac{a^2}{2}\cosh^{-1}\frac{x+a}{a}$.

3. $-\sqrt{9-x^2}$, $\sqrt{x^2-9}$, $-\frac{1}{4}\sqrt{9-4x^2}$, $\frac{1}{2}(\sin^{-1}x-x\sqrt{1-x^2})$,
$\frac{x\sqrt{1+x^2}}{2}-\frac{1}{2}\sinh^{-1}x$.

4. $\frac{1}{3}(x^2+a^2)^{\frac{3}{2}}$, $\frac{1}{3}(x^2+a^2)^{\frac{3}{2}}+\frac{b}{2}\left[x(x^2+a^2)^{\frac{1}{2}}+a^2\sinh^{-1}\frac{x}{a}\right]$,

$a\sqrt{x^2+c^2}+b\sinh^{-1}\frac{x}{c}$.

5. $\frac{1}{n+2}(x^2+a^2)^{\frac{n+2}{2}}$, $\frac{1}{n+2}(x^2+2ax+b)^{\frac{n+2}{2}}$, $\frac{1}{n+2}(ax^2-2bx+c)^{\frac{n+2}{2}}$.

6. $\frac{7}{2}\sin^{-1}x-\frac{x+4}{2}\sqrt{1-x^2}$, $\frac{5}{2}\sinh^{-1}x+\frac{x+4}{2}\sqrt{x^2+1}$,

$\frac{15}{8}\sinh^{-1}\frac{2x+1}{\sqrt{3}}+\frac{2x+5}{4}\sqrt{x^2+x+1}$;

$\frac{2x+4a-3c}{4}\sqrt{x^2+cx+d}+\frac{1}{8}(8b-4d-4ac+3c^2)\sinh^{-1}\frac{2x+c}{\sqrt{4d-c^2}}$,

if $c^2<4d$, with a similar result if $c^2>4d$.

7. $\frac{x+2}{2}\sqrt{x^2+4x+5}+\frac{1}{2}\sinh^{-1}(x+2)$, $\frac{x-2}{2}\sqrt{-x^2+4x+5}+\frac{9}{2}\sin^{-1}\frac{x-2}{3}$,

$\frac{2x+1}{4}\sqrt{4x^2+4x+5}+\sinh^{-1}\frac{2x+1}{2}$,

$\frac{2x-1}{4}\sqrt{-4x^2+4x+5}+\frac{3}{2}\sin^{-1}\frac{2x-1}{\sqrt{6}}$.

8. $\sqrt{x^2-a^2}+a\cosh^{-1}\frac{x}{a}$, $a\sin^{-1}\frac{x}{a}-\sqrt{a^2-x^2}$, $\frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x+2a}{2}\sqrt{a^2-x^2}$,

$(x+a+b)\sqrt{x^2-b^2}+\frac{b}{2}(2a+b)\cosh^{-1}\frac{x}{b}$, $\frac{x+4a}{2}\sqrt{x^2-a^2}+\frac{3a^2}{2}\cosh^{-1}\frac{x}{a}$.

9. $\frac{1}{n}\log\tan\frac{nx}{2}$, $\frac{1}{2}\log\tan\left(x+\frac{b}{2}\right)$, $\frac{1}{3}\log\tan\left(\frac{3x}{2}+\frac{\pi}{4}\right)$,

$\frac{1}{2}\log\tan\left(x+\frac{\pi}{4}\right)$, $\frac{1}{2}\log\tan x$.

10. $\frac{1}{\sqrt{a^2+b^2}}\log\tan\frac{1}{2}\left(x+\tan^{-1}\frac{b}{a}\right)$, $\frac{1}{2\sqrt{2}}\log\tan\left(x+\frac{\pi}{8}\right)$,

$\frac{ac+bd}{c^2+d^2}x+\frac{bc-ad}{c^2+d^2}\log(c\sin x+d\cos x)$.

13. $\log\{\operatorname{cosec}\theta(1-\sqrt{1-\sin^{2n}\theta})^{\frac{1}{n}}\}$.

Page 99.

2. $b^2\sin^{-1}\frac{x_1}{a}$, πb^2.

3. $\sqrt{e^{2x}+ae^x}+a\log(\sqrt{e^x+a}+\sqrt{e^x})$.

4. (i) $\sin^{-1}\dfrac{2x+3}{\sqrt{13}}$; (ii) $\dfrac{1}{\sqrt{6}}\cos^{-1}\dfrac{12-x}{5x}$;

(iii) $\dfrac{1}{2}\sqrt{2x^2+3x+4}+\dfrac{1}{4\sqrt{2}}\sinh^{-1}\dfrac{4x+3}{\sqrt{23}}$;

(iv) $\sqrt{x^2+2x-1}-2\cosh^{-1}\dfrac{x+1}{\sqrt{2}}$; (v) $3\sqrt{x^2+2x+5}+\sinh^{-1}\dfrac{x+1}{2}$

7. (i) $\dfrac{1}{3}\log\dfrac{\sqrt{1+x^6}-1}{x^3}$; (ii) $\sqrt{\dfrac{x-1}{x+1}}$.

9. $\sqrt{e^{2x}+e^x+1}+\frac{1}{2}\sinh^{-1}\dfrac{2e^x+1}{\sqrt{3}}-\sinh^{-1}\dfrac{2e^{-x}+1}{\sqrt{3}}$.

10. Mass $=\dfrac{4\pi k}{n+3}a^{n+3}$, where density $=kr^n$ and a is the radius.

(i) Mass $=4\pi ak$; (ii) $2\pi^2k$.

11. $\dfrac{ap^3}{12}$, a being BC and p the perpendicular from A upon BC.

13. $\log x=\pm\dfrac{2}{3b^2}\sqrt{a^2+b^2y^3}+\text{const.}$

15. (i) $-\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{\sqrt{-aR}}{\sqrt{b^2-ac}}\,(a-{}^{\text{ve}})$, $-\dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{\sqrt{aR}}{\sqrt{b^2-ac}}\,(b^2>ac,\ a+{}^{\text{ve}})$,

$-\dfrac{1}{\sqrt{a}}\cosh^{-1}\dfrac{\sqrt{aR}}{\sqrt{ac-b^2}}\,(b^2<ac,\ a+{}^{\text{ve}})$,

where $R\equiv a\cos^2\theta+2b\cos\theta+c$;

(ii) $\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{\sqrt{-aR}}{\sqrt{b^2-ac}}\,(a-{}^{\text{ve}})$, $\dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{\sqrt{aR}}{\sqrt{b^2-ac}}\,(b^2>ac,\ a+{}^{\text{ve}})$,

$\dfrac{1}{\sqrt{a}}\cosh^{-1}\dfrac{\sqrt{aR}}{\sqrt{ac-b^2}}\,(b^2<ac,\ a+{}^{\text{ve}})$,

where $R\equiv a\sin^2\theta+2b\sin\theta+c$;

(iii) $\dfrac{1}{\sqrt{-c}}\sin^{-1}\dfrac{\sqrt{-cR}}{\sqrt{b^2-ac}}\,(c-{}^{\text{ve}})$, $\dfrac{1}{\sqrt{c}}\sinh^{-1}\dfrac{\sqrt{cR}}{\sqrt{b^2-ac}}\,(b^2>ac,\ a+{}^{\text{ve}})$,

$\dfrac{1}{\sqrt{c}}\cosh^{-1}\dfrac{\sqrt{cR}}{\sqrt{ac-b^2}}\,(b^2<ac,\ c+{}^{\text{ve}})$,

where $R\equiv c\tan^2\theta+2b\tan\theta+a$;

(iv) $-\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{\sqrt{-aR}}{\sqrt{b^2-ac}}\,(a-{}^{\text{ve}})$, $-\dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{\sqrt{aR}}{\sqrt{b^2-ac}}\,(b^2>ac,\ a+{}^{\text{ve}})$,

$-\dfrac{1}{\sqrt{a}}\cosh^{-1}\dfrac{\sqrt{aR}}{\sqrt{ac-b^2}}\,(b^2<ac,\ a+{}^{\text{ve}})$,

where $R\equiv a\cot^2\theta+2b\cot\theta+c$;

(v) $-\dfrac{1}{\sqrt{a+c}}\sinh^{-1}\left(\sqrt{\dfrac{a+c}{b+c}}\cot\theta\right)$,

if $\dfrac{b+c}{a+c}$ be $+{}^{\text{ve}}$, and a modification (Art. 77) if $\dfrac{b+c}{a+c}$ be $-{}^{\text{ve}}$.

16. (i) $\frac{a^4}{8}\left[3\sin^{-1}\frac{x}{a}-\frac{x}{a^4}(2x^2+3a^2)\sqrt{a^2-x^2}\right]$;

(ii) $\frac{\theta}{b}+\frac{1}{b}\sqrt{\frac{a}{a+bc^2}}\tan^{-1}\left(\sqrt{\frac{a}{a+bc^2}}\cot\theta\right)$, where $\theta=\sin^{-1}\frac{x}{c}$,

provided $\frac{a}{a+bc^2}$ be positive, with a modification (Art. 89, 17 and 18) if negative.

17. (i) 48; (ii) $\frac{2b}{3a^2}(3ac-2b^2)$; (iii) $\frac{3bc}{a}$.

22. $\frac{2}{\sqrt{a-c}}\tan^{-1}\sqrt{\frac{c+x}{a-c}}\ (a>c)$; $\frac{1}{\sqrt{c-a}}\log\frac{\sqrt{c+x}-\sqrt{c-a}}{\sqrt{c+x}+\sqrt{c-a}}\ (a<c)$;

$-\frac{d}{da}\left\{\frac{2}{\sqrt{a-c}}\tan^{-1}\sqrt{\frac{c+x}{a-c}}\right\}$; $-2\frac{d}{dc}\left\{\frac{2}{\sqrt{a-c}}\tan^{-1}\sqrt{\frac{c+x}{a-c}}\right\}$.

23. (i) $a>c, \frac{1}{2a\sqrt{a^2-c^2}}\log\frac{a\sin\phi-\sqrt{a^2-c^2}}{a\sin\phi+\sqrt{a^2-c^2}}$, where $\phi=\cos^{-1}\frac{c}{x}$;

(ii) $a<c, \frac{1}{a\sqrt{c^2-a^2}}\tan^{-1}\frac{a\sin\phi}{\sqrt{c^2-a^2}}$.

26. (i) $\frac{1}{\sqrt2}\sinh^{-1}\frac{x\sqrt2}{1-x^2}$; (ii) $\frac{1}{\sqrt2}\sin^{-1}\frac{x\sqrt2}{1+x^2}$. **30.** $\frac{\alpha}{\sin\alpha\cos\alpha}$.

CHAPTER IV.

PAGE 113.

1. $\frac{e^{3x}}{9}(3x-1)$, $\frac{e^{ax}}{a^3}(a^2x^2-2ax+2)$,

$-e^{-x}(x^5+5x^4+5.4x^3+5.4.3x^2+5.4.3.2x+5.4.3.2.1)$,

$x\sinh x-\cosh x$, $(x^2+2)\cosh x-2x\sinh x$.

2. $x\sin x+\cos x$, $\left(\frac{x^5}{2}-\frac{5.4x^3}{2^3}+\frac{5.4.3.2x}{2^5}\right)\sin 2x$

$$+\left(\frac{5x^4}{2^2}-\frac{5.4.3x^2}{2^4}+\frac{5.4.3.2.1}{2^6}\right)\cos 2x.$$

$\frac{x^3}{6}+\frac{1}{8}(2x^2-1)\sin 2x+\frac{x}{4}\cos 2x$,

$\frac{1}{2}\left[x^2\left(\frac{\cos 2x}{2}-\frac{\cos 4x}{4}\right)-x\left(\frac{\sin 2x}{2}-\frac{\sin 4x}{8}\right)-\left(\frac{\cos 2x}{4}-\frac{\cos 4x}{32}\right)\right]$

$-\frac{x}{8}\left(\frac{\cos 2x}{1}+\frac{\cos 4x}{2}-\frac{\cos 6x}{3}\right)+\frac{1}{16}\left(\frac{\sin 2x}{1^2}+\frac{\sin 4x}{2^2}-\frac{\sin 6x}{3^2}\right)$.

3. $\frac{1}{\sqrt5}e^x\sin(2x-\tan^{-1}2)$, $\frac{e^x}{2}-\frac{e^x}{2\sqrt5}\cos(2x-\tan^{-1}2)$,

$\frac{1}{4}\frac{e^{3x}}{\sqrt{13}}\sin(2x-\tan^{-1}\frac{2}{3})-\frac{1}{40}e^{3x}\sin(4x-\tan^{-1}\frac{4}{3})$,

$\frac{e^{-5x}}{8}\left[+\frac{1}{5}+\frac{1}{\sqrt{29}}\cos(2x+\tan^{-1}\frac{2}{5})+\frac{1}{\sqrt{41}}\cos(4x+\tan^{-1}\frac{4}{5})\right.$

$$\left.-\frac{1}{\sqrt{61}}\cos(6x+\tan^{-1}\frac{6}{5})\right].$$

4. $\frac{x^4}{4}\log x - \frac{x^4}{16}$; $\frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}$;

$\frac{x^{n+1}}{n+1}\left[(\log x)^2 - \frac{2}{n+1}\log x + \frac{2}{(n+1)^2}\right]$;

$\frac{x^{n+1}}{n+1}\left[(\log x)^3 - \frac{3}{n+1}(\log x)^2 + \frac{6}{n+1}(\log x) - \frac{6}{(n+1)^3}\right]$.

5. $$\frac{e^{ax}}{4}\left\{\frac{\sin\left\{(q+r-p)x - \tan^{-1}\frac{q+r-p}{a}\right\}}{[(q+r-p)^2+a^2]^{\frac{1}{2}}} + \text{two similar terms} - \frac{\sin\left\{(p+q+r)\,x - \tan^{-1}\frac{p+q+r}{a}\right\}}{\sqrt{(p+q+r)^2+a^2}}\right\};$$

$$\frac{e^{ax}}{4}\left\{\frac{\cos\left\{(q+r-p)\,x - \tan^{-1}\frac{q+r-p}{a}\right\}}{\sqrt{(q+r-p)^2+a^2}} + \text{etc.} - \text{etc.} - \text{etc.}\right\}.$$

6. $8\sin px\sin qx\cos^2 rx = 2\cos(p-q)\,x + \cos(p-q+2r)\,x$
$+\cos(p-q-2r)x - 2\cos(p+q)\,x - \cos(p+q+2r)\,x - \cos(p+q-2r)\,x$.

Then apply rule for $\int e^{ax}\cos Nx\,dx$ to each term.

$8\cos px\cos qx\cos^2(p+q)\,x = 2\cos(p+q)\,x + 2\cos(p-q)\,x + \cos(p+q)\,x$
$+\cos 3(p+q)\,x + \cos(3p+q)\,x + \cos(3q+p)\,x = \sum A\cos Nx$, say.

Then $$\text{Integral} = \sum A\,\frac{e^{ax}\cos\left(Nx - \tan^{-1}\frac{N}{a}\right)}{\sqrt{a^2+N^2}}.$$

7. π; $\frac{1}{4}(\pi^2-8)$; $-\frac{\pi}{4}$.

8. $x\sin^{-1}x + \sqrt{1-x^2}$; $\frac{2x^2-1}{4}\sin^{-1}x - \frac{1}{4}x\sqrt{1-x^2}$;

$\frac{8x^4-3}{32}\sin^{-1}x + \frac{x(2x^2+3)}{32}\sqrt{1-x^2}$; $\frac{x^2}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1}x$.

PAGE 114.

1. $e^x(x^6 - 6x^5 + 6.5x^4 - 6.5.4x^3 + 6.5.4.3x^2$
$-6.5.4.3.2x + 6.5.4.3.2.1)$,

$(x^5 + 5.4x^3 + 5.4.3.2x)\cosh x - (5x^4 + 5.4.3x^2 + 5.4.3.2.1)\sinh x$,

$$\frac{x^6}{12} + \frac{\sinh 2x}{2}\left(\frac{x^5}{2} + \frac{5.4x^3}{2^3} + \frac{5.4.3.2x}{2^5}\right) - \frac{\cosh 2x}{2}\left(\frac{5x^4}{2^2} + \frac{5.4.3x^2}{2^4} + \frac{5.4.3.2.1}{2^6}\right).$$

2. $\frac{3}{4}(\pi^2-8)$; $\frac{\pi^4}{128} + \frac{3\pi^2}{32} - \frac{3}{8}$; $\frac{\pi^4}{2^6} - \frac{3\pi^2}{2^4} + \frac{3}{4}$.

3 $\pi^5 - 20\pi^3 + 120\pi$; $\frac{\pi^2}{24}(2\pi^4 + 15\pi^2 - 45)$; $-e - 8e^{-1} + 6$.

4. $\dfrac{\pi^4}{128}(a^2+b^2)-3\dfrac{a^2-b^2}{32}(\pi^2-4)$; $\frac{81}{4}\log 3-5$; $\dfrac{\pi-2}{4}$.

5. $\frac{1}{10}(e^{\frac{\pi}{2}}+2)$; $\dfrac{\pi}{96}$.

PAGE 130.

1. $(2x-\sin 2x)/4$; $(\cos 3x-9\cos x)/12$ or $-\cos x+\dfrac{\cos^3 x}{3}$

$(12x-8\sin 2x+\sin 4x)/32$;

$\dfrac{1}{2^4}\left(-\dfrac{\cos 5x}{5}+\dfrac{5}{3}\cos 3x-10\cos x\right)$ or $-\cos x+\dfrac{2}{3}\cos^3 x-\dfrac{1}{5}\cos^5 x$;

$\dfrac{1}{2^7}\left(\dfrac{\sin 8x}{8}-\dfrac{4}{3}\sin 6x+7\sin 4x-28\sin 2x+35x\right)$;

$\dfrac{1}{2^8}\left(-\dfrac{\cos 9x}{9}+9\dfrac{\cos 7x}{7}-36\dfrac{\cos 5x}{5}+84\dfrac{\cos 3x}{3}-126\cos x\right)$

or $-\cos x+4\dfrac{\cos^3 x}{3}-6\dfrac{\cos^5 x}{5}+4\dfrac{\cos^7 x}{7}-\dfrac{\cos^9 x}{9}$;

$\dfrac{(-1)^n}{2^{2n-1}}\left[\dfrac{\sin 2nx}{2n}-{}^{2n}C_1\dfrac{\sin(2n-2)x}{2n-2}+\ldots+\dfrac{(-1)^n}{2}{}^{2n}C_n x\right]$;

$\dfrac{(-1)^{n+1}}{2^{2n}}\left[\dfrac{\cos(2n+1)x}{2n+1}-{}^{2n+1}C_1\dfrac{\cos(2n-1)x}{2n-1}+\ldots+(-1)^n\,{}^{2n+1}C_n\cos x\right]$

or $-\cos x+{}^nC_1\dfrac{\cos^3 x}{3}-{}^nC_2\dfrac{\cos^5 x}{5}+\ldots$.

2. $\dfrac{1}{8}\left(x-\dfrac{\sin 4x}{4}\right)$; $\dfrac{\sin^4 x}{4}-\dfrac{\sin^6 x}{6}$; $\frac{1}{128}(3x-\sin 4x+\frac{1}{8}\sin 8x)$;

$-\dfrac{\cos^7 x}{7}+\dfrac{\cos^9 x}{9}$; $\dfrac{\sin^7 x}{7}-\dfrac{\sin^9 x}{9}$;

$-\dfrac{1}{2^9}\left[\dfrac{\sin 10x}{10}-\dfrac{\sin 8x}{4}-\dfrac{\sin 6x}{2}+2\sin 4x+\sin 2x-6x\right]$.

3. $\frac{1}{3}\tan^3 x$; $-\frac{1}{3}\cot^3 x$; $\tan x-\cot x$; $-\dfrac{1}{3\tan^3 x}-\dfrac{3}{\tan x}+3x+\dfrac{\tan^3 x}{3}$.

4. $(\pi-2)/8$; $43\sqrt{2}/120$; $(15\pi+44)/192$.

5. $-\dfrac{1}{4}\left[\dfrac{2\cos ax}{a}+\dfrac{\cos(a+2b)x}{a+2b}+\dfrac{\cos(a-2b)x}{a-2b}\right]$;

$\frac{3}{2}\sin^2 x-\frac{7}{4}\sin^4 x+\frac{2}{3}\sin^6 x$;

$-\dfrac{1}{4}\left[\dfrac{2\cos nx}{n}+\dfrac{\cos(n+2)x}{n+2}+\dfrac{\cos(n-2)x}{n-2}\right]$.

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2. (i) $x\cos^{-1}x-\sqrt{1-x^2}$; (ii) $x\sec^{-1}x-\log(x+\sqrt{x^2-1})$;

(iii) $\dfrac{x^4-1}{4}\tan^{-1}x+\dfrac{x}{4}-\dfrac{x^3}{12}$; (iv) $x\tan x+\log\cos x$;

(v) $x\sec x-\log\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$;

(vi) $\dfrac{c^2(ax+b)^2-(bc-ad)^2}{2ac^2}\log(cx+d)-\dfrac{a}{4c^2}(cx+d)^2-\dfrac{1}{c}(bc-ad)x$;

(vii) $x\tan^{-1}\sqrt{1-x^2}-\sin^{-1}x+\sqrt{2}\tan^{-1}\dfrac{x}{\sqrt{2}\sqrt{1-x^2}}$;

(viii) $\left(\dfrac{x^3}{3}-x\right)\tan^{-1}x+\frac{1}{2}(\tan^{-1}x)^2-\frac{1}{6}x^2+\frac{2}{3}\log(1+x^2)$;

(ix) $(a+x)\tan^{-1}\sqrt{\dfrac{x}{a}}-\sqrt{ax}$; (x) $\dfrac{1}{4}(x^2-2a^2)\cos^{-1}\dfrac{x}{2a}-\dfrac{1}{8}x\sqrt{4a^2-x^2}$;

(xi) $(2a+x)\tan^{-1}\sqrt{\dfrac{x}{2a}}-\sqrt{2ax}$; (xii) $\dfrac{x^{n+1}}{n+1}\left[\log x-\dfrac{1}{n+1}\right]$.

3. (i) $\dfrac{e^{a\sin^{-1}x}}{\sqrt{a^2+1}}\cos(\sin^{-1}x-\cot^{-1}a)$; (ii) $x-\sqrt{1-x^2}\sin^{-1}x$;

(iii) $\theta(\sec\theta+\cos\theta)-\sin\theta-\log\tan\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)$, where $x=\sin\theta$.

4. (i) $\dfrac{1}{m}e^{m\theta}$; (ii) $\dfrac{e^{m\theta}}{\sqrt{1+m^2}}\cos(\theta-\tan^{-1}m)$;

(iii) $\dfrac{e^{m\theta}}{2}\left\{\dfrac{1}{m}+\dfrac{1}{\sqrt{m^2+4}}\cos\left(2\theta-\tan^{-1}\dfrac{2}{m}\right)\right\}$;

(iv) $\dfrac{e^{m\theta}}{4}\left\{\dfrac{3}{\sqrt{m^2+1}}\cos\left(\theta-\tan^{-1}\dfrac{1}{m}\right)+\dfrac{1}{\sqrt{m^2+9}}\cos\left(3\theta-\tan^{-1}\dfrac{3}{m}\right)\right\}$;

(v) $$\frac{e^{m\theta}}{2^{n-2}}\left[\frac{\cos\left\{(n-1)\theta-\tan^{-1}\dfrac{n-1}{m}\right\}}{\sqrt{m^2+(n-1)^2}}\right.$$
$$\left.+{}^{n-1}C_1\frac{\cos\left\{(n-3)\theta-\tan^{-1}\dfrac{n-3}{m}\right\}}{\sqrt{m^2+(n-3)^2}}+\dots\right],$$
where $\tan\theta=x$.

5. (i) $x\dfrac{e^{bx}}{\sqrt{a^2+b^2}}\cos\left(ax-\tan^{-1}\dfrac{a}{b}\right)-\dfrac{e^{bx}}{a^2+b^2}\cos\left(ax-2\tan^{-1}\dfrac{a}{b}\right)$;

(ii) $$x^2\frac{e^{ax}}{(a^2+b^2)^{\frac{1}{2}}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)-2x\frac{e^{ax}}{a^2+b^2}\sin\left(bx-2\tan^{-1}\frac{b}{a}\right)$$
$$+2\frac{e^{ax}}{(a^2+b^2)^{\frac{3}{2}}}\sin\left(bx-3\tan^{-1}\frac{b}{a}\right);$$

(iii) $\frac{1}{2}e^x\left[x-1-\dfrac{x}{\sqrt{5}}\cos(2x-\tan^{-1}2)+\frac{1}{5}\cos(2x-2\tan^{-1}2)\right]$.

6. (i) $e^{ax}\dfrac{(a-b)\cos bx+(a+b)\sin bx}{a^2+b^2}$; (ii) $\dfrac{e^{(a+b)x}}{a+b}$;

(iii) $\dfrac{1}{4}\left[\dfrac{e^{(2a+b)x}}{2a+b}+\dfrac{e^{(2a-b)x}}{2a-b}+\dfrac{e^{bx}}{b}+\dfrac{e^{-bx}}{b}\right]$;

(iv) $-\dfrac{\cos bx}{2b}+\dfrac{1}{2}\dfrac{e^{2ax}}{\sqrt{4a^2+b^2}}\sin\left(bx-\tan^{-1}\dfrac{b}{2a}\right)$;

(v) $3^x(P\sin 4x-Q\cos 4x)$, where
$$P=\frac{x^2\cos\phi}{r}-\frac{2x\cos 2\phi}{r^2}+\frac{2\cos 3\phi}{r^3},$$
$$Q=\frac{x^2\sin\phi}{r}-\frac{2x\sin 2\phi}{r^2}+\frac{2\sin 3\phi}{r^3},$$
and $\phi=\tan^{-1}(4/\log 3)$, $r^2=4^2+(\log 3)^2$;

(vi) $\dfrac{x}{\sqrt{b^2+1}}\cos\left(b\log\dfrac{x}{a}-\tan^{-1}b\right)$;

(vii) $\dfrac{1}{2}\left[\dfrac{x}{1+b}\left(\dfrac{x}{a}\right)^b+\dfrac{a}{1-b}\left(\dfrac{a}{x}\right)^{b-1}\right]$; (viii) $\pi\sinh 1$.

7. (i) $\dfrac{e^x}{x+1}$; (ii) $e^x\tan\dfrac{x}{2}$; (iii) $-e^x\cot\dfrac{x}{2}$;

(iv) $\cosh x\tan\dfrac{x}{2}$; (v) $-\log(1+e^{-x})$;

(vi) $\dfrac{2}{n}\sqrt{1+e^{nx}}+\dfrac{1}{n}\log\dfrac{\sqrt{1+e^{nx}}-1}{\sqrt{1+e^{nx}}+1}$; (vii) $\dfrac{x-1}{x+1}e^x$.

8. (i) $x(\log x)^2-2x\log x+2x$;

(ii) $\frac{1}{2}(\log x)^2+\left(\dfrac{x^2}{2}-\dfrac{1}{x}\right)\log x-\left(\dfrac{x^2}{4}+\dfrac{1}{x}\right)$;

(iii) $-\dfrac{1}{x}\tan^{-1}x+\log x-\log\sqrt{1+x^2}$;

(iv) $x\log(x+\sqrt{a^2+x^2})-\sqrt{a^2+x^2}$;

(v) $\dfrac{2x^2+a^2}{4}\log(x+\sqrt{x^2+a^2})-\dfrac{x}{4}\sqrt{x^2+a^2}$;

(vi) $\dfrac{2x^2+3ax+2a^2}{6}\sqrt{a^2+x^2}+\dfrac{a^3}{2}\sinh^{-1}\dfrac{x}{a}$;

(vii) $\frac{2}{105}(x+a)^{\frac{3}{2}}(15x^2-12ax+43a^2)$;

(viii) $$e^{ax}\left[\frac{x^2}{(b^2+c^2)^{\frac{1}{2}}}\sin\left(bx+c-\tan^{-1}\frac{b}{a}\right)-\frac{2x}{b^2+c^2}\sin\left(bx+c-2\tan^{-1}\frac{b}{a}\right)\right.$$
$$\left.+\frac{2}{(b^2+c^2)^{\frac{3}{2}}}\sin\left(bx+c-3\tan^{-1}\frac{b}{a}\right)\right];$$

(ix) $$-9\left[\tfrac{1}{10}\cos^{\frac{10}{3}}\theta-\tfrac{5}{16}\cos^{\frac{16}{3}}\theta+\tfrac{10}{22}\cos^{\frac{22}{3}}\theta-\tfrac{10}{28}\cos^{\frac{28}{3}}\theta\right.$$
$$\left.+\tfrac{5}{34}\cos^{\frac{34}{3}}\theta-\tfrac{1}{40}\cos^{\frac{40}{3}}\theta\right],$$
where $\sin\theta=x^{\frac{1}{3}}$.

9. (i) $$\frac{e^{ax}}{2}\left[\frac{x\cos\left\{(b-c)x-\tan^{-1}\frac{b-c}{a}\right\}}{\sqrt{(b-c)^2+a^2}}-\frac{\cos\left\{(b-c)x-2\tan^{-1}\frac{b-c}{a}\right\}}{(b-c)^2+a^2}\right.$$
$$\left.-x\frac{\cos\left\{(b+c)x-\tan^{-1}\frac{b+c}{a}\right\}}{\sqrt{(b+c)^2+a^2}}+\frac{\cos\left\{(b+c)x-2\tan^{-1}\frac{b+c}{a}\right\}}{(b+c)^2+a^2}\right];$$

(ii) $$\frac{e^{ax}}{4}\left[\frac{2x}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)-\frac{2}{a^2+b^2}\sin\left(bx-2\tan^{-1}\frac{b}{a}\right)\right.$$
$$-\frac{x}{\sqrt{a^2+(b+2c)^2}}\sin\left\{(b+2c)x-\tan^{-1}\frac{b+2c}{a}\right\}$$
$$+\frac{1}{a^2+(b+2c)^2}\sin\left\{(b+2c)x-2\tan^{-1}\frac{b+2c}{a}\right\}$$
$$-\frac{x}{\sqrt{a^2+(b-2c)^2}}\sin\left\{(b-2c)x-\tan^{-1}\frac{b-2c}{a}\right\}$$
$$\left.+\frac{1}{a^2+(b-2c)^2}\sin\left\{(b-2c)x-2\tan^{-1}\frac{b-2c}{a}\right\}\right].$$

12. $\frac{x^3}{3}\log(1-x^2)+\frac{1}{3}\log\frac{1+x}{1-x}-\frac{2}{3}\left(x+\frac{x^3}{3}\right).$

13. $-\frac{5}{2}\cot^{\frac{2}{5}}\theta$; $-\frac{5}{2}\cos^{\frac{2}{5}}\theta.$ **14.** $uv^{(n-1)}-u'v^{(n-2)}+\ldots+(-1)^{n-1}u^{(n-1)}v.$

15. $\begin{vmatrix} u'', & v'', & w'' \\ u', & v', & w' \\ 1, & 1, & 1 \end{vmatrix}.$ **20.** $\cdot 78343.$ **22.** $\frac{(x^2+1)^2}{8}\tan^{-1}x-\frac{5x^3+3x}{24}.$

27. $\int_0^1 \frac{1-\sqrt{1-x}}{x\sqrt{1-x}}\,dx=2\int_0^{\frac{\pi}{2}}\tan\frac{\theta}{2}\,d\theta=2\log 2.$ **29.** $\frac{a_1 a_2}{2}T\cos\frac{2\pi\lambda}{T}.$

33. $2\sin\frac{4n-1}{2}\theta\cos^{\frac{3}{2}}\theta.$ **34.** $n^2\pi a^2.$

35. $2^{n+1}a^n\frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}.$ **39.** $A=\frac{518}{225}a,\quad V=\frac{\pi^2 a}{4}A.$

CHAPTER V.

Page 143.

1. $\frac{1}{2}\log(x^2+2x+3)-\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}.$ **2.** $\log(x+1)+\frac{1}{x+1}.$

3. $\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2).$ **4.** $-\log(3-x).$

5. $x-2\log(x^2+2x+2)+3\tan^{-1}(x+1).$

6. $2x-\frac{9}{2}\log(x^2+6x+10)+11\tan^{-1}(x+3).$

7. $\frac{1}{ad-bc}\tan^{-1}\frac{(a^2+c^2)x+(ab+cd)}{ad-bc}.$

8. $\frac{1}{2(bc-ad)}\log\frac{(a+c)x+(b+d)}{(a-c)x+(b-d)}.$

9. $\frac{1}{2(ad-bc)}\tan^{-1}\frac{(a^2+c^2)x^2+(ab+cd)}{ad-bc}.$

10. $$\frac{1}{2\sqrt{(ad-bc)^2+(cf-de)^2+(eb-af)^2}}$$
$$\times\tan^{-1}\frac{(a^2+c^2+e^2)x^2+(ab+cd+ef)}{\sqrt{(ad-bc)^2+(cf-de)^2+(eb-af)^2}}.$$

11. $\frac{1}{2(ad-bc)}\log\frac{ax^2+b}{cx^2+d}.$ **12.** $\frac{1}{2}\log(e^{2x}+2e^x+3)-\frac{1}{\sqrt{2}}\tan^{-1}\frac{e^x+1}{\sqrt{2}}.$

Page 161.

1. (i) $\log\frac{\sqrt{x^2-1}}{x}$; (ii) $\frac{1}{8}\log\frac{(x-1)(x-5)}{(x-3)^2}$;

(iii) $\frac{1}{3}\log\{x(3-x^2)^4\}$; (iv) $\frac{1}{12}\log\left\{\frac{(x+1)^2}{(x-1)^6}\cdot\frac{(x-2)^7}{(x+2)^3}\right\}$;

(v) $\frac{1}{7}[-\frac{1}{3}\log(x-3)+\frac{10}{3}\log(x+3)+\frac{3}{4}\log(x-4)-\frac{15}{4}\log(x+4)]$;

(vi) $x+\Sigma\frac{(a_1-a)(a_1-b)(a_1-c)}{(a_1-b_1)(a_1-c_1)}\log(x-a_1),$

where Σ refers to a cyclic interchange of the letters a_1, b_1, c_1;

(vii) $\frac{1}{2}\Sigma\dfrac{\left\{\begin{array}{l}(a_1-a)(a_1-b)(a_1-c)\log(x-a_1)\\ \qquad +(a_1+a)(a_1+b)(a_1+c)\log(x+a_1)\end{array}\right\}}{a_1(a_1^2-b_1^2)(a_1^2-c_1^2)}$,

where Σ refers to a cyclic interchange of a_1, b_1, c_1;.

(viii) $\frac{1}{10}\log\{(x-5)^3(x+15)^7\}$; (ix) $\frac{1}{3}\log\{(x-7)(x+17)^2\}$;

(x) $\frac{1}{6}\log\left\{\dfrac{(x-7)^2(x-13)^7}{(x-11)^9}\right\}$.

2. (i) $-\dfrac{1}{4}\dfrac{1}{(x-1)^2}+\dfrac{1}{4}\dfrac{1}{x-1}+\dfrac{1}{8}\log(x-1)-\dfrac{1}{8}\log(x+1)$;

(ii) $-\dfrac{1}{24}\dfrac{x(x^2+3)}{(x^2-1)^3}+\dfrac{1}{4}\dfrac{x}{(x^2-1)^2}-\dfrac{5}{16}\dfrac{x}{x^2-1}-\dfrac{5}{32}\log\left(\dfrac{x-1}{x+1}\right)$;

(iii) $-\dfrac{1}{3x^3}-\dfrac{5}{2x^2}-\dfrac{14}{x}+30\log x-\dfrac{2}{3(x-1)^3}+\dfrac{7}{2(x-1)^2}$

$$-\frac{16}{x-1}-30\log(x-1);$$

(iv) $-\dfrac{1}{ax}+\dfrac{b}{a^2}\log\dfrac{a+bx}{x}$; (v) $-\dfrac{1}{x-3}-\dfrac{1}{x-4}+2\log\dfrac{x-3}{x-4}$;

(vi) $x-\dfrac{a^3}{a-b}\dfrac{1}{x-a}+\dfrac{(2a-3b)a^2}{(a-b)^2}\log(x-a)+\dfrac{b^3}{(a-b)^2}\log(x-b)$;

(vii) $\dfrac{1}{x-2}+\log\dfrac{(x-3)^3}{(x-2)^2}$.

3. (i) $\dfrac{1}{b^2-a^2}\left(\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}-\dfrac{1}{b}\tan^{-1}\dfrac{x}{b}\right)$;

(ii) $x+\dfrac{1}{d^2-c^2}\left[\dfrac{(a^2-c^2)(b^2-c^2)}{c}\tan^{-1}\dfrac{x}{c}-\dfrac{(a^2-d^2)(b^2-d^2)}{d}\tan^{-1}\dfrac{x}{d}\right]$;

(iii) $\dfrac{x^3}{3}+(a^2-c^2)x-c(a^2-c^2)\tan^{-1}\dfrac{x}{c}$;

(iv) $\tan^{-1}x-\dfrac{1}{\sqrt{2}}\tan^{-1}x\sqrt{2}$;

(v) $\dfrac{ad-bc}{ed-fc}\dfrac{1}{\sqrt{cd}}\tan^{-1}\left(x\sqrt{\dfrac{c}{d}}\right)+\dfrac{af-be}{fc-ed}\dfrac{1}{\sqrt{ef}}\tan^{-1}\left(x\sqrt{\dfrac{e}{f}}\right)$;

(vi) $-\dfrac{b}{dfhx}+\Sigma\dfrac{(ad-bc)c^2}{(ed-fc)(gd-hc)}\dfrac{1}{\sqrt{cd}}\tan^{-1}\left(x\sqrt{\dfrac{c}{d}}\right)$.

4. (i) $\log\dfrac{x}{\sqrt{x^2+1}}$; (ii) $\frac{3}{4}\log(x^2-1)+\frac{1}{4}\log(x^2+1)-2\log x$;

(iii) $-\frac{1}{6}\log x+\frac{1}{4}\log(x^2-1)-\frac{1}{4}\log(x^2-2)+\frac{1}{12}\log(x^2-3)$;

(iv) $\dfrac{1}{4b\sqrt{b^2+4ac}}\log\dfrac{2a^2x^2+2ac+b^2-b\sqrt{b^2+4ac}}{2a^2x^2+2ac+b^2+b\sqrt{b^2+4ac}}$ $(b^2+4ac>0)$.

5. (i) $\frac{1}{4}\log\dfrac{x^2+x+1}{x^2-x+1}+\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{x\sqrt{3}}{1-x^2}$;

(ii) $\sqrt{3}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$

or $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x\sqrt{3}}{1-x^2}-\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{\sqrt{3}}{2x^2+1}$, which is the same thing;

(iii) $\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{1-x^2}$; (iv) $\tan^{-1}\dfrac{x}{1-x^2}$; (v) $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{ax\sqrt{3}}{a^2-x^2}$;

(vi) $\dfrac{1}{2a}\log\dfrac{x^2-ax+a^2}{x^2+ax+a^2}$;

(vii) $\dfrac{4}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}-\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$

or $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x\sqrt{3}}{1-x^2}-\sqrt{3}\tan^{-1}\dfrac{\sqrt{3}}{2x^2+1}$;

(viii) $\dfrac{1}{4\sqrt{2}}\log\dfrac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{1-x^2}$.

6. (i) $\frac{1}{2}\log(x-2)-\dfrac{1}{x-2}-\frac{1}{4}\log(x^2-2x+4)-\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{x-1}{\sqrt{3}}$;

(ii) $\frac{2}{3}\log(1+x)-\frac{1}{3}\log(1+2x+4x^2)-\dfrac{1}{3}\dfrac{1}{1+x}+\dfrac{2}{3\sqrt{3}}\tan^{-1}\dfrac{4x+1}{\sqrt{3}}$;

(iii) $x+\frac{18}{25}\log(x-1)+\frac{16}{25}\log(x^2+4)-\dfrac{1}{5}\dfrac{1}{x-1}-\dfrac{24}{25}\tan^{-1}\dfrac{x}{2}$;

(iv) $\dfrac{1}{4}\log\dfrac{(x+1)^2}{x^2+1}-\dfrac{1}{2}\dfrac{1}{x+1}$ (v) $\dfrac{1}{4}\log\dfrac{x^2+1}{(x-1)^2}-\dfrac{1}{2}\dfrac{1}{x-1}$;

(vi) $\log\dfrac{x}{x-1}-\dfrac{1}{2}\dfrac{1}{x-1}+\dfrac{1}{2}\tan^{-1}x$. (vii) $\dfrac{1}{a^4}\log\dfrac{\sqrt{a^2+x^2}}{x}-\dfrac{1}{2a^2x^2}$;

(viii) $-\dfrac{1}{2a^2b^2x^2}-\dfrac{a^2+b^2}{a^4b^4}\log x$

$-\dfrac{1}{2(a^2-b^2)}\left\{\dfrac{1}{a^4}\log(a^2+x^2)-\dfrac{1}{b^4}\log(b^2+x^2)\right\}$;

(ix) $-\dfrac{1}{6(x-1)^2}+\dfrac{1}{3(x-1)}+\frac{2}{9}\log(x-1)-\frac{1}{9}\log(x^2+x+1)$;

(x) $-\dfrac{1}{28}\dfrac{1}{2x-3}-\dfrac{3}{196}\log(2x-3)+\dfrac{3}{392}\log(4x^2+5)+\dfrac{1}{98\sqrt{5}}\tan^{-1}\dfrac{2x}{5}$.

7. (i) $\log\dfrac{x}{\sqrt{x^2+1}}+\dfrac{1}{2}\dfrac{1}{x^2+1}$;

(ii) $-\dfrac{1}{2}\log(x-1)-\dfrac{1}{4}\dfrac{1}{x-1}+\dfrac{1}{4}\log(x^2+1)+\dfrac{1}{4}\tan^{-1}x-\dfrac{1}{4}\dfrac{1}{x^2+1}$;

(iii) $\dfrac{1}{2}\tan^{-1}x+\dfrac{1}{2}\dfrac{x-1}{1+x^2}$;

(iv) $\dfrac{c^2+3ab}{8c^5}\tan^{-1}\dfrac{x}{c}+\dfrac{ab}{2c^4}\dfrac{x}{c^2+x^2}+\dfrac{ab-c^2}{8c^4}\dfrac{x(c^2-x^2)}{(c^2+x^2)^2}-\dfrac{a+b}{4}\dfrac{1}{(c^2+x^2)^2}$.

8. $\dfrac{1}{2\sqrt{2}}\{\pi+2\log(\sqrt{2}-1)\}$; $\dfrac{1}{2\sqrt{2}}\{\pi+2\log(\sqrt{2}+1)\}$.

9. (i) $\dfrac{\pi}{2}$; (ii) $\dfrac{\pi}{2\sqrt{3}}$. 10. $\log\frac{4}{3}$.

14. (i) $4\log(2x-1)-\log(x+2)-\frac{3}{2}\log(x^2+1)-4\tan^{-1}x$;

(ii) $x-2\log x+\frac{3}{4}\log(x-1)+\frac{1}{4}\log(x+1)+\frac{1}{2}\log(x^2+1)-\frac{1}{2}\tan^{-1}x$;

(iii) $\dfrac{1}{2}\,\dfrac{1}{2\sin\frac{a}{2}}\tan^{-1}\dfrac{2x\sin\frac{a}{2}}{1-x^2}$;

(iv) $$\frac{1}{5}\Bigg[\log(x+1)-\cos\frac{\pi}{5}\log\left(x^2-2ax\cos\frac{\pi}{5}+a^2\right)-\cos\frac{3\pi}{5}\log\left(x^2-2ax\cos\frac{3\pi}{5}+a^2\right)+2\sin\frac{\pi}{5}\tan^{-1}\frac{x-a\cos\frac{\pi}{5}}{a\sin\frac{\pi}{5}}+2\sin\frac{3\pi}{5}\tan^{-1}\frac{x-a\cos\frac{3\pi}{5}}{a\sin\frac{3\pi}{5}}\Bigg];$$

(v) $\dfrac{\pi}{2a}$.

17. (i) $$\frac{1}{9x}+\frac{1}{8}\log\frac{x-1}{x+1}-\frac{5}{72}\sqrt{\frac{5}{3}}\log\frac{x\sqrt{5}-\sqrt{3}}{x\sqrt{5}+\sqrt{3}}+\frac{5\sqrt{5}}{36\sqrt{3}}\left[\cot\theta\,\mathrm{cosec}\,\theta-\log\cot\frac{\theta}{2}\right],\ \text{where}\ \theta=\sec^{-1}\frac{x\sqrt{5}}{\sqrt{3}};$$

(ii) $$\frac{1}{27x^3}+\frac{5}{27x^2}+\frac{28}{27x}-\frac{590}{243}\log x+\frac{5^5}{2^4\cdot 3^4}\,\frac{1}{5x-3}+\frac{5^5\cdot 23}{2^7\cdot 3^5}\log(5x-3)+\frac{1}{8}\log(x-1)-\frac{1}{128}\log(x+1);$$

(iii) $(2\sqrt{2}-\sqrt{3}-1)\dfrac{\pi}{2}$.

19. $-\frac{1}{3}\log(x+1)+\frac{1}{6}\log(x^2-x+1)+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}$.

20. $-\tan^{-1}\frac{1}{2}(\sqrt{\tan x}+\sqrt{\cot x})$.

21. (i) $\left(x-\dfrac{2}{a^2+b^2}\right)\tan^{-1}\sqrt{\dfrac{a^2x-1}{b^2x-1}}+\dfrac{1}{ab}\,\dfrac{a^2-b^2}{a^2+b^2}\log\{a\sqrt{b^2x-1}+b\sqrt{a^2x-1}\}$;

(ii) $\dfrac{c}{4b}\log\left\{\left(\dfrac{z+\rho_1}{z-\rho_1}\right)^{\frac{1}{\rho_1}}\left(\dfrac{z-\rho_2}{z+\rho_2}\right)^{\frac{1}{\rho_2}}\right\}+\dfrac{cz}{(z^2-\rho_1^2)(z^2-\rho_2^2)}$,

where $\quad (z^2-a^2)^2=b^2+\dfrac{c}{x},\quad a^2+b=\rho_1^2,\quad a^2-b=\rho_2^2.$

22. $\dfrac{2\sqrt{3}}{\sqrt{a}}\tan^{-1}\dfrac{2\sqrt{x}+\sqrt{a}}{\sqrt{3a}}-\dfrac{2}{\sqrt{3a}}\tan^{-1}\dfrac{2\sqrt{x}-\sqrt{a}}{\sqrt{3a}}$.

23. $\dfrac{-x}{(x^3+3x+1)^2}$.

24. $\left[\dfrac{1}{4}\,\dfrac{\sin x}{\cos^4 x}-\dfrac{5}{8}\,\dfrac{\sin x}{\cos^2 x}+\dfrac{3}{16}\log\dfrac{1+\sin x}{1-\sin x}\right]_0^{\frac{\pi}{4}}=\dfrac{3}{16}\log\dfrac{\sqrt{2}+1}{\sqrt{2}-1}-\dfrac{\sqrt{2}}{8}$.

25. $$-\frac{1}{(n-1)a}\,\frac{1}{(x-a)^{n-1}}+\frac{1}{(n-2)a^2}\,\frac{1}{(x-a)^{n-2}}-\frac{1}{(n-3)a^3}\,\frac{1}{(x-a)^{n-3}}+\dots+\frac{(-1)^{n-1}}{a^{n-1}}\,\frac{1}{x-a}+\frac{(-1)^{n-1}}{a^n}\log(x-a)+\frac{(-1)^n}{a^n}\log x.$$

26. If n be even, $=2m$,

$$x+{}^mC_1(a-b)\log(x-a)-{}^mC_2\frac{(a-b)^2}{x-a}-\frac{{}^mC_3}{2}\frac{(a-b)^3}{(x-a)^2}-\dots$$
$$-\frac{{}^mC_m}{m-1}\frac{(a-b)^m}{(x-a)^{m-1}}.$$

If n be odd, $=2m+1$,

$$2(b-a)\left[\frac{1}{2m-1}\left(\frac{x-b}{x-a}\right)^{\frac{2m-1}{2}}+\frac{2}{2m-3}\left(\frac{x-b}{x-a}\right)^{\frac{2m-3}{2}}+\frac{3}{2m-5}\left(\frac{x-b}{x-a}\right)^{\frac{2m-5}{2}}+\dots\right.$$
$$\left.+\frac{m}{1}\left(\frac{x-b}{x-a}\right)^{\frac{1}{2}}+\frac{1}{2}\frac{(x-a)^{\frac{1}{2}}(x-b)^{\frac{1}{2}}}{b-a}-\frac{2m+1}{2}\tanh^{-1}\left(\frac{x-b}{x-a}\right)^{\frac{1}{2}}\right].$$

27. $\log\dfrac{e^x(e^x+1)}{(2e^x+1)^2}$.　　　**28.** $\dfrac{x^2-1}{4}\log\dfrac{1+x}{1-x}+\dfrac{1}{2}x$.

30. $\dfrac{x^3}{3}\log(1-x^2)-\dfrac{2x}{3}-\dfrac{2x^3}{9}+\dfrac{1}{3}\log\dfrac{1+x}{1-x}$.

45. Let $A\equiv a\alpha^2+b\alpha+c$, $B=a\beta^2+b\beta+c$, $C=a\gamma^2+b\gamma+c$,

$$P=-\frac{A^2}{(\alpha-\beta)^2(\alpha-\gamma)^2},\quad P'=\frac{2A}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}\left\{\frac{B}{(\alpha-\beta)^2}+\frac{C}{(\alpha-\gamma)^2}\right\},$$

and Q, Q'; R, R' similar expressions obtained by a cyclic interchange of letters,

$$I=\frac{P}{x-\alpha}+\frac{Q}{x-\beta}+\frac{R}{x-\gamma}+P'\log(x-\alpha)+Q'\log(x-\beta)+R'\log(x-\gamma).$$

CHAPTER VI.

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1. (i) $[(ac+be)\theta+(bc-ae)\log(c\sin\theta+e\cos\theta)]/(c^2+e^2)$;

(ii) $\dfrac{1}{\sqrt{2}}\log\tan\left(\dfrac{\theta}{2}+\dfrac{3\pi}{8}\right)$;

(iii) $aK-\dfrac{\beta}{b}\log(a+b\cos\theta)$, where

$$K=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\quad(a>b)$$

or $$=\frac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\sqrt{\frac{b-a}{b+a}}\tan\frac{\theta}{2}\quad(a<b);$$

(iv) $\dfrac{1}{\sin\alpha}\cosh^{-1}\dfrac{1+\cos\alpha\cos x}{\cos\alpha+\cos x}=\dfrac{2}{\sin\alpha}\tanh^{-1}\left(\tan\dfrac{\alpha}{2}\tan\dfrac{x}{2}\right)$;

(v) $\dfrac{1}{\sqrt{a^2+b^2}}\log\tan\dfrac{1}{2}\left(x+\tan^{-1}\dfrac{a}{b}\right)$;　　(vi) $\log(\cos\theta+\sin\theta)$;

(vii) $\dfrac{1}{4}\log\dfrac{1+\sin\theta}{1-\sin\theta}+\dfrac{1}{2}\dfrac{1}{1+\sin\theta}=\dfrac{1}{2}\left[\log(\sec\theta+\tan\theta)+\dfrac{1}{1+\sin\theta}\right]$;

(viii) $\cosh^{-1}\dfrac{3\cos(x-\tan^{-1}3)-\sqrt{10}}{3-\sqrt{10}\cos(x-\tan^{-1}3)}$;

(ix) $\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{1}{\sqrt{3}}\tan\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)$;

(x) $[ax+b\log(a\cos x+b\sin x)]/(a^2+b^2)$.

2. (i) $\frac{2}{3}\log 2$; (ii) $\dfrac{\pi}{\sqrt{a^2-c^2}}\,(a>c)$; (iii) $\dfrac{\pi}{3\sqrt{3}}$; (iv) $\dfrac{\pi}{1-a^2}$.

3. $x\cos\alpha+\sin\alpha\cosh^{-1}\dfrac{1+\cos\alpha\cos x}{\cos\alpha+\cos x}$.

4. (i) $\dfrac{1}{\alpha\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\dfrac{\alpha}{\sqrt{\alpha^2-\beta^2}}\tan x\right)$;

(ii) $\dfrac{1}{2}\left[\cosh^{-1}\dfrac{1}{\sqrt{2}}\,\dfrac{2-\cos x-\sin x}{1-\cos x-\sin x}+\dfrac{1}{\sqrt{7}}\cos^{-1}\dfrac{1}{\sqrt{2}}\,\dfrac{2+3\cos x+3\sin x}{3+\cos x+\sin x}\right]$;

(iii) $\dfrac{1}{3}\log\dfrac{\sin x(1+\cos x)}{(1+2\cos x)^2}$.

5. $\frac{1}{4}\tanh x$.

6. (i) $\dfrac{5}{9}\,\dfrac{\sin x}{4+5\cos x}-\dfrac{4}{27}\cosh^{-1}\dfrac{5+4\cos x}{4+5\cos x}$;

(ii) $\dfrac{a}{a^2-b^2}\displaystyle\int\dfrac{dx}{a+b\cos x}-\dfrac{b}{a^2-b^2}\,\dfrac{\sin x}{a+b\cos x}=$ etc., by Art. 173;

(iii) $\dfrac{1}{a^2+b^2}\tan\left(\theta-\tan^{-1}\dfrac{b}{a}\right)$;

(iv) $I=\displaystyle\int\left[\dfrac{dx}{a+\sqrt{b^2+c^2}\cos(x-\gamma)}\right]^2$, where $\tan\gamma=\dfrac{c}{b}$, and then use (ii).

7. (iii) $\dfrac{\pi}{2\sin^4\alpha\cos\alpha}\{(1+\cos\alpha)^2-\sin\alpha\}$;

(iv) $\dfrac{\pi}{6\sin^6\alpha\cos\alpha}\{2(1+\cos\alpha)^3-\sin\alpha(2+\cos^2\alpha)\}$.

8. $\sin\theta\cos\theta\log(1+\tan\theta)-\dfrac{\theta}{2}+\dfrac{1}{2}\log\sin\left(\theta+\dfrac{\pi}{4}\right)$.

9. (i) $\alpha/2\sin\alpha$; (ii) $\tanh^{-1}\left(\tan\dfrac{\alpha}{2}\right)\Big/\sin\alpha$.

10. (i) $\pi/2ab$; (ii) $\pi/12$; (iii) $\dfrac{\pi}{2}\left(\dfrac{a-b}{c-d}+\dfrac{bc-ad}{c-d}\,\dfrac{1}{\sqrt{cd}}\right)$;

(iv) $\pi(\alpha^2+\beta^2)/4\alpha^3\beta^3$; (v) $\pi/4$.

11. $\dfrac{\pi}{2}\,\dfrac{2a^2+b^2}{(a^2-b^2)^{\frac{5}{2}}}$. **12.** $\dfrac{\pi}{2}\,\dfrac{2+3e^2}{(1-e^2)^{\frac{7}{2}}}$. **13.** $\dfrac{2}{\sqrt{4bc-a^2}}\tan^{-1}\dfrac{2be^x+a}{\sqrt{4bc-a^2}}$.

16. (i) $2\sqrt{\tan x}$;

(ii) $I=\dfrac{a}{a^2-b^2}\displaystyle\int\dfrac{dx}{a+b\cos x}-\dfrac{b}{a^2-b^2}\,\dfrac{\sin x}{a+b\cos x}=$ etc. (Art. 173);

(iii) $\dfrac{1}{a^2+b^2}\tan\left(\theta-\tan^{-1}\dfrac{b}{a}\right)=\dfrac{1}{a^2+b^2}\,\dfrac{a\sin\theta-b\cos\theta}{a\cos\theta+b\sin\theta}$.

17. (i) $\dfrac{3}{68}\tan^{-1}\left(\dfrac{1}{2}\tan\dfrac{\theta}{2}\right)-\dfrac{5}{68}\tanh^{-1}\left(2\tan\dfrac{\theta}{2}\right)$; (ii) π;

(iii) $\theta+\dfrac{1}{\sqrt{3}}\log\dfrac{\tan\theta-\sqrt{3}}{\tan\theta+\sqrt{3}}$; (iv) $-\dfrac{2}{b^2}\left\{\log(a+b\cos x)+\dfrac{a}{a+b\cos x}\right\}$.

18. (i) $\dfrac{\sin 2\theta}{2}\log\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}-\dfrac{1}{2}\log\sec 2\theta$;

(ii) $-\cosh^{-1}(\cos\theta+\sin\theta)$; (iii) $\operatorname{cosec}^{-1}\left(2\cos^2\dfrac{\theta}{2}\right)$;

19. $\cos^{-1}\left(\dfrac{\sin x}{2}\right)+2\sqrt{3}\tanh^{-1}\left[\sqrt{3}\tan\dfrac{1}{2}\left\{\cos^{-1}\left(\dfrac{\sin x}{2}\right)\right\}\right]$;

20. $\operatorname{cosec}^{-1}(1+\sin 2\theta)$. **21.** $\sec^{-1}(\cos\theta+\sec\theta)$.

22. (i) $-2\sqrt{1-\sin x}$; (ii) $-2\sqrt{1-\sin x}-\sqrt{2}\log\tan\left(\dfrac{x}{4}+\dfrac{\pi}{8}\right)$;

(iii) $\dfrac{1}{\sqrt{b-a}}\cos^{-1}\left[\sqrt{\dfrac{b-a}{b}}\cos x\right]$.

23. $\cosh x\cot\dfrac{x}{2}$.

24. $\dfrac{\sin x-x\cos x}{\cos x+x\sin x}$. **25.** $\log\log\tan x$.

26. (i) $2x\tan^{-1}x-\log(1+x^2)$; (ii) $3x\tan^{-1}x-\frac{3}{2}\log(1+x^2)$;
(iii) $\frac{1}{2}x\tan^{-1}x-\frac{1}{4}\log(1+x^2)$.

27. $\dfrac{1}{2}\log\dfrac{1-\sin\theta}{1+\sin\theta}-\dfrac{1}{\sqrt{2}}\log\dfrac{1-\sqrt{2}\sin\theta}{1+\sqrt{2}\sin\theta}$, where $x=\tan\theta$.

28. $\dfrac{1}{2}\log\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$, $\dfrac{1}{2\sqrt{3}}\log\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}$,

$\dfrac{1}{8}\log\dfrac{1-\sin x}{1+\sin x}-\dfrac{1}{4\sqrt{2}}\log\dfrac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}$.

29. (i) $\dfrac{1}{\sin 2a}\log\dfrac{\sin(\theta-a)}{\sin(\theta+a)}$; (ii) $\dfrac{1}{2\sin a}\log\dfrac{\sin\theta-\sin a}{\sin\theta+\sin a}$.

30. (i) $\dfrac{8a^2+4ab-b^2}{ab^2(a+b)}-\dfrac{8a}{b^3}\log\dfrac{a+b}{a}$;

(ii) $\dfrac{4a^2-2ab-b^2}{2b^3}-\dfrac{a^2-b^2}{b^3}\log\dfrac{a+b}{a}$; (iii) $\dfrac{b-2a}{ab^2}+\dfrac{2a}{b^3}\log\dfrac{a+b}{a}$;

(iv) $$\frac{1}{b^3}\left[\frac{4}{n-3}\left\{\frac{1}{a^{n-3}}-\frac{1}{(a+b)^{n-3}}\right\}-\frac{8a}{n-2}\left\{\frac{1}{a^{n-2}}-\frac{1}{(a+b)^{n-2}}\right\}\right.$$
$$\left.+\frac{4a^2-b^2}{n-1}\left\{\frac{1}{a^{n-1}}-\frac{1}{(a+b)^{n-1}}\right\}\right],$$

unless $n=1$, 2 or 3, when a logarithmic term occurs from one of the integrations.

32. $-x+\cot(a-\beta)\log\dfrac{\sin(x-a)}{\sin(x-\beta)}$.

42. $\dfrac{2}{1-ab}\left[\dfrac{1}{\sqrt{1-a^2}}\tan^{-1}\sqrt{\dfrac{1+a}{1-a}}\tan\dfrac{x}{2}-\dfrac{b}{\sqrt{b^2-1}}\tan^{-1}\sqrt{\dfrac{b+1}{b-1}}\tan\dfrac{x}{2}\right]$.

43. (i) $\frac{1}{2}e^x\{x\sin x+(x-1)\cos x\}$;

(ii) $(3x+2x^3)/3(1+x^2)^{\frac{3}{2}}$;

(iii) $\frac{\sqrt{3}}{18}(\sin 4\theta-4\sin 2\theta-12\cos^4\theta)$, where $\tan\theta=(2x+1)/\sqrt{3}$.

44. $\Sigma\dfrac{\cos^3 a}{\sin(a-b)\sin(a-c)}\log\sin(x-a)-x\Sigma\dfrac{\sin a\cos^2 a}{\sin(a-b)\sin(a-c)}$.

46. (iii) Put $x+a\log x=xy$.

48. (i) $\dfrac{\cos x-\sin x}{(x-1)\cos x-(x+1)\sin x}$. (ii) $\frac{1}{2}\dfrac{(x+1)\cos x+(x-1)\sin x}{(x-1)\cos x-(x+1)\sin x}$.

CHAPTER VII.

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6. (i) $\dfrac{x^5}{5b}-\dfrac{ax^2}{2b^2}+\dfrac{a^2}{b^2}I_1$,

where $I_1=\dfrac{1}{3bk}\left[\log\dfrac{\sqrt{x^2-kx+k^2}}{x+k}+\sqrt{3}\tan^{-1}\dfrac{2x-k}{k\sqrt{3}}\right]$ and $k^3=\dfrac{a}{b}$;

(ii) $-\dfrac{x}{3b(a+bx^3)}+\dfrac{1}{3b}I$,

where $I=\dfrac{1}{3bk^2}\left[\log\dfrac{x+k}{\sqrt{x^2-kx+k^2}}+\sqrt{3}\tan^{-1}\dfrac{2x-k}{k\sqrt{3}}\right]$;

(iii) $-\dfrac{1}{2ax^2}-\dfrac{b}{a}I$.

7. (i) $\dfrac{x}{12a(a+bx^4)^3}+\dfrac{11}{12a}\left[\dfrac{x}{8a(a+bx^4)^2}+\dfrac{7}{8}\left\{\dfrac{x}{4a(a+bx^4)}+\frac{3}{4}I_0\right\}\right]$,

where $I_0=\displaystyle\int\frac{dx}{a+bx^4}$; and if a, b be of like sign and $k^4=\dfrac{a}{b}$,

$I_0=\dfrac{1}{2k^3}\dfrac{1}{b\sqrt{2}}\left[\tanh^{-1}\dfrac{kx\sqrt{2}}{k^2+x^2}+\tan^{-1}\dfrac{kx\sqrt{2}}{k^2-x^2}\right]$;

or if of unlike sign and $k'^4=-\dfrac{a}{b}$,

$I_0=-\dfrac{1}{2bk'^3}\left(\tanh^{-1}\dfrac{k'}{x}+\tan^{-1}\dfrac{x}{k'}\right)$;

(ii) $-\dfrac{1}{a^3x}-\dfrac{13b}{32a^3}\dfrac{x^3}{a+bx^4}-\dfrac{b}{8a^2}\dfrac{x^3}{(a+bx^4)^2}-\dfrac{45}{32}\dfrac{b}{a^3}J_1$,

and $J_1=\dfrac{1}{2bk\sqrt{2}}\left[-\tanh^{-1}\dfrac{kx\sqrt{2}}{k^2+x^2}+\tan^{-1}\dfrac{kx\sqrt{2}}{k^2-x^2}\right]$, if $\dfrac{a}{b}$ be $+^{\text{ve}}=k^4$,

or $=\dfrac{1}{2bk'}\left[-\tanh^{-1}\dfrac{k'}{x}+\tan^{-1}\dfrac{x}{k'}\right]$, if $\dfrac{a}{b}$ be $-^{\text{ve}}=-k'^4$.

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4. If $I_{m,n}$ denote the given integral,

$$I_{m,n}=\frac{x^{m-1}(1+x^2)^{\frac{n}{2}+1}}{m+n+1}-\frac{m-1}{m+n+1}I_{m-2,n},$$

$$I_{5,7}=(1+x^2)^{\frac{9}{2}}\left\{\frac{x^4}{13}-\frac{4x^2}{13.11}+\frac{4.2}{13.11.9}\right\}.$$

6. With a similar notation,

(a) $I_n=\dfrac{x}{(n-2)a^2(a^2+x^2)^{\frac{n-2}{2}}}+\dfrac{n-3}{n-2}\dfrac{1}{a^2}I_{n-1}$;

(b) $I_{n,p}=\dfrac{x^n(a+bx)^{p+\frac{3}{2}}}{(p+n+\frac{3}{2})b}-\dfrac{an}{(p+n+\frac{3}{2})}I_{n-1,p}$;

(c) $mI_m=x^{m-1}(a^2+x^2)^{\frac{1}{2}}-(m-1)a^2I_{m-2}$;

(d) $(m-n+1)I_{m,n}=\dfrac{x^{m-2}}{(a^3+x^3)^{\frac{n}{3}-1}}-(m-2)a^3I_{m-3,n}$;

(e) $mI_m=x^{m-2}(x^3-1)^{\frac{2}{3}}+(m-2)I_{m-3}$;

(f) $I_{n,p}=\dfrac{x^{2n-1}(x^2+a^2)^{p+\frac{3}{2}}}{2n+2p+2}-\dfrac{(2n-1)a^2}{2n+2p+2}I_{n-1,p}$,

$$(x^3-1)^{\frac{2}{3}}\left(\frac{x^6}{8}+\frac{6x^3}{8.5}+\frac{6.3}{8.5.2}\right).$$

7. $x^{2n}(1-x^2)^{\frac{1}{2}}=2nI_{2n-1}-(2n+1)I_{2n+1}$, where the integral $\equiv I_{2n+1}$.

8. $I_n=\dfrac{2}{2n+1}x^n\sqrt{x-1}+\dfrac{2n}{2n+1}I_{n-1}$.

11. $I_n=e^{ax}\cos^{n-1}x\dfrac{a\cos x+n\sin x}{a^2+n^2}+\dfrac{n(n-1)}{a^2+n^2}I_{n-2}$,

$$I_4=\frac{e^{ax}}{a^2+4^2}\left[\cos^3x(a\cos x+4\sin x)+\frac{4.3}{a^2+2^2}\left\{\cos x(a\cos x+2\sin x)+2.1.\frac{1}{a}\right\}\right].$$

12. (1) $I_n=-x^n\cos x+nx^{n-1}\sin x-n(n-1)I_{n-2}$;

(2) $I_n=e^{ax}\sin^{n-1}x\dfrac{a\sin x-n\cos x}{n^2+a^2}+\dfrac{n(n-1)}{n^2+a^2}I_{n-2}$,

$$I_n=-\sin^{n-1}x\frac{a\sin x\sin ax+n\cos x\cos ax}{n^2-a^2}+\frac{n(n-1)}{n^2-a^2}I_{n-2}.$$

16. (m even) $\dfrac{m(m-1)(m-2)(m-3)\dots2.1}{(n^2+m^2)\{n^2+(m-2)^2\}\dots(n^2+2^2)}\dfrac{2\sinh\frac{n\pi}{2}}{n}$;

(m odd) $\dfrac{m(m-1)(m-2)(m-3)\dots3.2}{(n^2+m^2)\{n^2+(m-2)^2\}\dots(n^2+3^2)}\dfrac{2\cosh\frac{n\pi}{2}}{n^2+1^2}$.

18. $\frac{1}{3m}+\frac{m}{3m(3m-2)}+\frac{m(m-1)}{3m(3m-2)(3m-4)}+\dots$

$$+\frac{m(m-1)\dots 2}{3m(3m-2)\dots(m+2)}+\frac{m(m-1)\dots 1}{3m(3m-2)\dots(m+2)}\cdot\frac{1}{m}\left(1-\cos\frac{m\pi}{2}\right).$$

34. If $m^2\equiv\frac{b+\sqrt{b^2-4ac}}{2c}$, $n^2\equiv\frac{b-\sqrt{b^2-4ac}}{2c}$, and $b^2>4ac$,

$$\int\frac{dx}{a+bx^2+cx^4}=\frac{1}{\sqrt{b^2-4ac}}\left[\frac{1}{n}\tan^{-1}\frac{x}{n}-\frac{1}{m}\tan^{-1}\frac{x}{m}\right];$$

or $$=\frac{1}{4ck^3}\left[\sec\phi\tanh^{-1}\frac{2kx\cos\phi}{k^2+x^2}+\operatorname{cosec}\phi\tan^{-1}\frac{2kx\sin\phi}{k^2-x^2}\right],$$

where $a=ck^4$;

and $\cos 2\phi=-\frac{b}{2\sqrt{ac}}$, where $b^2<4ac$.

If $b^2=4ac$, the integral $=\frac{x}{bx^2+2a}+\frac{1}{\sqrt{2ab}}\tan^{-1}x\sqrt{\frac{b}{a}}$,

$$\int\frac{x^2dx}{a+bx^2+cx^4}=\frac{1}{\sqrt{b^2-4ac}}\left(m\tan^{-1}\frac{x}{m}-n\tan^{-1}\frac{x}{n}\right),\text{ if } b^2>4ac,$$

$$=\frac{1}{4kc}\left(-\sec\phi\tanh^{-1}\frac{2kx\cos\phi}{k^2+x^2}+\operatorname{cosec}\phi\tan^{-1}\frac{2kx\sin\phi}{k^2-x^2}\right),$$

if $b^2<4ac$,

$$=\frac{2a}{b}\left(\frac{1}{\sqrt{2ab}}\tan^{-1}x\sqrt{\frac{b}{2a}}-\frac{x}{2a+bx^2}\right),\text{ if } b^2=4ac.$$

[BERTRAND, *I. C.*, p. 36]

36. (α) $I_n=I_{n-2}-\frac{\tanh^{n-1}x}{n-1}$;

(β) $I_n=-\frac{(n-2)x\cos x+\sin x}{(n-1)(n-2)\sin^{n-1}x}+\frac{n-2}{n-1}I_{n-2}$;

(γ) $\frac{be^x-ce^{-x}}{(a+be^x+ce^{-x})^n}=-(n-2)I_{n-2}+(2n-3)aI_{n-1}+(n-1)(4bc-a^2)I_n$.

40. $\frac{b+cx}{(a+2bx+cx^2)^{n-1}}=-2(n-1)(b^2-ac)I_n-(2n-3)cI_{n-1}$.

43. $\frac{\pi}{2^4}(a+b)(5a^2-2ab+5b^2)$.

44. $I_n-2I_{n-1}+I_{n-2}=-\frac{2}{n-1}\sin 2(n-1)x$,

$$n(2x-\pi)+\cot x+2\left[(n-1)\sin 2x+(n-2)\frac{\sin 4x}{2}+\dots+\frac{\sin 2(n-1)x}{n-1}\right].$$

49. See Art. 202.

CHAPTER VIII.

PAGE 286.

1. (i) $\log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$; (ii) $\frac{1}{\sqrt{3}}\log\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$;

(iii) $2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$; (iv) $\frac{2}{3}(x-1)^{\frac{3}{2}}+2\sqrt{3}\tan^{-1}\sqrt{\frac{x-1}{3}}$.

2. (i) $\dfrac{1}{\sqrt{2}}\left(\tanh^{-1}\dfrac{\sqrt{2x}}{1+x}+\tan^{-1}\dfrac{\sqrt{2x}}{1-x}\right)$;

(ii) $\dfrac{1}{\sqrt{2}}\left(\tanh^{-1}\dfrac{\sqrt{2}\sqrt{x+1}}{x+2}+\tan^{-1}\dfrac{\sqrt{2}\sqrt{x+1}}{-x}\right)$;

(iii) $-\sqrt{2}\tanh^{-1}\sqrt{2}\,\dfrac{\sqrt{x+1}}{x+2}$;

(iv) $2\sqrt{x+1}+\dfrac{3}{\sqrt{2}}\tanh^{-1}\sqrt{2}\,\dfrac{\sqrt{x+1}}{x+2}-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{\sqrt{2(x+1)}}{-x}$.

3. (i) $-\operatorname{cosech}^{-1}x$; (ii) $-\dfrac{1}{\sqrt{2}}\sinh^{-1}\dfrac{1-x}{1+x}$;

(iii) $\sinh^{-1}x+\dfrac{1}{\sqrt{2}}\sinh^{-1}\dfrac{1-x}{1+x}$;

(iv) $\sqrt{x^2+2x+3}-\sinh^{-1}\dfrac{x+1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}\sinh^{-1}\dfrac{\sqrt{2}}{x+1}$.

5. $\log\dfrac{\sqrt{2\cot\theta+3}-1}{\sqrt{2\cot\theta+3}+1}-\dfrac{1}{\sqrt{3}}\log\dfrac{\sqrt{2\cot\theta+3}-\sqrt{3}}{\sqrt{2\cot\theta+3}+\sqrt{3}}$.

6. $$\frac{2^{\frac{1}{4}}}{ab}\Bigg[\sqrt{\frac{a}{a+b}}\coth^{-1}\left\{\sqrt{\frac{a}{a+b}}(\cot\theta+1)\right\}-\sqrt{\frac{a}{a-b}}\coth^{-1}\left\{\sqrt{\frac{a}{a-b}}(\cot\theta+1)\right\}$$
$$+\sqrt{\frac{b}{a+b}}\tanh^{-1}\left\{\sqrt{\frac{b}{a+b}}(\tan\theta+1)\right\}+\sqrt{\frac{b}{a-b}}\tan^{-1}\left\{\sqrt{\frac{b}{a-b}}(\tan\theta+1)\right\}\Bigg].$$

7. $\sinh^{-1}\left(\dfrac{1}{\sqrt{3}}\sec 2\theta\right)$.

8. $\sqrt{x^2+1}\left[\dfrac{x^4}{5}+\dfrac{x^3}{4}+\dfrac{x^2}{15}+\dfrac{9x}{8}+\dfrac{43}{15}\right]+\dfrac{15}{8}\sinh^{-1}x-2\sqrt{2}\sinh^{-1}\dfrac{x+1}{x-1}$.

9. (i) $\sin^{-1}\dfrac{2x-a-b}{a-b}$;

(ii) $\dfrac{1}{\sqrt{a-b}}\log\dfrac{\sqrt{a-b}+\sqrt{x-b}}{\sqrt{a-b}-\sqrt{x-b}}\ (a>b)$, $-\dfrac{2}{\sqrt{b-a}}\tan^{-1}\dfrac{\sqrt{x-b}}{\sqrt{b-a}}\ (a<b)$;

(iii) $\dfrac{1}{\sqrt{a-b}}\log\dfrac{\sqrt{a-b}-\sqrt{a-x}}{\sqrt{a-b}+\sqrt{a-x}}\ (b<a)$, $\dfrac{2}{\sqrt{b-a}}\tan^{-1}\sqrt{\dfrac{a-x}{b-a}}\ (b>a)$;

(iv) (a) $-\dfrac{1}{a\sqrt{2}}\sinh^{-1}\dfrac{a-x}{a+x}$; ($b$) $\dfrac{1}{a\sqrt{2}}\sinh^{-1}\dfrac{a+x}{a-x}$;

(c) $-\dfrac{1}{a}\sqrt{\dfrac{a-x}{a+x}}$; ($d$) $\dfrac{1}{a}\sqrt{\dfrac{a+x}{a-x}}$.

10. $-\dfrac{1}{\sqrt{ab}}\cosh^{-1}\dfrac{2ab-(a+b)x}{(a-b)x}$.

12. $\cosh^{-1}\dfrac{x+p}{\sqrt{p^2-q}}+\dfrac{a-\beta}{\sqrt{\beta^2+2p\beta+q}}\cosh^{-1}\dfrac{(\beta+p)x+p\beta+q}{\sqrt{p^2-q}(a-\beta)}$, if $p^2>q$, with a modification if $p^2<q$.

13. (i) $-\sinh^{-1}\dfrac{2+x}{x\sqrt{3}}$; (ii) $\dfrac{1}{a}\sec^{-1}\dfrac{x}{a}$; (iii) $\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{\sqrt{1-x^2}}$.

14. (i) $\sqrt{\dfrac{1+x}{1-x}}$; (ii) $\dfrac{1}{\sqrt{3}}\tanh^{-1}\sqrt{\dfrac{1+4x}{3}}$.

15. (i) $\dfrac{2}{\sqrt{\mu-\lambda}}\tan^{-1}\sqrt{\dfrac{x-\mu}{\mu-\lambda}}\ (\lambda<\mu)$, $-\dfrac{2}{\sqrt{\lambda-\mu}}\coth^{-1}\sqrt{\dfrac{x-\mu}{\lambda-\mu}}\ (\lambda>\mu)$;

(ii) $\dfrac{x^2}{2}-\dfrac{\lambda^2}{2}\log(x^2+\lambda^2)$; (iii) $\dfrac{1}{2}\log\dfrac{(x+1)(x+3)}{(x+2)^2}$.

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1. (i) $2\tan^{-1}\sqrt{x}$ (ii) $2\tan^{-1}\sqrt{1+2x}$;

(iii) $-\dfrac{1}{\sqrt{2}}\cosh^{-1}\dfrac{4-3x}{x}$; (iv) $-\sinh^{-1}\dfrac{1}{\sqrt{3}}\dfrac{1-x}{1+x}$;

(v) $\sqrt{x^2+x+1}-\dfrac{1}{2}\sinh^{-1}\dfrac{2x+1}{\sqrt{3}}-\sinh^{-1}\dfrac{1}{\sqrt{3}}\dfrac{1-x}{1+x}$; (vi) $\dfrac{x\sqrt{x-1}}{\sqrt{x+1}}$;

(vii) $-\dfrac{2}{na^{\frac{n}{2}}}\sinh^{-1}\left(\dfrac{a}{x}\right)^{\frac{n}{2}}$; (viii) $2\operatorname{cosec}^{-1}\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)$.

2. (i) $-\dfrac{1}{\sqrt{3}}\cosh^{-1}\left\{-\dfrac{2x+1}{x+2}\right\}$;

(ii) $\sqrt{x^2-1}-2\cosh^{-1}x+\sqrt{3}\cosh^{-1}\left\{-\dfrac{2x+1}{x+2}\right\}$.

6. (i) $\dfrac{10}{19}\cosh^{-1}\dfrac{4}{\sqrt{3}}\sqrt{\dfrac{4x^2-2x+1}{5x^2+8x}}-\dfrac{9}{19}\sinh^{-1}\sqrt{\dfrac{4x^2-2x+1}{5x^2+8x}}$;

(ii) $\dfrac{1}{\sqrt{(b^2-a^2)(c^2-b^2)}}\cos^{-1}\sqrt{\dfrac{b^2-a^2}{c^2-a^2}}\sqrt{\dfrac{x^2+2ax+c^2}{x^2+2ax+b^2}}\ (a<b<c)$,

with similar results for other cases.

7. (i) $\dfrac{1}{2ab}\sin^{-1}\dfrac{(a^2+b^2)x^2-(a^4+b^4)}{(a^2-b^2)(a^2+b^2-x^2)}$;

(ii) $\dfrac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\dfrac{x^2+c^2}{x^2+a^2}}+\dfrac{b}{a\sqrt{a^2-c^2}}\cosh^{-1}\dfrac{a}{c}\sqrt{\dfrac{x^2+c^2}{x^2+a^2}}$;

(iii) $-\dfrac{1}{\sqrt{a+c}}\sinh^{-1}\left\{\sqrt{\dfrac{a+c}{b+c}}\cot\theta\right\}$.

8. (i) $$\frac{1}{\sqrt{(\cos a-\cos\beta)(\cos a-\cos\gamma)}}\times\cosh^{-1}\left(\frac{\dfrac{2}{\cos x+\cos a}-\dfrac{1}{\cos a-\cos\beta}-\dfrac{1}{\cos a-\cos\gamma}}{\dfrac{1}{\cos a-\cos\beta}-\dfrac{1}{\cos a-\cos\gamma}}\right)$$

for the case $\cos a>\cos\beta$ or $\cos\gamma$, with modifications for other cases;

(ii) $$-\frac{1}{\sqrt{\sin(a-\beta)\sin(a-\gamma)}}\times\cosh^{-1}\left(\frac{\dfrac{2}{\tan x-\cot a}+\dfrac{1}{\cot a-\cot\beta}+\dfrac{1}{\cot a-\cot\gamma}}{\dfrac{1}{\cot\beta-\cot a}-\dfrac{1}{\cot\gamma-\cot a}}\right).$$

9. $\dfrac{1}{a}\sqrt{\dfrac{x^2+ax+a^2}{x^2-ax+a^2}}.$

10. (i) $$-\frac{1}{2\sqrt{5}}\left[3\sqrt{2}\sin^{-1}\sqrt{\frac{1}{3}\,\frac{-x^2+10x-13}{3x^2-10x+9}}+5\sinh^{-1}\sqrt{\frac{1}{2}\,\frac{-x^2+10x-13}{3x^2-10x+9}}\right];$$

(ii) $\dfrac{1}{\sqrt{6}}\cosh^{-1}\dfrac{17-5x}{x-1}-\dfrac{1}{\sqrt{2}}\cosh^{-1}\dfrac{10-3x}{(x-2)}$;

(iii) $\dfrac{10}{3}\sinh^{-1}\dfrac{1}{x-1}-\dfrac{13}{3\sqrt{10}}\sinh^{-1}\dfrac{3x-2}{x-4}$;

(iv) $$-\frac{b-a}{(b-c)(b-d)}\frac{2}{\sqrt{b-e}}\sinh^{-1}\sqrt{\frac{b-e}{x-b}}-\frac{c-a}{(c-b)(c-d)}\frac{2}{\sqrt{c-a}}\sinh^{-1}\sqrt{\frac{c-e}{x-c}}-\frac{d-a}{(d-b)(d-c)}\frac{2}{\sqrt{d-e}}\sinh^{-1}\sqrt{\frac{d-e}{x-d}};$$

(v) $-\cosh^{-1}\sqrt{\dfrac{x^2+x+2}{x^2+x+1}}+\dfrac{5}{\sqrt{3}}\cos^{-1}\sqrt{\dfrac{3}{7}\cdot\dfrac{x^2+x+2}{x^2+x+1}}.$

11. (i) $\dfrac{\sqrt{x^4+x^2+1}}{x}$; (ii) $\cosh^{-1}\left(x+\dfrac{1}{x}\right).$

13. (i) $\dfrac{1}{2}\sin\theta-\dfrac{1}{\sqrt{3}}\tanh^{-1}\left(\dfrac{1}{\sqrt{3}}\tan\dfrac{\theta}{2}\right)$, where $\cos\theta=x^2$;

(ii) $\tan^{-1}\{x(\sqrt{1+x^4}+x^2)^{\frac{1}{2}}\}$; (iii) $\dfrac{2}{\sqrt{5}}\cosh^{-1}\sqrt{5\,.\,\dfrac{x^2+ax}{x^2+ax-a^2}}.$

14. $$\frac{1}{(a^2+1)}\left\{\frac{1}{\sqrt{b^2+1}}\sin^{-1}\left(\frac{\sqrt{b^2+1}}{b}\sin x\right)+\frac{1}{a\sqrt{b^2-a^2}}\sinh^{-1}\sqrt{\frac{b^2-a^2}{b^2}\,\frac{\tan^2 x}{a^2-\tan^2 x}}\right\},$$

if $b^2>a^2$, with other forms for other cases.

18. $-\frac{\sqrt{2}}{18}\left[4\cos^{-1}\sqrt{\frac{1}{3}\frac{4x^2-26x+49}{2x^2-10x+17}}+7\cosh^{-1}\sqrt{\frac{4x^2-26x+49}{2x^2-10x+17}}\right]$.

20. $\frac{1}{ab}\tan^{-1}\frac{a}{b}\frac{x}{\sqrt{a^2+b^2+x^2}}$.

21. (i) $\sec^{-1}(\cos x+\sec x)$; (ii) $\frac{\pi}{a\sqrt{a^2+c^2}}$.

25. If $s_1=s_2$, $-\frac{1}{\sqrt{s_1-s_3}}\sinh^{-1}\sqrt{\frac{s_1-s_3}{s-s_1}}$.

If $s_2=s_3$, $\frac{1}{\sqrt{s_1-s_3}}\cos^{-1}\sqrt{\frac{s_1-s_3}{s-s_3}}$.

30. $\frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{x^2+1}$.

31. (i) $\frac{1}{2\sqrt{2}}\log\frac{\sqrt{1+x^4}+x\sqrt{2}}{1-x^2}+\frac{1}{2\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{1+x^2}$;

(ii) $\frac{1}{4\sqrt{2}}\log\frac{\sqrt{1+x^4}+x\sqrt{2}}{1-x^2}-\frac{1}{4\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{1+x^2}$.

34. (ii) $e^x\sqrt{\frac{1+x^n}{1-x^n}}$.

35. (i) $\sin\theta-\frac{1}{3}\theta-\frac{4}{3\sqrt{5}}\log\frac{\sqrt{5}+\tan\frac{\theta}{2}}{\sqrt{5}-\tan\frac{\theta}{2}}$, where $x=\cos\theta$;

(ii) $-\frac{1}{4}[\tan\theta-2\log\tan\theta+\frac{3}{4}\log(\tan\theta-1)+\frac{1}{4}\log(\tan\theta+1)$
$+\frac{1}{2}\log(\tan^2\theta+1)-\frac{1}{2}\theta]$.

41. (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\sin^2\theta\right)$; (ii) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\frac{z^2+1}{z^2+z+1}\right)$.

45. (i) $\log\sqrt{\frac{x^3-x+1}{x^3+x+1}}$; (ii) $\frac{2}{\sqrt{3a}}\left\{3\tan^{-1}\frac{2\sqrt{x}+\sqrt{a}}{\sqrt{3a}}-\tan^{-1}\frac{2\sqrt{x}-\sqrt{a}}{\sqrt{3a}}\right\}$

52. (i) $\frac{1}{2}\frac{x^2}{b^4-x^4}\sin^{-1}\frac{x^2}{b^2}-\frac{1}{2}\frac{1}{(b^4-x^4)^{\frac{1}{2}}}$;

(ii) $\frac{1}{2\sqrt{2}}\log\tan\left(\theta+\frac{\pi}{4}\right)+\frac{\theta}{\sqrt{2}}$, where $\sin\phi=\sqrt{2}\sin\theta$.

CHAPTER IX.

Page 326.

1. (i) $\log_e 2$; (ii) $\frac{\pi}{4}$; (iii) $\frac{\pi}{2}$; (iv) $\frac{(2k-1)(2k-3)\dots 1}{2k(2k-2)\dots 2}$.

3. 2; 4. $\sqrt{2}/a$; 5. $1/\sqrt{2}$.

Page 353.

1. (i) $\frac{\cos x}{x\cos x-\sin x}$; (ii) $\frac{1}{x(1-\log x)}$.

3. (i) $2(n-1)(ac-b^2)\int\frac{dx}{X^n}=\frac{b+cx}{X^{n-1}}+(2n-3)c\int\frac{dx}{X^{n-1}}$.

(ii) $\int \cos mx \sin^4 x\, dx = \dfrac{\cos^2 mx}{m^2-4^2}\dfrac{d}{dx}\dfrac{\sin^4 x}{\cos mx}$

$$-\frac{4.3}{(m^2-4^2)(m^2-2^2)}\cos^2 mx\frac{d}{dx}\frac{\sin^2 x}{\cos mx}+\frac{4.3.2.1}{(m^2-4^2)(m^2-2^2)}\frac{\sin mx}{m}.$$

4. $\dfrac{1}{\sqrt{2}}\cosh^{-1}\sqrt{2\dfrac{3x^2-10x+9}{5x^2-16x+14}}-\dfrac{1}{\sqrt{3}}\cos^{-1}\sqrt{\dfrac{3}{2}\cdot\dfrac{3x^2-10x+9}{5x^2-16x+14}}.$

6. $a\dfrac{\sqrt{p^2n^2+q^2}}{\sqrt{P^2n^2+Q^2}}\cos\left(nt+e+\tan^{-1}\dfrac{pn}{q}-\tan^{-1}\dfrac{Pn}{Q}\right);$

$$a\frac{(PP'+QQ'n^2)\sin(nx+a)+(P'Q-Q'P)n\cos(nx+a)}{P^2+Q^2n^2},$$

where $\left.\begin{array}{l}P=\alpha-\gamma n^2+\ldots,\\ Q=\beta-\delta n^2+\ldots,\end{array}\right\}$

and P', Q' are the corresponding expressions, with Capitals instead of Greek letters.

8. $\dfrac{1}{e}$. **9.** $\dfrac{3}{2}$, $\dfrac{4}{e}$. **12.** 2. **13.** 1.

15. $\dfrac{1}{x}\log\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$. If $\pi>x>\dfrac{\pi}{2}$,

Principal Value $=\dfrac{1}{x}\log\left\{-\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)\right\}=\dfrac{1}{x}\log\tan\left(\dfrac{3\pi}{4}-\dfrac{x}{2}\right).$

16. $2-\log 2-\pi$. **32.** $\dfrac{n\pi}{2ab}$

41. Principal Value $=\dfrac{1}{2c}\log\left(\dfrac{b+c}{b-c}\cdot\dfrac{c-a}{c+a}\right)$. [See Art. 347 (c).]

47. (i) $\dfrac{e^x}{x+1}$; (ii) $e^x\dfrac{x-1}{x+1}$; (iii) $e^x\sqrt{\dfrac{1+x}{1-x}}$;

(iv) $\dfrac{1}{2}\left[\log\dfrac{e^x}{e^x-1}-\dfrac{1}{e^x-1}\left\{1+\dfrac{x}{(e^x-1)}\right\}\right]$;

(v) $\dfrac{\sin x}{\cos x+x\sin x}$; (vi) $\log\left(\dfrac{\log\tan e^2}{\log\tan e}\right)$;

(vii) $-2\sqrt{1-x}\log(1+x^2)+8\sqrt{1-x}$

$$-4\left\{R\tanh^{-1}\frac{2R\sqrt{1-x}}{2R^2-x}+S\tan^{-1}\frac{2S\sqrt{1-x}}{2S^2+x}\right\},$$

where $R^2=\dfrac{\sqrt{2}+1}{2}$, $S^2=\dfrac{\sqrt{2}-1}{2}$.

CHAPTER X.

Page 377.

12. The integrand becomes ∞ at the limit $\theta=\alpha$, but remains real and finite from $\theta=0$ to $\theta=\alpha$, and the rule of differentiation is not established for this case. But putting $\sin\dfrac{\theta}{2}=\sin\dfrac{\alpha}{2}\sin\dfrac{\phi}{2}$, the difficulty disappears.

14. $y = Ax^{\frac{2\lambda-1}{1-\lambda}}$, where $\dfrac{\int_0^x \xi\eta\, d\xi}{\int_0^x \eta\, d\xi} = \lambda x.$

16. $y = Ax^{\frac{2-n}{2(n-1)}}$, the height of the centroid being $\frac{1}{n}$ of the height of the segment.

17. A straight line through the origin.

19. The density at each point varies inversely as the square of the abscissa.

20. $y=(Ax+B)^k$, A, B, k being constants. 21. If $F(x)=A/\sqrt{x}$.

38. The first $=\frac{\pi}{4}$. The second $=-\frac{\pi}{4}$. The rule for the reversal of the order of integration is not established when the subject of integration becomes infinite at any point of the range of integration. For $a=0$, $\int_0^1 \frac{a^2-x^2}{(a^2+x^2)^2}\,dx$ is infinite.

39. The case reduces to $\int_0^\infty e^{-x^2}\cos 2\beta x\, dx = e^{-\beta^2}\int_0^\infty e^{-x^2}\,dx.$

CHAPTER XII.

Page 415.

1. $\frac{8}{3}a^2$.

2. (a) $c^2\sinh\frac{h}{c}$; (b) e^h-1; (c) $h(\log h-1)+1$, $(h>1)$;

(d) $\frac{\pi ab}{4}-\frac{b^2}{2a}\sqrt{a^2-b^2}-\frac{ab}{2}\cos^{-1}\frac{b}{a}$;

(e) (i) $k^2\log\frac{b}{a}$, (ii) $k^2\sin\omega\log\frac{b}{a}$; (f) $\frac{1}{2}(e^{h^2}-1)$.

3. (1) $\frac{16}{3}a^2$; (2) $\frac{16}{3}ab$. Area bisected in either case.

4. (1) $\frac{\pi ab}{2}\pm\frac{a}{b}\left(c\sqrt{b^2-c^2}+b^2\sin^{-1}\frac{c}{b}\right)$;

(2) If $A_1=\frac{a}{2b}\left[c\sqrt{b^2-c^2}+b^2\sin^{-1}\frac{c}{b}\right]$, $A_2=\frac{b}{2a}\left[d\sqrt{a^2-d^2}+a^2\sin^{-1}\frac{d}{a}\right]$,

the four regions are
$$\frac{\pi ab}{4}-A_1-A_2+cd,$$
$$\frac{\pi ab}{4}+A_1-A_2-cd,$$
$$\frac{\pi ab}{4}-A_1+A_2-cd,$$
$$\frac{\pi ab}{4}+A_1+A_2+cd.$$

5. $4a^2$. 6. $3\pi a^2$. 7. $\frac{a^2}{2}(4-\pi)$. 11. $\frac{352}{15}a^2\sqrt{2}$.

13. (i) $\frac{8a^2}{15}$; (ii) $\frac{4}{3}a^2$. **16.** $\frac{\pi}{4}+\frac{1}{2}\log 2-\frac{1}{2}$. **17.** (i) $\frac{3\pi a^2}{4}$.

19. $a^2\left(\frac{16}{3}+4\sqrt{3}-\frac{\pi}{2}\right)$. **21.** $c^2\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\mp\frac{\pi}{2}\right)$. **24.** $\frac{2}{3}$.

Page 428.

1. $(a^2-b^2)\tan^{-1}\frac{a}{b}+ab$. **2.** $\frac{\pi a^2}{16}$, $\frac{\pi a^2}{2}$. **3.** $\frac{\pi a^2}{20}$, $\frac{\pi a^2}{4}$.

4. $\frac{\pi a^2}{4n}$; n even, $\frac{\pi a^2}{2}$; n odd, $\frac{\pi a^2}{4}$. **5.** $\frac{a^2}{4}\tan\alpha\, e^{2\beta\cot\alpha}(e^{2\gamma\cot\alpha}-1)$.

6. $\frac{a^2}{6}\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)$. **7.** $\frac{a^2}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$. **8.** $\frac{3}{2}\pi a^2$.

9. (i) $\pi\left(a^2+\frac{1}{2}b^2\right)$; (ii) $A_o=\frac{2a^2+b^2}{2}\cos^{-1}\left(\frac{-a}{b}\right)+\frac{3a}{2}\sqrt{b^2-a^2}$,

$$A_i=\frac{2a^2+b^2}{2}\cos^{-1}\left(\frac{a}{b}\right)-\frac{3a}{2}\sqrt{b^2-a^2}.$$

10. $\frac{a^2}{3}(10\pi+9\sqrt{3})$. **12.** $\frac{3a^2}{2}$.

14. $\frac{a^2}{4}\log\left(\frac{1+\sqrt{\sin\alpha}}{1-\sqrt{\sin\alpha}}\,\frac{1-\sqrt{\sin\beta}}{1+\sqrt{\sin\beta}}\right)-\frac{a^2}{2}[\tan^{-1}\sqrt{\sin\alpha}-\tan^{-1}\sqrt{\sin\beta}]$.

15. Area of lozenge $=\frac{a^2}{16}(16-9\sqrt{3})$.

17. $\frac{5}{4}\pi a^2$. **19.** $\frac{5}{2}\pi a^2$. **20.** $\frac{\pi a^2}{16}\left(\frac{\pi^2}{6}-1\right)$.

Page 429.

1. $\left(\frac{\sqrt{3}}{2}+\frac{\pi}{12}-1\right)a^2$, $\left(\frac{\sqrt{3}}{2}+\frac{25\pi}{12}+1\right)a^2$, $\frac{8a^2}{15}\sqrt{\frac{a}{b}}$ **2.** $(\pi-2)a^2$.

3. $\left\{2\log(\sqrt{2}+1)-\frac{11\sqrt{2}}{24}\right\}a^2$. **4.** a^2, $\pi a^2\sqrt{2}$.

7. $\frac{b^2-a^2}{2}\log\frac{b-a}{b+a}+(b^2+a^2)\cot^{-1}\frac{b}{a}$.

8. $3\pi a^2$. **17.** $16\pi a^2/3\sqrt{3}$. **18.** $\pi a^2/2$.

19. $\frac{l^2}{2(1-e^2)^{\frac{3}{2}}}\left[\cos^{-1}\frac{e+\cos\theta}{1+e\cos\theta}-e\sqrt{1-e^2}\frac{\sin\theta}{1+e\cos\theta}\right]_{-\alpha}^{\pi-\alpha}$.

21. $\frac{a^2}{4b^3}[2b^2\{(\alpha+2\pi)^2e^{2b\alpha}-(\beta+2\pi)^2e^{2b\beta}\}-2b\{(\alpha+2\pi)e^{2b\alpha}-(\beta+2\pi)e^{2b\beta}\}$

$$+(e^{2b\alpha}-e^{2b\beta})]e^{2b\pi}$$

$$-\frac{a^2}{4b^3}[2b^2(\alpha^2e^{2b\alpha}-\beta^2e^{2b\beta})-2b(\alpha e^{2b\alpha}-\beta e^{2b\beta})+(e^{2b\alpha}-e^{2b\beta})].$$

22. $\frac{\pi a^2}{16}\left(\frac{\pi^2}{6}-1\right)$ **23.** $\frac{\pi a^2}{2}\sqrt{2}$. **24.** $2:1$.

25. $\frac{19}{12}\sqrt{7}-\frac{7}{12}+\frac{5}{4}\log\frac{5+2\sqrt{7}}{3}$. **26.** $\frac{2\pi}{\sqrt{3}}$. **27.** (i) $\frac{3\pi a^2}{8}$.

30. $(\pi+2)a^2$. **31.** $\pi a^2-a^2\cos^{-1}\frac{b^2}{a^2}+b^2\cosh^{-1}\frac{a^2}{b^2}$. **33.** $a^2\left(1-\frac{\pi}{4}\right)$.

35. $A=\sqrt{R^2-b^2}-b\cos^{-1}\frac{b}{R}$, where $R^2=(p-a)^2+q^2$.

43. $$\frac{a_1b_1}{2}\tan^{-1}\frac{a_1b_1\sin(\theta_2-\theta_1)}{a_1^2\sin\theta_1\sin\theta_2+b_1^2\cos\theta_1\cos\theta_2}$$
$$-\frac{ab}{2}\tan^{-1}\frac{ab\sin(\theta_2-\theta_1)}{a^2\sin\theta_1\sin\theta_2+b^2\cos\theta_1\cos\theta_2},\text{ where } a_1^2-a^2=b_1^2-b^2=\lambda.$$

52. πa^2. **53.** $v_1+\frac{c}{v_1^2}\left[\frac{1}{2}\log 3-\frac{\pi b}{6\sqrt{3}}\right]$. **54.** $\frac{ab}{2}\sinh c\,[\sinh 2c+c]$.

55. $\pi c(\sqrt{a}-\sqrt{b})^2$. **56.** At the cusps.

57. $$\left.\begin{array}{l}\text{Area of loop of first } =\frac{\pi a^2}{2} =157 \text{ sq. cm., about,}\\ \text{Area of loop of second} =\frac{\pi a^2}{2}\sqrt{2}=222 \text{ sq. cm., about}\end{array}\right\}(a=10).$$

58. $(\pi+1)a^2$.

CHAPTER XIII.

PAGE 466.

1. Double the area swept out by the portion of the tangent intercepted between the original curve and the first positive pedal.

3. $\frac{\pi ab}{4}-\frac{b}{2}\sqrt{a^2-b^2}$.

4. $\frac{3a^4+2a^2b^2+3b^4}{16ab}\tan^{-1}\frac{b}{a}-\frac{(3a^2+b^2)(a^2+3b^2)(a^2-b^2)}{16(a^2+b^2)^2}$.

7. $\pi a(a-b)$. **13.** $\frac{\pi^3a^2}{24}+\frac{\pi}{8}\{(h-a)^2+a^2\}$, and is least if $h=a$.

14. $x^2y^2=(a^2-y^2)(y^2-b^2)$. **20.** πc^2, c being the constant.

25. $\left[a^2\theta+\frac{a^4}{2c^2}\tan\theta\right]_{\theta_1}^{\theta_2}$, where c is the diameter of the circle.

31. The vertex. **34.** A circle of radius a; πa^2.

CHAPTER XIV.

PAGE 478.

(i) $\frac{\mu a^4}{8}$. Density$=\mu xy$; (ii) $\bar{x}=\bar{y}=\frac{8}{15}a$; (iii) $B=\frac{1}{3}Ma^2$.

2. (i) $\mu \dfrac{2^{q+2}a^{p+q+2}}{(q+1)(2p+q+3)}$;

(ii) $\bar{x}=\dfrac{2p+q+3}{2p+q+5}a$; $\bar{y}=2\dfrac{q+1}{q+2}\cdot\dfrac{2p+q+3}{2p+q+4}a$; (iii) $B=\dfrac{2p+q+3}{2p+q+7}Ma^2$.

3. (i) $\bar{x}=\dfrac{n+1}{n+2}l$, ($l$ = length); (ii) $\dfrac{n+1}{n+3}Ml^2$.

(iii) $\dfrac{2}{(n+2)(n+3)}Ml^2$; (iv) $\dfrac{1}{4}\dfrac{n^2+n+2}{(n+2)(n+3)}Ml^2$.

4. (i) $\bar{x}=\frac{4}{5}a$, $\bar{y}=\dfrac{3}{5}\dfrac{2+m^2}{3+m^2}ma$; (ii) $B=\frac{2}{3}Ma^2$.

5. (i) $\bar{x}=\dfrac{a}{5}\dfrac{15\pi-44}{3\pi-8}$, $\bar{y}=\dfrac{a}{3\pi-8}$; (ii) $\bar{x}=\frac{9}{5}a^{\frac{1}{3}}b^{\frac{2}{3}}$, $\bar{y}=\frac{9}{5}a^{\frac{2}{3}}b^{\frac{1}{3}}$;

(iii) $\bar{x}=\frac{2}{5}a$; $\bar{y}=a$.

6. (i) Moment of Inertia about base $=\dfrac{Mh^2}{6}$, h being the perpendicular from the vertex to the base;

$\dfrac{\Delta}{3}(AL^2+AM^2+AN^2)$, where A is the angular point and L, M, N the mid-points of the sides.

PAGE 484.

1. (α) $\left.\begin{array}{l}\bar{x}=\dfrac{2}{3}\dfrac{a\sin\alpha}{\alpha},\\ \bar{y}=0,\end{array}\right\}$ 2α being the angle of the sector, and a the radius;

(β) $\bar{x}=\dfrac{n+2}{n+3}\dfrac{a\sin\alpha}{\alpha}$, $\bar{y}=0$.

2. $\bar{x}=\dfrac{n+2}{n+4}a$, $\bar{y}=0$, a being the diameter;

(i) $\dfrac{(n+2)(n+3)(n+5)}{(n+4)^2(n+6)}Ma^2$; (ii) $\dfrac{(n+2)(n+3)}{(n+4)^2(n+6)}Ma^2$;

(iii) $\dfrac{(n+2)(n+3)}{(n+4)^2}Ma^2$.

3. (b) If (p_1, q_1), (p_2, q_2), (p_3, q_3) be the coordinates of A, B, C, viz.

$$p_1=-\frac{c_2-c_3}{m_2-m_3},\qquad q_1=\frac{m_2c_3-m_3c_2}{m_2-m_3},\text{ etc.,}$$

$$A=\frac{M}{12}\Sigma(q_2+q_3)^2,\quad B=\frac{M}{12}\Sigma(p_2+p_3)^2.$$

4. $\bar{x}=\dfrac{p+1}{p+2}\dfrac{a_2^{p+2}-a_1^{p+2}}{a_2^{p+1}-a_1^{p+1}}$, $\bar{y}=\dfrac{q+1}{q+2}\dfrac{b_2^{p+2}-b_1^{p+2}}{b_2^{p+1}-b_1^{p+1}}$,

$A=\dfrac{q+1}{q+3}M\dfrac{b_2^{q+3}-b_1^{q+3}}{b_2^{q+1}-b_1^{q+1}}$, $B=\dfrac{p+1}{p+3}M\dfrac{a_2^{p+3}-a_1^{p+3}}{a_2^{p+1}-a_1^{p+1}}$.

7. Area $=\dfrac{a^2}{6}(2\pi+3\sqrt{3})$;

(1) $\bar{x}=\dfrac{3a\sqrt{3}}{2(3\sqrt{3}-\pi)}$, $\bar{y}=0$; (2) $Ma^2\dfrac{9\sqrt{3}-\pi}{9\sqrt{3}-3\pi}$.

8. (i) $A=\dfrac{2^4 . 3^2}{35} M a^{\frac{4}{3}} b^{\frac{2}{3}}$, $B=\dfrac{2^4 . 3^2}{35} a^{\frac{2}{3}} b^{\frac{4}{3}}$;

(ii) $C=\dfrac{2^4 . 3^2}{35} M a^{\frac{2}{3}} b^{\frac{2}{3}} (a^{\frac{2}{3}}+b^{\frac{2}{3}})$.

9. $\bar{x}=x-\dfrac{c}{s}(y-c)$, $\bar{y}=\dfrac{1}{4}\left(y+\dfrac{cx}{s}\right)$.

Page 492.

1. (i) $2a^2\left(1-\dfrac{\pi}{4}\right)$; (ii) $\dfrac{a^2n^2-b^2m^2}{2m^3n^3}\tan^{-1}\dfrac{an}{bm}+\dfrac{ab}{2m^2n^2}$.

2. (i) $7\pi a^2/2^9$; (ii) $7\pi a^2\sqrt{2}/2^{14}$.

7. $\pi\left[c^2+\dfrac{l^2}{(1-e^2)^{\frac{3}{2}}}\right]$. 9. $15\pi ab/2^7$.

15. (i) ab. (ii) $\pi ab/2$.

17. $11\pi a^2/2^{15} . 3^{12}$. 21. $\bar{x}=8a\sqrt{2}\{\log(\sqrt{2}+1)-\frac{7}{12}\sqrt{2}\}/\pi(4\sqrt{\ }$ [illegible] $5)$.

25. $2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2$. 26. $\pi(a^2+b^2)c$ [illegible].

CHAPTER XV.

Page 521.

7. $\dfrac{\Sigma[A]\sin 2A}{4\sin A\sin B\sin C}-\pi R^2$, R being the radius of the circumcircle.

CHAPTER XVI.

Page 533.

1. $a\left[2\sqrt{\dfrac{8a-3x}{2a-x}}+\sqrt{3}\cosh^{-1}\dfrac{3x-7a}{a}\right]_{x_1}^{x_2}$. 2. A cycloid.

4. $\left(x_2^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{\frac{3}{2}}-\left(x_1^{\frac{2}{3}}+y_1^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Page. 538.

1. (i) $a(\theta_2-\theta_1)$; (ii) $\dfrac{a\sqrt{1+m^2}}{m}(e^{m\theta_2}-e^{m\theta_1})$;

(iii) $2a\left(\cos\dfrac{\theta_1}{2}-\cos\dfrac{\theta_2}{2}\right)$;

(iv) $2a\left\{\left(\tan\dfrac{\theta_2}{2}-\tan\dfrac{\theta_1}{2}\right)+\dfrac{1}{3}\left(\tan^3\dfrac{\theta_2}{2}-\tan^3\dfrac{\theta_1}{2}\right)\right\}$;

(v) $a\left[\dfrac{\sqrt{1+3\cos^2\theta}}{\cos\theta}-\dfrac{\sqrt{3}}{2}\cosh^{-1}(1+6\cos^2\theta)\right]$ (cf. Ex. 1, p. 533);

(vi) $\dfrac{a}{18}\left[(4+9\tan^2\theta)^{\frac{3}{2}}\right]_{\theta_1}^{\theta_2}$.

PAGE 541.

1. (i) A circle; (ii) A catenary;
(iii) An involute of a circle; (iv) The tractrix;
(v) An equiangular spiral; (vi) A cycloid;
(vii) $\theta+2\sin^{-1}\sqrt{\frac{r}{2a}}+2\sqrt{\frac{2a-r}{r}}=\text{const.}$

PAGE 546.

2. $\frac{8a}{3}$.

PAGE 570.

2. $4a/\sqrt{3}$.

5. (i) $-$the area;
(ii) the area;
(iii) 0 or 2π, according as the origin lies within or without the area, there being one convolution about the pole; or if there be n convolutions, $2n\pi$.

10. Equiangular spirals. 12. $5a$. 13. Involute of a circle.

15. $2a[3\sqrt{3}+3\sqrt{2}+\log(\sqrt{2}+1)]$, $4a$ being the latus rectum.

17. Epicycloid. $2\frac{c^2-a^2}{a}$. 19. $4a$.

25. $\bar{x}=a\frac{\sqrt{2}}{3}(B+C)/A$, $\bar{y}=a\frac{\sqrt{2}}{3}(B-C)/A$,

where $A=\Big[\tan\psi-\psi\Big]_{\psi_1}^{\psi_2}$, $B=\Big[\sec\psi+\cos\psi\Big]_{\psi_1}^{\psi_2}$,

$$C=\left[\frac{\sin^3\psi}{\cos^2\psi}-\frac{3}{2}\frac{\sin\psi}{\cos^2\psi}-\frac{3}{2}\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right]_{\psi_1}^{\psi_2}, \text{ and } \psi=\frac{\pi}{4}-\theta.$$

[*E. T.*]

27. $s=\dfrac{2r\cos\frac{a}{2}}{\cos a}$. 28. $s=\dfrac{\pi}{16}\dfrac{1}{a^{\frac{5}{2}}b^{\frac{5}{2}}}\{3a^2+2ab+3b^2\}$.

29. Area $=\pi(a^2+2b^2)$. 30. $\frac{1}{2}\pi a^2$.

31. $s=\dfrac{1-m^2}{n}\displaystyle\int\frac{\sin\phi\,d\phi}{(\sin^2\phi+m^2\cos^2\phi)^{\frac{2n-1}{m}}}$.

39. $s=2a(\sec^3\psi-1)$. If $c=0$, the involute is $y^2=4a(x+2a)$.

40. $\dfrac{a^3-b^3}{a^2-b^2}$.

CHAPTER XVII.

PAGE 600.

2. $A=\frac{1}{\sqrt{2}}F_1$, $\left(\text{mod. }\frac{1}{\sqrt{2}}\right)=1{\cdot}31102\ldots$ square units.

PAGE 636.

2. With notation in *Diff. Calc.*, Art. 458,
$$\begin{cases} b=a, & A=2a^2, \\ b>a, & A=2b^2E_1, \text{ mod. } \dfrac{a^2}{b^2}. \\ b<a, & A=2a^2\left[E_1-\dfrac{a^4-b^4}{a^4}F_1\right], \text{ mod. } \dfrac{b^2}{a^2}. \end{cases}$$

10.
$$x=\frac{(y+\sqrt{y^2-4a^6})^{\frac13}+(y-\sqrt{y^2-4a^6})^{\frac13}}{2^{\frac13}},$$
$$x=11 \text{ or } -4\pm3\sqrt{-3}.$$

24. (i) $\tanh^{-1}\dfrac{\sqrt{x^4+2x^3-3x^2-4x+3}}{x^2+x-2}$;

(ii) $\dfrac{1}{3}\tanh^{-1}\dfrac{(x+2)\sqrt{R}}{x^3+3x^2-2-\dfrac{a}{2}}$, where $R=x^4+2x^3-3x^2-ax+a$;

(iii) $2\tanh^{-1}\dfrac{x}{x+3}\sqrt{\dfrac{x+2}{x+1}}$; (iv) $\tanh^{-1}\dfrac{x}{x+1}\sqrt{\dfrac{1+6x+4x^2}{1-2x+4x^2}}$;

(v) $\cosh^{-1}\dfrac{x^2+ax+a^2}{a\sqrt{2}}$; (vi) $\tanh^{-1}\dfrac{x}{x^3+1}\sqrt{x^4+1}$;

(vii) $2\tanh^{-1}(x+1)\sqrt{x}$; (viii) $\tanh^{-1}x\sqrt{x^4+1}$;

(ix) $2\tanh^{-1}\dfrac{x+b}{x+a}\sqrt{\dfrac{x^2+a^2}{x^2+b^2}}$; (x) $\tanh^{-1}x\sqrt{\dfrac{x+1}{x-1}}$;

(xi) $\tanh^{-1}\dfrac{x}{\sqrt{x^4-1}}$; (xii) $\tanh^{-1}\dfrac{x\sqrt{1+x^4}}{1+x}$.

CHAPTER XVIII.

PAGE 669.

4.
$$s=2ak^2\int\frac{\sqrt{1+m^2}\,dm}{(am^2-h)^2+4a^2m^2},$$
$y^2=4ax$ being the parabola, k^2 the const. of inversion, and $(h, 0)$ the pole.

10. $\dfrac{1}{r-1}\dfrac{1}{ca^{r-1}}$.

12. $I=\dfrac{2}{\sqrt{(1+\cos v)(\cosh u-\cos v)}}\sin^{-1}\sqrt{\dfrac{1+\cos v}{1+\cosh u}\cdot\dfrac{x^2-2x\cosh u+1}{x^2-2x\cos v+1}}$.

$$I_{e^u}^{\infty}=\frac{\sqrt{2}}{\cos\frac{v}{2}(\cosh u-\cos v)^{\frac12}}\sin^{-1}\left(\frac{\cos\frac{v}{2}}{\cosh\frac{u}{2}}\right).$$

14. $F_1(x-\sqrt{x^2-1})+\log F_2(x-\sqrt{x^2-1})$ **15.** $\dfrac{3a^2}{2}$.

17. (i) $\dfrac{1}{a-b}\sin^{-1}\dfrac{(x-a)^a}{(x-b)^b}$; (ii) $\dfrac{1}{a^q}\tan^{-1}\left(\dfrac{x^p}{a^q}+\dfrac{a^p}{x^q}\right)$.

19. $a=\frac{\pi}{2}-\frac{1}{2}\cos^{-1}e.$ **24.** $x\cos a+\sin a\log\sin(x-a).$

25. (i) $I=\frac{7}{25}\log(x+1)-\frac{1}{5}\frac{1}{x+1}-\frac{7}{50}\log(x^2+4)+\frac{1}{25}\tan^{-1}\frac{x}{2},$

$[I]_0^\infty=(\pi+14\log 2+10)/50$;

(ii) $I=\frac{1}{2^5}\left\{\frac{\sin 6\theta}{6}+\frac{3\sin 4\theta}{2}+\frac{15\sin 2\theta}{2}+10\theta\right\}$, where $\theta=\tan^{-1}x$,

$[I]_0^\infty=\frac{5\pi}{32}$;

(iii) $I=\frac{3}{16}\left\{\frac{\sin x}{5-3\cos x}+\frac{5}{6}\tan^{-1}2\tan\frac{x}{2}\right\},$

$[I]_0^\pi=\frac{5\pi}{64}.$

CHAPTER XIX.

Page 723.

3. $\frac{1}{2}(r_2^2-r_1^2)\tan a\sin^2 a$ $(r_1=OP_1,\ r_2=OP_2).$

6. Evolute of roulette of the cusp is a four cusped hypocycloid Intrinsic equation of envelope of axis with notation of Ex. 2, Art. 670, is

$$s=a\sin^2\frac{\chi}{3}\left(5+7\cos^2\frac{\chi}{3}\right).$$

20. See Art. 657.

25. The rolling of a catenary upon a straight line.

30. $s=a\psi-3a\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)+\text{const.}$

CHAPTER XX.

Page 772.

6. $\text{Arc}=\frac{a}{\sqrt{2}}\left[\frac{1}{R}\log\frac{z^2-Rz+\sqrt{2}}{z^2+Rz+\sqrt{2}}-R\tan^{-1}\frac{2z}{R(\sqrt{2}-z^2)}\right]_{\theta_1}^{\theta_2},$

where $R^2=2(\sqrt{2}+1)$, $z=\cos\frac{\theta}{2}$, and θ is the azimuthal angle of a point on the curve.

CHAPTER XXI.

Page 790.

2. $\pi^2a^3.$ **3.** $\frac{8\sqrt{2}}{15}\pi a^3.$ **5.** $\frac{4}{3}\pi a^3(3\log 2-2).$

6. $\begin{pmatrix}x=a\cos\theta\\ y=b\sin\theta\end{pmatrix}\Big\}.$ For surface from $\theta=\theta_1$ to $\theta=\theta_2$, revolution about the y-axis,

$$S=\pi a\left[\sin\theta\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}+\frac{1-e^2}{e}a\log\left\{ae\sin\theta+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\right\}\right]_{\theta_1}^{\theta_2}.$$

8. $\dfrac{\pi a^3}{12}$. **10.** About axis, $\frac{8}{3}\pi a^2(3\pi-4)$; about base, $\frac{64}{3}\pi a^2$.

11. $\dfrac{\pi^2 a^3}{4\sqrt{2}}$. **14.** $\dfrac{\pi^2 a^3}{2}$. **16.** $\dfrac{4\pi n^3 a^3 \sin\frac{\pi}{n}}{(n^2-1)(9n^2-1)}$. **22.** A circular cylinder.

27. $\dfrac{\pi}{3c^3}(\sqrt{a+c}-\sqrt{a-c})\{a(c-2a)\sqrt{a+c}+(2a^2+ac+2c^2)\sqrt{a-c}\}$.

29. $\dfrac{2\pi}{3}\{(1+2h)^{\frac{3}{2}}-1\}$.

CHAPTER XXII.

Page 862.

1. In each case $V=\dfrac{h}{3}(A+\sqrt{AB}+B)$, where $h=$ height of frustum and A, B the areas of the ends.

2. $\frac{1}{3}Ea^3$, a being the radius of the sphere and E the spherical excess.

8. $\displaystyle\int\frac{dS}{p^3}=\frac{4}{3}\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2$. **9.** $\frac{4}{3}\pi abc$.

21. $\dfrac{d_1 d_2 d_3}{\Delta}$, where $\Delta=\begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix}$.

29. $\dfrac{\pi}{2}(x_2-x_1)\left\{\alpha^2+\alpha'^2+(\beta+\beta')\dfrac{x_2+x_1}{2}\right\}$.

31. $\dfrac{1}{4}\dfrac{1+\gamma}{1-\gamma}\left(\alpha_1^{\frac{2}{1+\gamma}}-\alpha_2^{\frac{2}{1+\gamma}}\right)\left\{(4\beta_1)^{\frac{1-\gamma}{1+\gamma}}-(4\beta_2)^{\frac{1-\gamma}{1+\gamma}}\right\}(\tan^{-1}b_1-\tan^{-1}b_2)$.

39. $\dfrac{4\pi bc}{a^2}(a\cosh a-\sinh a)$, $\dfrac{4\pi abc}{(a^2+b^2)^{\frac{3}{2}}}(\sqrt{a^2+b^2}\cosh\sqrt{a^2+b^2}-\sinh\sqrt{a^2+b^2})$,

$\dfrac{4\pi abc}{(a^2+b^2+c^2)^{\frac{3}{2}}}(\sqrt{a^2+b^2+c^2}\cosh\sqrt{a^2+b^2+c^2}-\sinh\sqrt{a^2+b^2+c^2})$.

43. Envelope $y=\pm x$, $y^4(x^4-a^4)+a^4x^4=0$.

50. $\dfrac{\pi}{2}-1$. **51.** $\dfrac{\pi}{(p-1)c^{2(p-1)}}$.

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